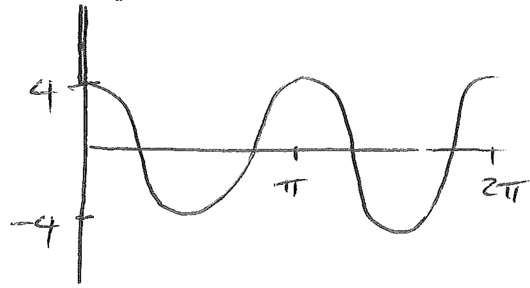
Day 9/10: Review and Thinking Questions

1. Determine the values of x, $0 \leq x \leq 2\pi$, that will result in a y value greater than or equal to 3 for the equation $y = 4 - 8\sin^2 x$. Round answers to 4 decimal places.

$y = 4(1 - 2\sin^2 x) = 4\cos 2x$



$4\cos 2x = 3$

$\cos 2x = \frac{3}{4}$ period $= \pi$

$2x_1 = 0.7227$

$x_1 = 0.3614$

$x_3 = 0.3614 + \pi$

$= 3.5030$

$2x_2 = 2\pi - 0.7227$

$x_2 = 2.7802$

$x_4 = 2.7802 + \pi$

$= 5.9218$

$\therefore y \geq 3$ if

$x \in [0, 0.3614] \cup [2.7802, 3.5030]$

$\cup [5.9218, 2\pi]$

2. The Quadratic trigonometric equation $a\cos^2 x + b\cos x - 1 = 0$ has the solutions $\frac{\pi}{3}, \pi, \frac{5\pi}{3}$ in the interval $0 \leq x \leq 2\pi$. What are the values of a and b?

$x = \pi \Rightarrow$ $a\cos^2\pi + b\cos\pi - 1 = 0$

$a - b - 1 = 0$

$\boxed{a - b = 1}$ (1)

$x = \frac{\pi}{3}$: $a\cos^2\left(\frac{\pi}{3}\right) + b\cos\frac{\pi}{3} - 1 = 0$

$a\left(\frac{1}{4}\right) + b\left(\frac{1}{2}\right) - 1 = 0$

$\frac{a}{4} + \frac{b}{2} = 1$

$\boxed{a + 2b = 4}$ (2)

use substitution or elimination to solve for 'a' and 'b'

(1) $a - b = 1$

(2) $a + 2b = 4$

(1)-(2) $-3b = -3$

$\boxed{b = +1}$

sub $b = +1$ in (1)

$a - b = 1$

$a - (+1) = 1$

$\boxed{a = 2}$

3. Solve $6\sin^3 x - 13\sin^2 x + 2\sin x + 5 = 0$. Determine exact values if possible, otherwise round to 2 decimal places. $P(x) = 6x^3 - 13x^2 + 2x + 5$ $P(1) = 0 \Rightarrow x - 1$ is a factor

$$\begin{array}{r|rrrr} 1 & 6 & -13 & 2 & 5 \\ & \downarrow & 6 & -7 & -5 \\ \hline & 6 & -7 & -5 & 0 \end{array}$$

$P(x) = (x-1)(6x^2 - 7x - 5)$

$= (x-1)(3x-5)(2x+1)$

$x = 1, \ 5/3, \ -\frac{1}{2}$

$\therefore 6\sin^3 x - 13\sin^2 x + 2\sin x + 5 = 0$

$\therefore \sin x = 1$ $\quad \sin x = \frac{5}{3}$ $\quad \sin x = -\frac{1}{2}$ $\quad \alpha = \frac{\pi}{6}$

