Day 5: 4.5 An algorithm for Curve Sketching

When sketching a curve, a graphing calculator is insufficient because it can only show a portion of the graph. It does not show the whole picture. Sometimes important features of the graph cannot be noticed because of an inappropriate x, y scale.
A table of values is insufficient because a lot of important information may be overlooked between the points. Furthermore, how many points can we make up?

THE THREE-STEP METHOD OF SKETCHING A CURVE – ALL TOGETHER!

Step 1 Analyze y

a) Domain.
b) y-intercept ($x = 0$).
c) Special value(s) of x that :
- makes $y = 0$ (these values of x are the **zeroes or x-intercepts**);
- makes $y = \#/0$ = undefined (the **vertical asymptotes**)
- makes $y = 0/0$ = undefined (**point of discontinuity**).

e) Evaluate $\lim_{x\to\infty} y$ to find **Horizontal Asymptotes**.

d) Others --- **symmetry** (odd or even functions), extra points .
- f(x) = f(-x) the function is **even**, symmetry over the y-axis
- *f(x) = -f(-x) the function is odd, symmetry over the origin.*

Step 2 Analyze y'

Information you may be able to get are
- $y' = f'(x) > 0$, the graph is increasing.
- *$y' = f'(x) < 0$, the graph is decreasing.*
- ***local max/min points*** *(slope = 0)*
- ***points of inflection*** *(slope = undefined)*
- ***"sharp points" called "cusps"*** *(slope = undefined)*
- ***points with vertical tangents*** *(slope = undefined)*

Step 3 Analyze y''

Information you may be able to get are
- $y'' = f''(x) > 0$, intervals of **concave up**
- *$y'' = f''(x) < 0$, intervals of* ***concave down****.*
- ***points of inflection*** *(points where the concavity changes from [$\cap$ to $\cup$] or [$\cup$ to $\cap$]).*

Step 4 Compile and Sketch

Example: Sketch $y = x^3 - 6x^2 + 9x$

1) Domain $\{x \in \mathbb{R}\}$ since it is a polynomial.

2) Intercepts x-int: 0, 3 y-int: 0

$x(x^2 - 6x + 9) = 0$

$x(x-3)^2 = 0$

3) Any Special points, VA? HA? NO.

4) Critical values (Analyze f′(x))

$y' = 3x^2 - 12x + 9$

$3x^2 - 12x + 9 = 0$

$3(x^2 - 4x + 3) = 0$

$3(x-3)(x-1) = 0$

y': + | 1 | − | 3 | +

(1, 4) local max

(3, 0) local min

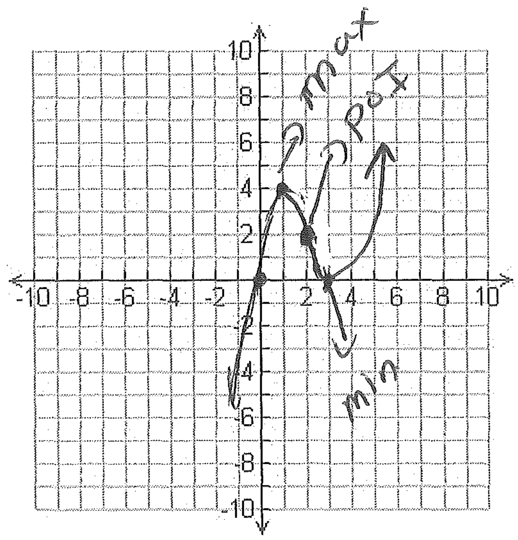
5) Concavity, Points of Inflection?

$y'' = 0 \Rightarrow 6x - 12 = 0$

$x = 2$

y'': − | 2 | +

(2, 2) POI



6) Any special value? x-ints are 0, 3

No hole, VA.

7) Chart & Sketch

x	$(-\infty, 1)$	(1, 2)	(2, 3)	(3, 4)
$f'(x)$	+	−	−	+
$f''(x)$	−	−	+	+
$f(x)$	incre CD	dec CD	dec CU	inc CU

Example: Sketch $y = \frac{4-x^2}{x^2-3}$

1) Domain $\{x \in \mathbb{R} \mid x \neq \pm\sqrt{3}\}$

2) Intercepts x-int: $x = \pm 2$

y-int: $-\frac{4}{3}$

3) Special points, VA? HA?

