UNIT 4: CURVE SKETCHING

Date	*Day*	*Topic*	*Homework*
	1	4.1 Increasing and Decreasing Functions	Pg 169 # 1, 3(a,b), 5-6, 9, 11
	2	4.2 Critical points, local maxima, local minima	Pg 178 #2-5, 7 (a,c,f), 8, 15
	3	4.3 Vertical and Horizontal Asymptotes	Pg 193 #1-3 4(a,c), 5(d), 7(d), 10 ab, 17
	4	4.4 Concavity and Points of Inflection	Pg 205 #1-5, 8, 9 14
	5	4.5 Algorithm for curve sketching	Pg 212 # 3, 4, 6, 7, 10(b)
	7	Chapter Review	Pg. 196 #2, 4-17 Pg. 216 #3 – 7, 10, 12 – 15, 16a, 17, 19
	8	*Chapter Test*	

Learning Goals

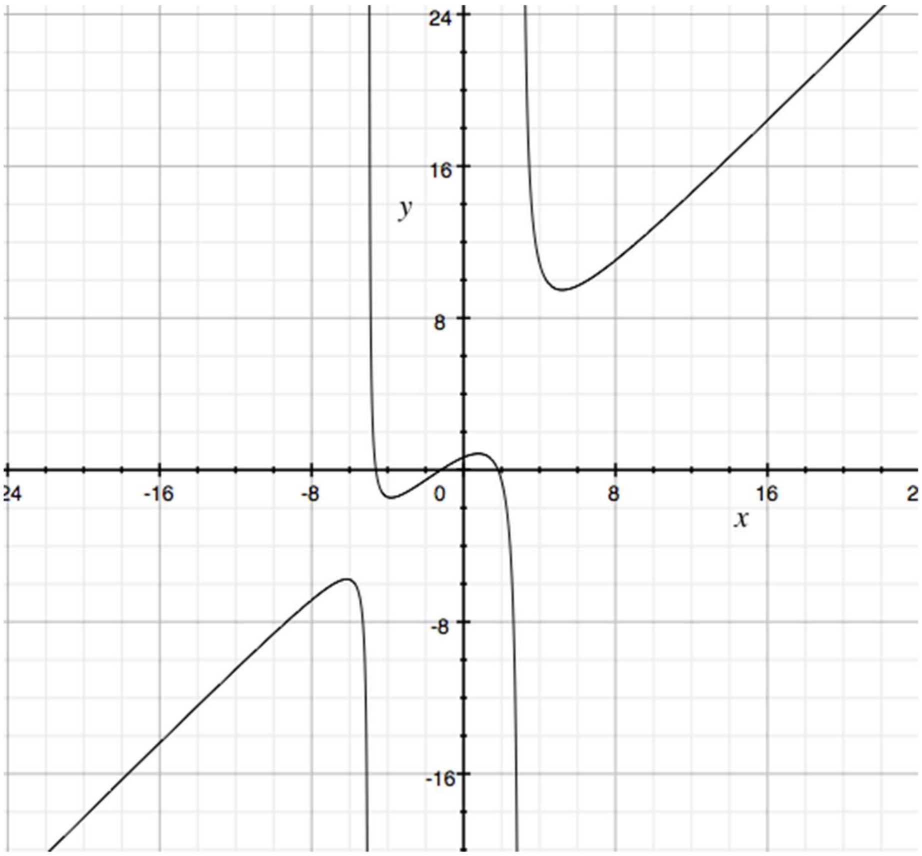
In this unit, we will learn:

- *Domain, Range, Increasing, Decreasing, Critical Points*
- *Vertical, Horizontal Asymptotes, Concavity of a curve*
- *Algorithm for curve sketching*

Day 1: 4.1 Increasing and Decreasing Functions

INVESTIGATION:

The graph of the function $y = \frac{x^3 + 4x^2 - 5x - 10}{x^2 + 2x - 1}$



Discussion: IMPORTANT INFORMATION

Domain:

Range

Asymptote(s):

End Behaviour:

Special Points

- *Intercepts*
- *Local (Relative) Maxima/Minima*
- *Inflection*

Intervals:

Increase:

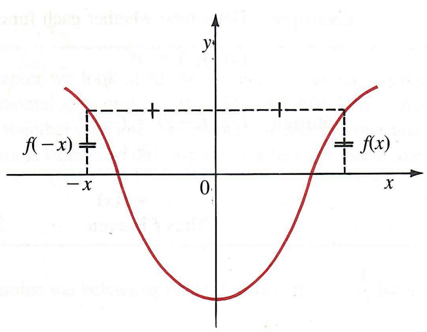
Decrease:

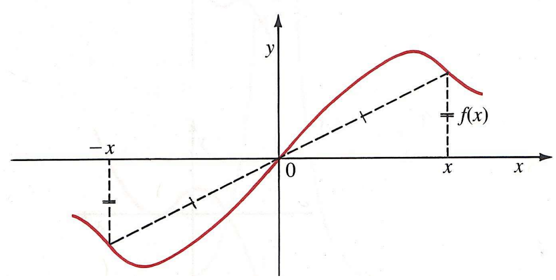
INTERVAL NOTATIONS

Remember this table?