$x = \frac{\pi}{2}$ $\quad$ No soln since $\sin x \le 1$

$x = \pi + \frac{\pi}{6}, \ 2\pi - \frac{\pi}{6}$

$= \frac{7\pi}{6}, \ \frac{11\pi}{6}$

$\therefore x = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$

4. Solve $\cos 2x - 3\cos x - 3 - \cos^2 x = \sin^2 x$ for $0 \le x \le 2\pi$

$2\cos^2 x - 1 - 3\cos x - 3 - \cos^2 x - \sin^2 x = 0$

$2\cos^2 x - 1 - 3\cos x - 3 - \underbrace{(\cos^2 x + \sin^2 x)}_{1} = 0$

$2\cos^2 x - 3\cos x - 5 = 0$

$(2\cos x - 5)(\cos x + 1)$

$\cos x = \frac{5}{2}$ $\quad \cos x = -1$

No soln $\quad \boxed{\therefore x = \pi}$

5. a) Show that $4 - \cos 2\theta + 5\sin\theta = 2\sin^2\theta + 5\sin\theta + 3$

b) Hence, solve the equation $4 - \cos 2\theta + 5\sin\theta = 0$ for $0 \le x \le 2\pi$

a) $LS = 4 - \cos 2\theta + 5\sin\theta$

$= 4 - [1 - 2\sin^2\theta] + 5\sin\theta$

$= 2\sin^2\theta + 5\sin\theta + 3$

b) $2\sin^2\theta + 5\sin\theta + 3 = 0$

$(2\sin\theta + 3)(\sin\theta + 1) = 0$

$\sin\theta = -\frac{3}{2}$ $\quad \sin\theta = -1$

No soln $\quad \boxed{\theta = \frac{3\pi}{2}}$

6. Solve $\frac{\sin 4x}{2\sin x} = \frac{1}{2}$ for $0 \le x \le 2\pi$

See the next page!

7. If x satisfies the equation $\sin\left(x + \frac{\pi}{3}\right) = 2\sin x \sin\left(\frac{\pi}{3}\right)$, show that $11\tan x = a + b\sqrt{3}$, where a, b are positive integers.

$\sin x \cos\frac{\pi}{3} + \sin\frac{\pi}{3}\cos x = 2\sin x \sin\frac{\pi}{3}$

$(\sin x)\left(\frac{1}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\cos x = (2\sin x)\left(\frac{\sqrt{3}}{2}\right)$

$\sin x + \sqrt{3}\cos x = 2\sqrt{3}\sin x$

$(2\sqrt{3} - 1)\sin x = \sqrt{3}\cos x$

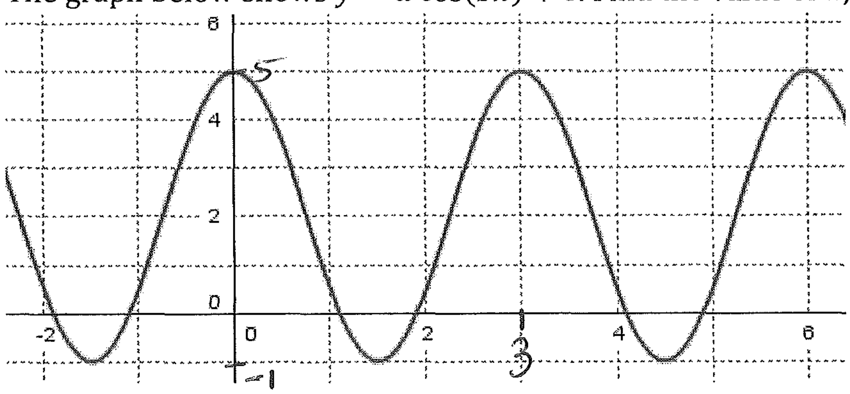
$\frac{\sin x}{\cos x} = \frac{\sqrt{3}}{2\sqrt{3}-1} \cdot \frac{2\sqrt{3}+1}{2\sqrt{3}+1} \Rightarrow \tan x = \frac{6+\sqrt{3}}{11}$