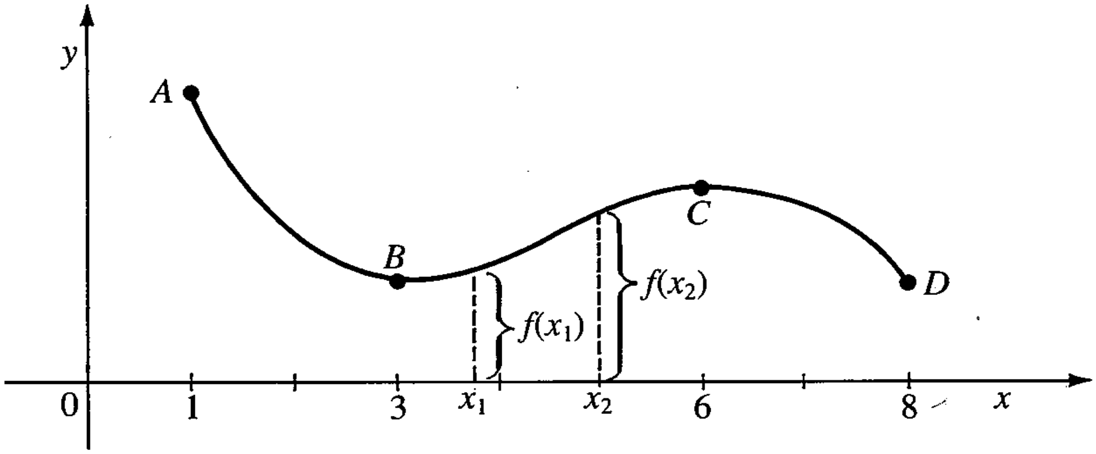
Bracket Interval	Inequality	Number Line	In Words
			The set of all real numbers x such that
(a, b)	$a < x < b$	a b $\mathbb{R}$	x is greater than a and less than b
$(a, b]$	$a < x \leq b$	a b $\mathbb{R}$	x is greater than a and less than or equal to b
$[a, b)$	$a \leq x < b$	a b $\mathbb{R}$	x is greater than or equal to a and less than b
$[a, b]$	$a \leq x \leq b$	a b $\mathbb{R}$	x is greater than or equal to a and less than or equal to b
$[a, \infty)$	$x \geq a$	a $\mathbb{R}$	x is greater than or equal to a
$(-\infty, a]$	$x \leq a$	a $\mathbb{R}$	x is less than or equal to a
(a, ∞)	$x > a$	a $\mathbb{R}$	x is greater than a
$(-\infty, a)$	$x < a$	a $\mathbb{R}$	x is less than a
$(-\infty, \infty)$	$-\infty < x < \infty$	$\mathbb{R}$	x is an element of the real numbers

EVEN OR ODD FUNCTION?





INCREASING AND DECREASING FUNCTION



In general, a function f is called **increasing on an interval I** if

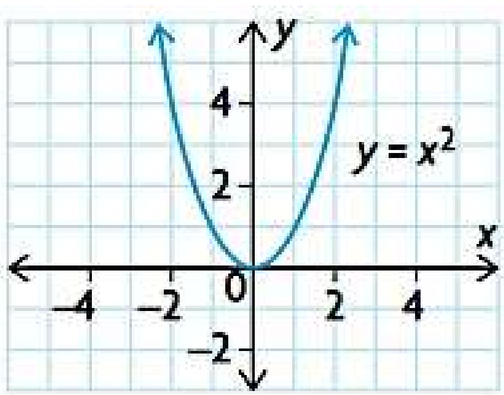
$$f(x_1) < f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ in I}$$

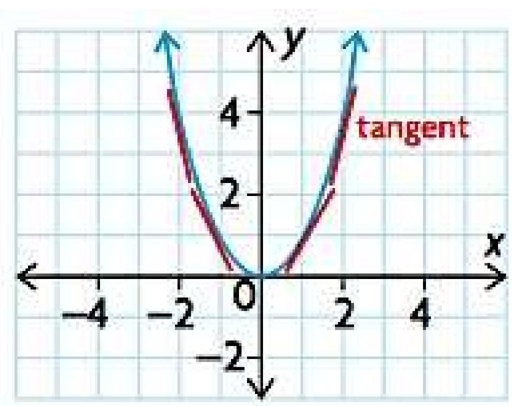
It is called **decreasing on I** if

$$f(x_1) > f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ in I}$$

Which interval the above function is increasing? Which interval it is decreasing?

State the interval that y = x^2 is increasing; the interval it is decreasing.





What do you notice about the tangent line in each interval?

Test for Increasing & Decreasing Intervals

1. *If $f'(x) > 0$ for all x in an interval I, then $f(x)$ is increasing on I*
2. *If $f'(x) < 0$ for all x in I, then $f(x)$ is decreasing on I*

Example 1: Find the intervals on which the function $f(x) = 1 - 5x + 4x^2$ is increasing and decreasing

x		
$f'(x)$		
$f(x)$		

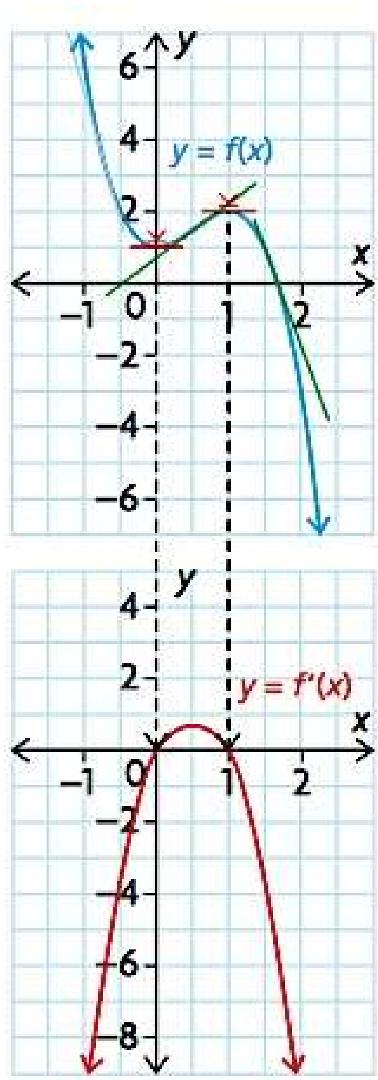
Example 2: Where is the function $y = x^3 + 6x^2 + 9x + 2$ increasing?