$11\tan x = a + b\sqrt{3}$

$11\tan x = 6 + \sqrt{3}$

$\boxed{\therefore\ a = 6 \quad b = 1}$

8. The graph below shows $y = a\cos(bx) + c$. Find the value of a, b, and c.



$a = \frac{max - min}{2}$

$= 3$

$b = \frac{2\pi}{period} = \frac{2\pi}{3}$

$c = \frac{max + (min)}{2}$

$= 2$

$\therefore a = 3,\quad b = \frac{2\pi}{3},\quad c = 2$

(6) $\dfrac{\sin 4x}{2\sin x} = \dfrac{1}{2}$ $\quad 0 \le x \le 2\pi$

$\dfrac{\sin 4x}{\sin x} = 1$

$\sin 4x / \sin x = 1$

$2\sin 2x\cos 2x / \sin x = 1$

$\dfrac{2(2\cancel{\sin x}\cos x)\cos 2x}{\cancel{\sin x}} = 1$

$[4\cos x\cos 2x - 1] = 0$

$[4\cos x(2\cos^2 x - 1) - 1] = 0$

$[8\cos^3 x - 4\cos x - 1] = 0$

Let $P(x) = 8x^3 - 4x - 1$

$P(-\frac{1}{2}) = 0 \Rightarrow 2x+1$ is a factor

$$\begin{array}{r|rrrr} -\frac{1}{2} & 8 & 0 & -4 & -1 \\ & \downarrow & -4 & 2 & 1 \\ \hline & 8 & -4 & -2 & 0 \end{array}$$

$\therefore P(x) = (2x+1)(4x^2 - 2x - 1)$

$\therefore x = -\frac{1}{2}$

$x = \dfrac{2 \pm \sqrt{(-2)^2 - 4(4)(-1)}}{8}$

$= \dfrac{2 \pm \sqrt{20}}{8}$

$x_1 = 0.81 \quad x_2 = -0.31$

$\cos x = -\frac{1}{2} \quad \alpha = \frac{\pi}{3}$

$x = \pi - \frac{\pi}{3}, \ \pi + \frac{\pi}{3}$

$= \frac{2\pi}{3}, \ \frac{4\pi}{3}$

$\cos x = +0.81$

$x \doteq 0.63$ or $2\pi - 0.63$

$\doteq 0.63$ or 5.65

$\cos x = -0.31 \quad \alpha = 1.26$

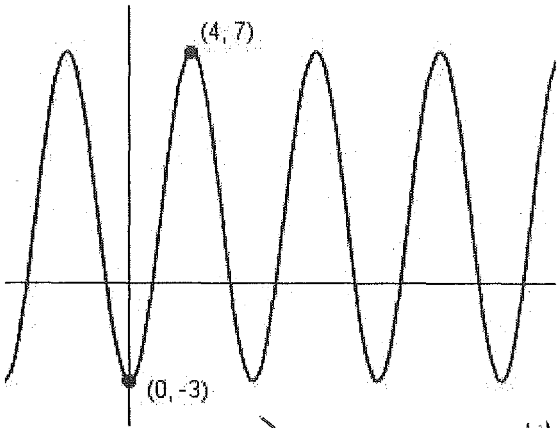
$x \doteq \pi - 1.26, \ \pi + 1.26$

$= 1.88, \ 4.40$

$\therefore x = \frac{2\pi}{3}, \frac{4\pi}{3},$

$0.63, 5.65, 1.88, 4.40$

9. The graph of $y = p\cos qx + r$, for $-5 \leq x \leq 14$, is shown below.
   a) Find the value of p, q, and r.
   b) The equation $y = k$ has exactly two solutions. Write down the value of k.



$\therefore \text{amp} = \frac{\max - \min}{2} = \frac{7-(-3)}{2} = 5$

$r = \text{a.o.c} = \frac{\max + \min}{2} = \frac{7+(-3)}{2} = 2$

Period $= 8 \Rightarrow k = q = \frac{2\pi}{8} = \frac{\pi}{4}$

a) $p = -\text{amplitude} = -5$, $q = \frac{\pi}{4}$, $r = 2$

b) $k = -3$

10. Simplify $\dfrac{\cos(\pi+x)\cos\left(\dfrac{\pi}{2}+x\right)}{\cos(\pi-x)} - \dfrac{\sin\left(\dfrac{3\pi}{2}\right)}{\sec(\pi+x)}$