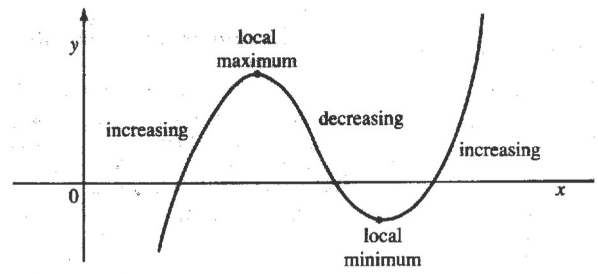
x			
$f'(x)$			
$f(x)$			

Example 3: Find the intervals of increase and decrease for the function $g(x) = x^4 - 4x^3 - 8x^2 - 1$

x				
$f'(x)$				
$f(x)$				

Day 2: 4.2 Critical Points, Local Maxima, Local Minima





Observe the graph of $f(x) = -2x^3 + 3x^2 + 1$ and $f'(x) = -6x(x-1)$

What do you notice about the two graphs?

CRITICAL NUMBERS

A **critical number** of a function is a number c such that either $f'(c) = 0$ or $f'(c)$ is undefined.

For (i) Rational functions of the form $y = \dfrac{f(x)}{g(x)}$:

We need to find the x's that make the "top" zero and the x's that make the "bottom" zero.
In other words, we need to solve the two equations of $f(x) = 0$ and $g(x) = 0$

(ii) For Non-Rational functions of $y = f(x)$:
the job is to solve the equation of $f(x) = 0$

Examples: Find all the critical numbers for the following functions:

1. $y = x^2 - 5$.

2. $y = 6x^3 - 26x^2 - 20x$.

3. $y = x^3 - 4x^2 + 2x - 5$

4. $y = \dfrac{x^2 - 16}{x^2 + 6x + 8}$

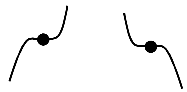
5. $y = \dfrac{x^2 + 25}{x^3 + 8}$

Graphical Meaning At The Critical Numbers

Case 1:
$f'(c) = 0$ means the slope of the graph at $x = c$ is zero. A zero slope can be one of the following situations.

- a local maximum or local minimum.

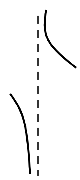
- a point of inflection

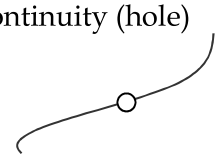


Case 2:
$f'(c) =$ undefined means the slope at $x = c$ is undefined. It can be one of the following situations.

- the graph is **discontinuous** at $x = c$.

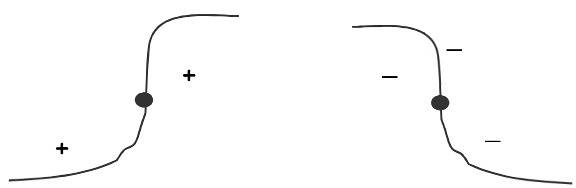
e.g Vertical Asymptote or Point of Discontinuity (hole)





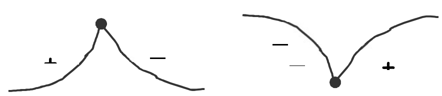
These have been discovered when you analyze $y = f(x)$.

- the graph has a **vertical tangent** or slope at $x = c$.



POINTS WITH VERTICAL SLOPE

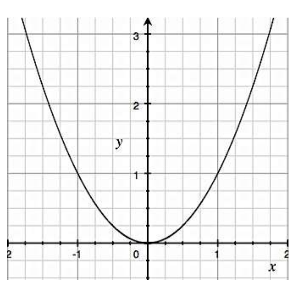
- the graph has a **cusp** at $x = c$.



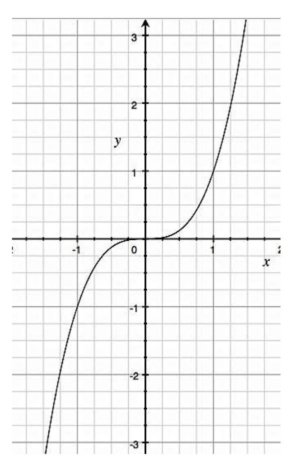
CUSPS

Examples:

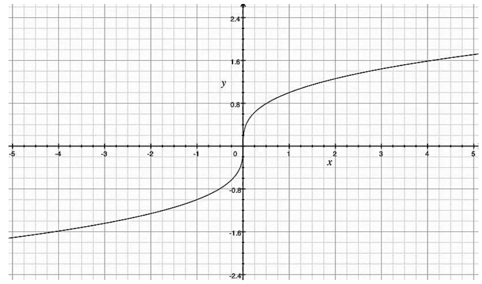
$f(x) = x^2$
$f'(x) = 2x$



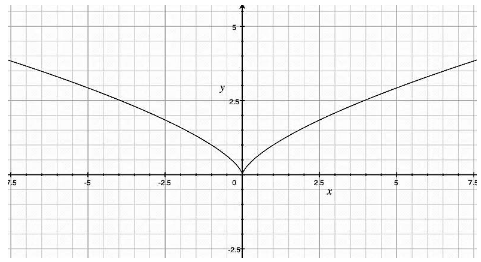
$f(x) = x^3$
$f'(x) = 3x^2$



A vertical tangent $f(x)=\sqrt[3]{x}$



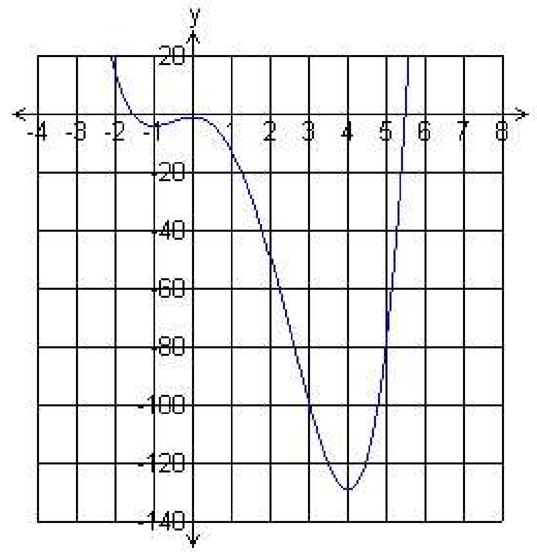
A cusp $f(x)=x^{\frac{2}{3}}$



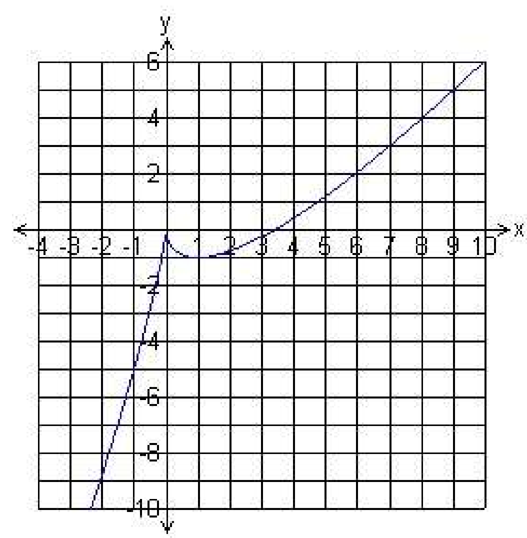
FIRST DERIVATIVE TEST
Let c be a critical number of a continuous function *f.*
1. If f'*(x)* changes from positive to negative at c, then f has a local maximum at c.
2. If f'(*x*) changes from negative to positive at c, then f has local minimum at c.
3. If f'(*x*) does not change sign at c, then f has no maximum or minimum at c.

Ex 1: Find the local maximum and minimum values of $f(x) = x^3 - 3x + 1$

Ex 2: Find the local maximum and minimum of $g(x)=x^4 - 4x^3 - 8x^2 - 1$ and use this information to sketch the graph of g(x).



Ex 3: Find the critical numbers, interval of increase and decrease, and local max and min of $f(x)= 2x - 3x^{2/3}$



Day 3: 4.3 Vertical and Horizontal Asymptotes

Vertical Asymptotes

Example 1: Find $\lim_{x\to 0^+} \frac{1}{x}$ and $\lim_{x\to 0^-} \frac{1}{x}$.

Example 2: Find $\lim_{x\to 6}\left[2-\frac{5}{(x-6)^2}\right]$.

To find vertical asymptotes of rational functions, we find the values of x where the **denominator** is zero and compute the limits of the function from the right and left.

Example 3 a) Find the vertical asymptotes of the function $y=\frac{x}{x^2-x-6}$.

b) Sketch the graph near the asymptotes.

Horizontal Asymptotes

The line $y = L$ is called a horizontal asymptote of the curve if either $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$.

To find H.A., we need to consider x approaches $+\infty$ and $-\infty$.

Example 4 Find $\lim_{x\to\infty} \frac{1}{x}$ and $\lim_{x\to-\infty} \frac{1}{x}$.

Example 5 Evaluate $\lim_{x\to\infty} \frac{4x^2 - x + 2}{6x^2 + 5x + 1}$.

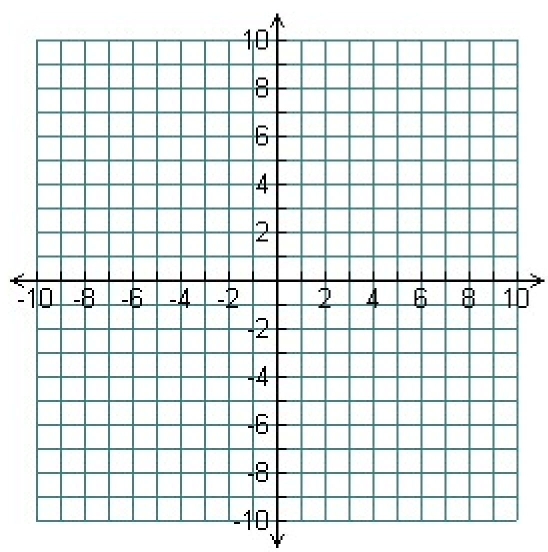
Example 6

Determine the value of each of the following:

a. $\lim_{x\to+\infty} \frac{2x-3}{x+1}$ b. $\lim_{x\to-\infty} \frac{x}{x^2+1}$ c. $\lim_{x\to+\infty} \frac{2x^2+3}{3x^2-x+4}$

Example 7 Find the HA & VA's of the function and sketch its graph near the asymptotes.

$$y = \frac{x+1}{x-2}$$



Vertical Asymptote, Point of Discontinuity & *X*-intercept Of A Rational Function

Find all 'special' values of x which make the denominator or numerator zero.