$= \dfrac{(\cos\pi\cos x - \sin\pi\sin x)(\cos\frac{\pi}{2}\cos x - \sin\frac{\pi}{2}\sin x)}{\cos\pi\cos x - \sin\pi\sin x} - \dfrac{\sin\frac{3\pi}{2}}{\dfrac{1}{\cos\pi\cos x - \sin\pi\sin x}}$ $\quad = -1$

$= \dfrac{(-\cos x - \cancel{0\sin x})(\cancel{0\cos x} - 1\sin x)}{-1\cos x - 0\sin x} + (-\cos x - 0\sin x)$

$= \dfrac{(\cancel{-\cos x})(-\sin x)}{\cancel{-\cos x}} - \cos x$

$= -\sin x - \cos x$

$= -(\sin x + \cos x)$

11. If $\tan\left(\frac{\pi}{4}+x\right)=3\tan\left(\frac{\pi}{4}-x\right)$, find the value of $\tan x$.

(1) $\tan\left(\frac{\pi}{4}+x\right)=\frac{\tan\frac{\pi}{4}+\tan x}{1-\tan\frac{\pi}{4}\tan x}=\frac{1+\tan x}{1-\tan x}$

(2) $3\tan\left(\frac{\pi}{4}-x\right)=3\left[\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}\right]=3\left[\frac{1-\tan x}{1+\tan x}\right]$

(3)

$$\frac{1+\tan x}{1-\tan x}=3\left[\frac{1-\tan x}{1+\tan x}\right]$$

$$(1+\tan x)^2=3(1-\tan x)^2$$

$$1+2\tan x+\tan^2 x=3(1-2\tan x+\tan^2 x)$$

$$2\tan^2 x-8\tan x+2=0$$

$$\tan x=\frac{+8\pm\sqrt{(-8)^2-4(2)(2)}}{4}$$

(4)

$$\tan x=\frac{8\pm\sqrt{48}}{4}$$

$$=\frac{8\pm\sqrt{16}\sqrt{3}}{4}$$

$$\boxed{\tan x = 2\pm\sqrt{2}}$$

12. If $\cos\theta+\sin\theta=\frac{2}{3}$, find the value of $\sin 2\theta$.

$$(\cos\theta+\sin\theta)^2=\left(\frac{2}{3}\right)^2$$

$$\cos^2\theta+2\sin\theta\cos\theta+\sin^2\theta=\frac{4}{9}$$

$$2\sin\theta\cos\theta+1=\frac{4}{9}\Rightarrow \sin 2\theta=\frac{4}{9}-1\Rightarrow \sin 2\theta=\frac{-5}{9}$$

13. If $\cos\theta+\sin\theta=\frac{1+\sqrt{3}}{2}$ and $\cos\theta-\sin\theta=\frac{1-\sqrt{3}}{2}$, find the value of $\sin 2\theta$.

$$(\cos\theta+\sin\theta)(\cos\theta-\sin\theta)=\left(\frac{1+\sqrt{3}}{2}\right)\left(\frac{1-\sqrt{3}}{2}\right)$$

$$\cos^2\theta-\sin^2\theta=\frac{1}{4}(1-3)=-\frac{2}{4}=\frac{-1}{2}$$

$$\cos 2\theta=\frac{-1}{2}\rightarrow$$

$$2\cos^2\theta-1=\frac{-1}{2}$$

$$2\cos^2\theta=\frac{1}{2}$$

$$\cos^2\theta=\frac{1}{4}$$

$$\cos\theta=\pm\frac{1}{2}$$

$$1-2\sin^2\theta=\frac{-1}{2}$$

$$-2\sin^2\theta=-\frac{3}{2}$$

$$\sin^2\theta=\frac{-3}{-4}$$

$$\sin\theta=\pm\sqrt{\frac{3}{4}}=\frac{\pm\sqrt{3}}{2}$$

$$\sin 2\theta=2\sin\theta\cos\theta$$

$$=2\left(\pm\frac{1}{2}\right)\left(\pm\frac{\sqrt{3}}{2}\right)=\pm\frac{\sqrt{3}}{2}$$