Example 8 $\quad y = \frac{x+1}{x-2}$

- *when $x = 2$, $y = \frac{3}{0}$ (undefined). There is a vertical asymptote at $x = 2$.*
- *when $x = -1$, $y = \frac{0}{-3} = 0$. There is an x-intercept at $(-1, 0)$.*

Example 9 $\quad y = \frac{(x+3)(x-5)}{(x-2)(x+3)}$

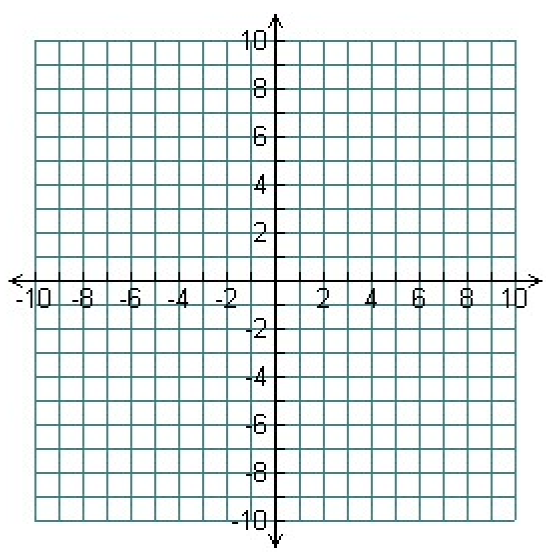
- *when $x = 2$, $y = \frac{-3}{0}$ (undefined). There is a* ***vertical asymptote*** *at $x=2$*
- *when $x = -3$, $y = \frac{0}{0}$ (undefined) . There is a* ***point of discontinuity*** *('hole') on the curve.*

To find the location of this 'hole', we can sub in $x = -3$ **after** reducing (cross out) the common factor (x+3) from the numerator and the denominator.
The 'hole' is located at $\left(-3, \frac{8}{5}\right) = (-3, 1.6)$.

- *when $x = 5$, $y = \frac{0}{3} = 0$. There is an x-intercept at $(5, 0)$.*
- *The chart to determine whether the graph is above or below the x-axis.*

-3 2 5

x				
y				



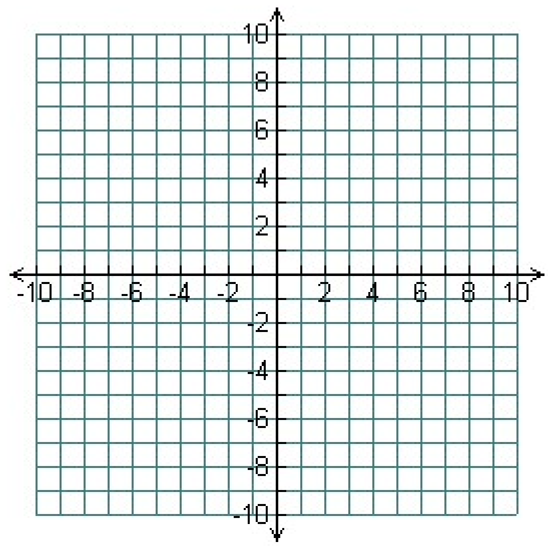
Use limits to describe the behaviour of the graph near the vertical asymptote.

$\lim_{x\to 2^-} f(x) = +\infty \qquad \lim_{x\to 2^+} f(x) = -\infty$

SUMMARY

- y = $\frac{0}{\#} = 0$ x-intercept
- y = $\frac{0}{0}$ = undefined a hole'
- y = $\frac{\#}{0}$ = undefined vertical asymptote

Practice: Sketch the rational function $f(x) = \dfrac{x^2 - 16}{2x^2 - x - 28}$.

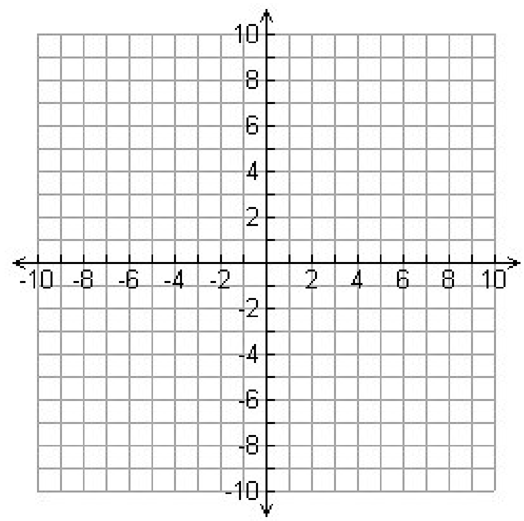


Oblique Asymptotes

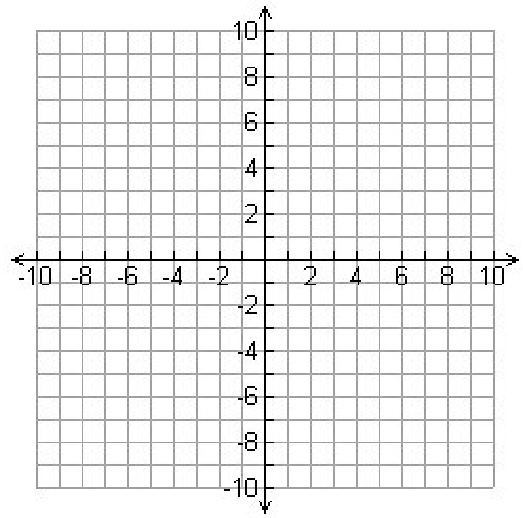
Oblique asymptotes are straight lines (not parallel to any axes) that the curve can cross locally but that the curve approaches infinitely closely at the extremes (same behaviour as the horizontal asymptote). They occur with rational functions in which the degree of the numerator polynomial ix exactly one degree more than the degree of the denominator polynomial. When this happens, there is NO horizontal asymptote, although vertical asymptotes are still possible.

Find the Oblique Asymptote for each of the following:

Ex 1: $\boldsymbol{f(x) = \frac{x^2 - 3x + 6}{x - 1}}$



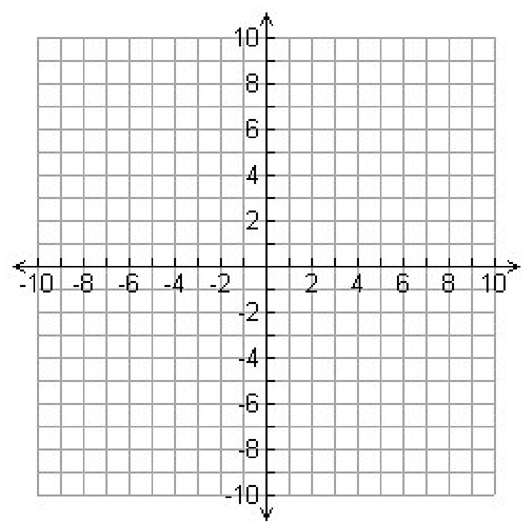
Ex 2: $f(x) = \frac{2x^3 - 3x^2 + x - 3}{x^2 + 1}$



Go back to page 2 for the graph.

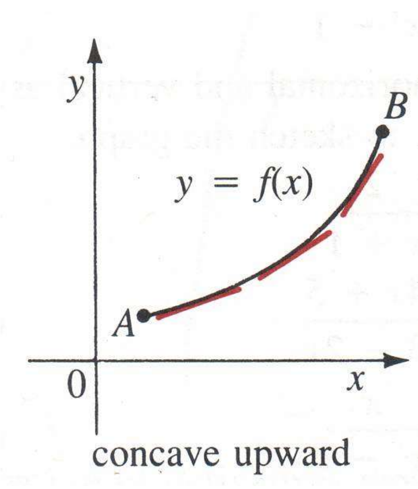
Special Cases

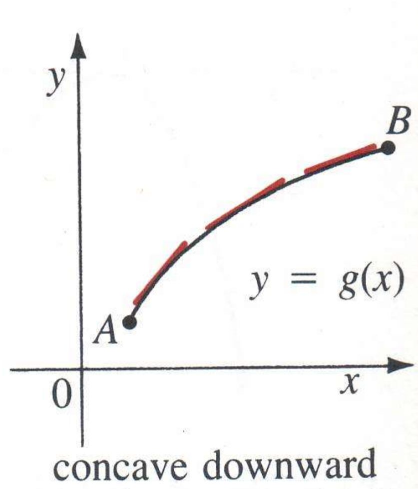
$$\boldsymbol{f(x) = \frac{4x^2 + 5x - 1}{2x^2 + 3}}$$



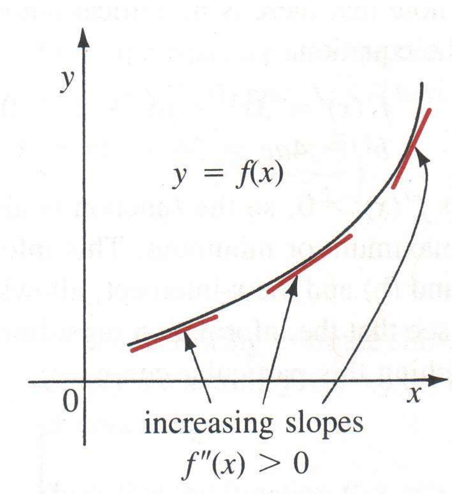
Day 4: 4.4 Concavities and Points of Inflection

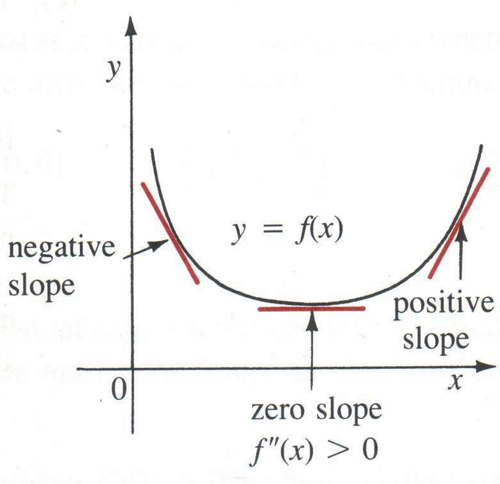
INVESTIGATION:



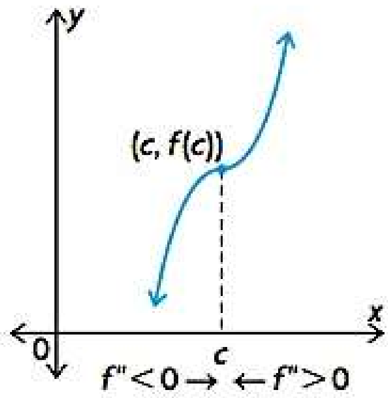


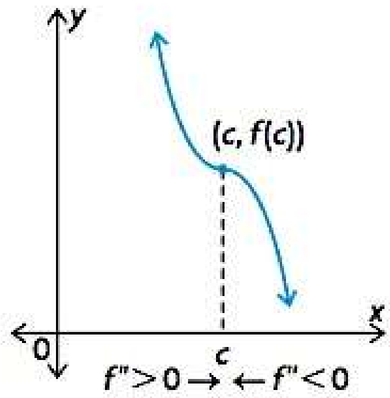
What do we notice?





What do we notice?