$$(a+b)^2-(a-b)^2$$
$$=a^2+2ab+b^2-(a^2-2ab+b^2)$$
$$=4ab$$

14. Simplify $\sin^2\left(\frac{\pi}{8}+\frac{\theta}{2}\right)-\sin^2\left(\frac{\pi}{8}-\frac{\theta}{2}\right)$.

$$=\Big(\underbrace{\sin\tfrac{\pi}{8}\cos\tfrac{\theta}{2}}_{a}+\underbrace{\sin\tfrac{\theta}{2}\cos\tfrac{\pi}{8}}_{b}\Big)^2-\Big(\underbrace{\sin\tfrac{\pi}{8}\cos\tfrac{\theta}{2}}_{a}-\underbrace{\sin\tfrac{\theta}{2}\cos\tfrac{\pi}{8}}_{b}\Big)^2$$

$$=4ab$$

$$=4\left(\sin\tfrac{\pi}{8}\cos\tfrac{\theta}{2}\right)\left(\sin\tfrac{\theta}{2}\right)\cos\tfrac{\pi}{8}$$

$$=\cancel{4}\left[2\sin\tfrac{\pi}{8}\cos\tfrac{\pi}{8}\right]\left[2\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}\right]=\sin\left(\tfrac{2\pi}{8}\right)\sin\left(2\tfrac{\theta}{2}\right)$$

$$=\frac{\sin\theta}{\sqrt{2}}=\frac{\sqrt{2}\sin\theta}{2}$$

15. Prove that $\tan(x+y+z)=\dfrac{\tan x+\tan y+\tan z-\tan x\tan y\tan z}{1-\tan x\tan y-\tan x\tan z-\tan y\tan z}$.

$$LS=\tan[(x+y)+z]$$

$$=\frac{\tan(x+y)+\tan z}{1-\tan(x+y)(\tan z)}=\frac{\dfrac{\tan x+\tan y}{1-\tan x\tan y}+\tan z}{1-\left[\dfrac{\tan x+\tan y}{1-(\tan x)(\tan y)}\right]\tan z}$$

$\rightarrow$ multiply by $1-\tan x\tan y$

$$=\frac{\tan x+\tan y+\tan z(1-\tan x\tan y)}{(1-\tan x\tan y)-(\tan x+\tan y)(\tan z)}$$

$\leftarrow$ expand

$$=RS$$

$LS=RS$

QED.

16. Find the value of $\cos^2 1^0+\cos^2 2^0+\cos^2 3^0+\ldots+\cos^2 89^0+\cos^2 90^0$.

$[\sin x=\cos(90-x)]$

$[\cos x=\sin(90-x)]$

$$=\cos^2 1^\circ+\cos^2 2^\circ+\cos^2 3^\circ+\ldots+\sin^2 1^\circ+0$$

$$=\underbrace{\cos^2 1+\sin^2 1}_{1}+\underbrace{\cos^2 2+\sin^2 2}_{1}+\ldots+\cos^2 45$$

$$=(1)(44)+\left(\frac{1}{\sqrt{2}}\right)^2$$

$$=44.5$$

OR

$x = \cos^2 1 + \cos^2 2 + \dots + \cos^2 45 + \dots + \cos^2 89 + \cos^2 90$

$x = \cos 90 + \cos^2 89 + \cos^2 88 + \dots + \cos^2 45 + \dots + \cos^2 1$

$\therefore 2x = [\cos^2 1 + \cos^2 89] + [\cos^2 2 + \cos^2 88] + \dots + 2\cos^2 45$

$2x = [\cos^2 1 + \sin^2 1] + [\cos^2 2 + \sin^2 2] + \dots + 2\cos^2 45$

$2x = (1) + 1 + \dots + 2\left(\frac{1}{\sqrt{2}}\right)^2$

1

$2x = 89$

$x = 44.5$