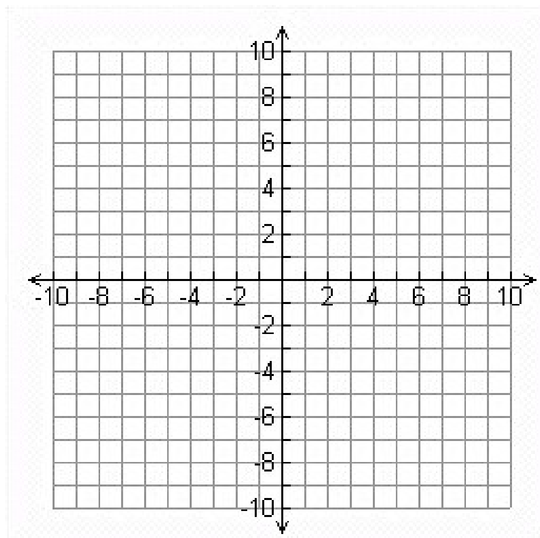
What do we notice?

Example 1

(a) Determine where the curve $y = x^3 - 3x^2 + 4x - 5$ is concave upward and where it is concave downward.

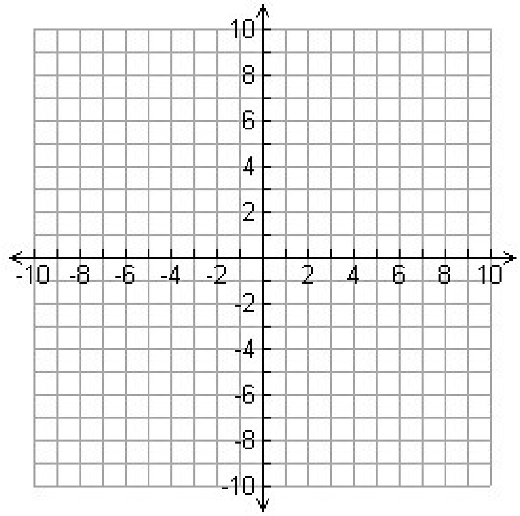
(b) Find the points of inflection.

(c) Use this information to sketch the curve.



x		
$f''(x)$		
$f(x)$		

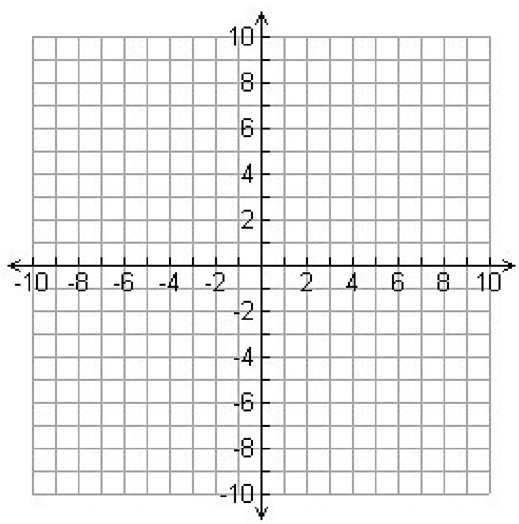
Example 2:
a) Determine where the curve y = $x^4 - 8x^3$ is concave upward and where it is concave downward.
b) Find the points of inflection.
c) Use this information to sketch the curve.



x			
f″(x)			
f(x)			

Example 3: Discuss the $y = \dfrac{x}{x^2+1}$ curve with respect to concavity and points of inflection if it is given that

$$y' = \frac{1-x^2}{(x^2+1)^2} \quad \text{and} \quad y'' = \frac{2x(x^2-3)}{(x^2+1)^3}$$



x				
f″(x)				
f(x)				

Day 5: 4.5 An algorithm for Curve Sketching

When sketching a curve, a graphing calculator is insufficient because it can only show a portion of the graph. It does not show the whole picture. Sometimes important features of the graph cannot be noticed because of an inappropriate x, y scale.
A table of values is insufficient because a lot of important information may be overlooked between the points. Furthermore, how many points can we make up?

THE THREE-STEP METHOD OF SKETCHING A CURVE – ALL TOGETHER!

Step 1 Analyze *y*

a) D**omain**.
b) ***y*-intercept** ($x = 0$).
c) Special value(s) of x that :
- makes $y = 0$ (these values of x are the **zeroes or *x*-intercepts**);
- makes $y = \#/0$ = undefined (the **vertical asymptotes**)
- makes $y = 0/0$ = undefined (**point of discontinuity**).

e) Evaluate $\lim\limits_{x \to \infty} y$ to find **Horizontal Asymptotes**.
d) Others --- **symmetry** (odd or even functions), extra points .
- f(x) = f(-x) the function is **even**, symmetry over the y-axis
- *f(x) = -f(-x) the function is* ***odd****, symmetry over the origin.*

Step 2 Analyze *y′*

Information you may be able to get are
- y′ = f′(x) > 0, the graph is **increasing**.
- *y′ = f′(x) < 0, the graph is* ***decreasing****.*
- ***local max/min points*** *(slope = 0)*
- ***points of inflection*** *(slope = undefined)*
- ***"sharp points" called "cusps"*** *(slope = undefined)*
- ***points with vertical tangents*** *(slope = undefined)*

Step 3 Analyze y″

Information you may be able to get are
- y″ = f″(x) > 0, intervals of **concave up**
- *y″ = f″(x) < 0, intervals of* ***concave down****.*
- ***points of inflection*** *(points where the concavity changes from [$\cap$ to $\cup$] or [$\cup$ to $\cap$]).*

Step 4 Compile and Sketch

Example: Sketch $y = x^3 - 6x^2 + 9x$

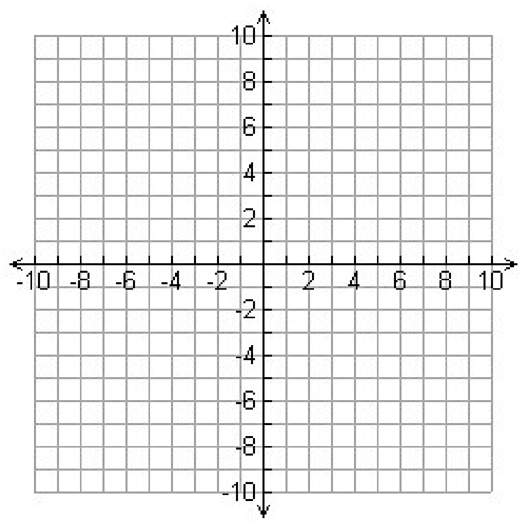
1) Domain

2) Intercepts

3) Any Special points, VA? HA?

4) Critical values (Analyze $f'(x)$)

5) Concavity, Points of Inflection?



6) Any special value?

7) Chart & Sketch

x				
$f'(x)$				
$f''(x)$				
$f(x)$				

Example: Sketch $y = \frac{4 - x^2}{x^2 - 3}$

1) Domain

2) Intercepts

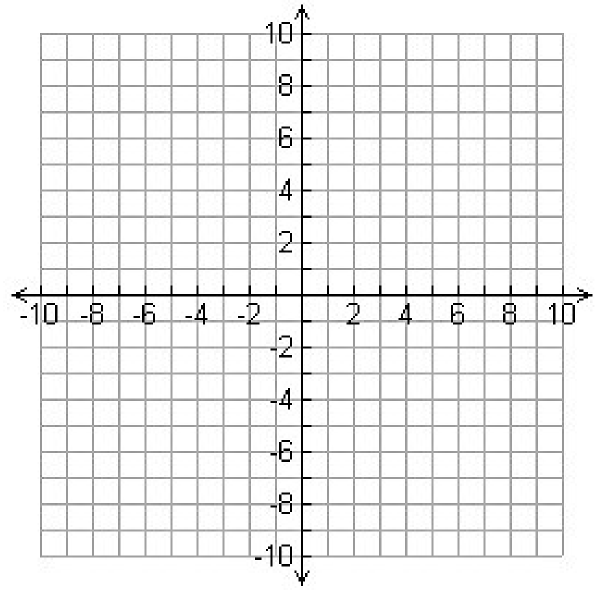
3) Special points, VA? HA?

4) Critical values (Analyze f′(x))

5) Concavity, Points of Inflection

6) Any special value?

7) Chart & Sketch



x				
$f'(x)$				
$f''(x)$				
$f(x)$				

Example: Sketch $y = \dfrac{x^2 - 3x + 6}{x - 3}$

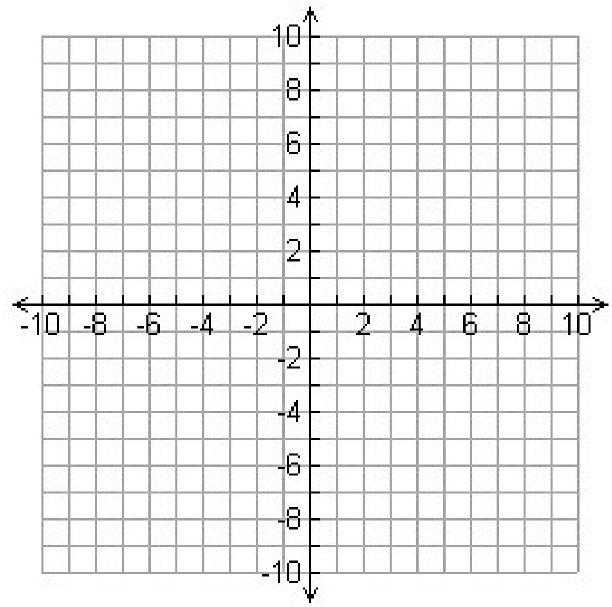
1) Domain

2) Intercepts

3) Special points, VA? HA?

4) Critical values (Analyze f′(x))

5) Concavity, Points of Inflection



6) Any special value?

7) Chart & Sketch

x				
f′(x)				
f″(x)				
f(x)				

Review

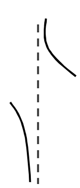
When The Slope is Undefined

Suppose slope is undefined at $x = 4$

Sub $x = 4$ back into the $f(x)$ or the y equation

Found $y = \frac{\#}{0}$

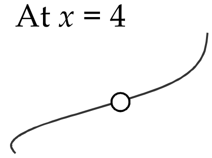
Vertical Asymptote at $x = 4$



$y = \frac{0}{0}$

A point of Discontinuity

At $x = 4$

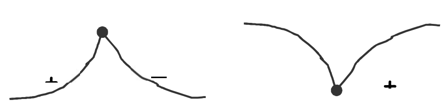


$y =$ a real number

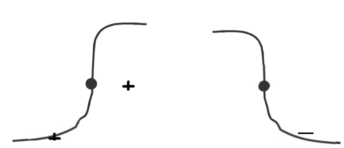
A CUSP or a point with VERTICAL SLOPE

at $x = 4$ depending on the change of increasing/decreasing situation on the left and right side of $x = 4$.

CUSPS



POINTS WITH VERTICAL SLOPE



Sketching the Graph of a Polynomial or Rational Function

1. Use the function (y value) to

- determine the domain and any discontinuities
- determine the intercepts
- find any asymptotes, and determine function behaviour relative to these asymptotes

2. Use the first derivative ($f'(x)$)to

- find the critical numbers
- determine where the function is increasing and where it is decreasing
- identify any local maxima or minima

3. Use the second derivative ($f''(x)$) to

- determine where the graph is concave up and where it is concave down
- find any points of inflection

The second derivative can also be used to identify local maxima and minima.

4. Calculate the values of y that correspond to critical points and points of inflection.

- Put all the information into a chart
- Use the information above to sketch the graph.

Remember that you will not use all the steps in every situation!

Use only the steps that are necessary to give you a good idea of what the graph will look like.