VA: $x = \sqrt{3}$ $\quad$ $x = -\sqrt{3}$ $\quad$ HA: $y = -1$

$\lim_{x \to \sqrt{3}^+} f(x) = +\infty$ $\quad$ $\lim_{x \to -\sqrt{3}^+} f = -\infty$ $\quad$ $\lim_{x \to \infty} f(x) = -1^+$

$\lim_{x \to \sqrt{3}^-} f(x) = -\infty$ $\quad$ $\lim_{x \to -\sqrt{3}^-} f = +\infty$ $\quad$ $\lim_{x \to -\infty} f(x) = -1^+$

4) Critical values (Analyze f'(x))

$$y' = \frac{(-2x)(x^2-3) - (2x)(4-x^2)}{(x^2-3)^2}$$

$$= \frac{-2x\left[(x^2-3) + (4-x^2)\right]}{(x^2-3)^2}$$

$$= \frac{-2x}{(x^2-3)^2}$$

$-\sqrt{3}$ $\quad$ 0 $\quad$ $\sqrt{3}$

y' $\quad$ + $\quad$ + $\quad$ − $\quad$ −

y $\quad$ inc $\quad$ inc $\quad$ dec $\quad$ dec

$(0, -4/3)$ local max

5) Concavity, Points of Inflection

$$y'' = \frac{-2(x^2-3)^2 - (-2x)(2)(x^2-3)(2x)}{(x^2-3)^4}$$

$$= \frac{-2(x^2-3)\left[(x^2-3) - x(2)(\cancel{x^2-3})(2x)\right]}{(x^2-3)^4}$$

$$= \frac{-2(x^2-3)\left[x^2-3-4x^2\right]}{(x^2-3)^4}$$

$$= \frac{-2(-x^2-3)}{\quad} = \frac{2(x^2+3)}{\quad^3}$$

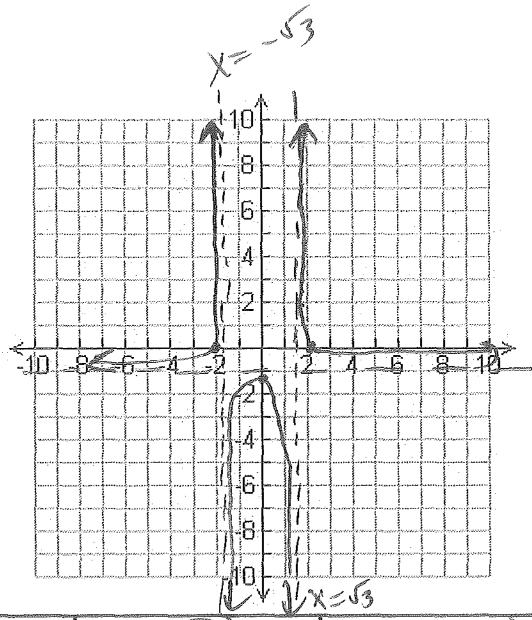
$y'' \neq 0$ $\quad$ y'' d.n.e when $x = \pm\sqrt{3}$

$-\sqrt{3}$ $\quad$ $\sqrt{3}$

y'' $\quad$ + $\quad$ − $\quad$ +

$\therefore$ Concave up $x \in (-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$

CD: $x \in (-\sqrt{3}, \sqrt{3})$



6) Any special value?

7) Chart & Sketch

x	$(-\infty, -\sqrt{3})$	$(-\sqrt{3}, 0)$	$(0, \sqrt{3})$	$(\sqrt{3}, \infty)$
$f'(x)$	+	+	−	−
$f''(x)$	+	−	−	+
$f(x)$	inc, CU	inc, CD	dec, CD	dec, CU

Example: Sketch $y = \dfrac{x^2 - 3x + 6}{x - 3} = x + \dfrac{6}{x-3}$

1) Domain $\{x \in \mathbb{R} \mid x \neq 3\}$

2) Intercepts

x-int: $x^2 - 3x + 6 = 0$

Quadratic formula: $x = \dfrac{3 \pm \sqrt{(-3)^2 - 4(1)6}}{2}$

NO sol^n

y-int: -2

3) Special points, VA? HA?

OBLIQUE Asymptote: $y = x$

$$\begin{array}{r|rrr} 3 & 1 & -3 & 6 \\ & \downarrow & 3 & 0 \\ \hline & 1 & 0 & 6 \end{array}$$

4) Critical values (Analyze f′(x)) $y' = \dfrac{(2x-3)(x-3) - (x^2 - 3x + 6)}{(x-3)^2}$

$y' = \dfrac{2x^2 - 9x + 9 - x^2 + 3x - 6}{(x-3)^2}$

$= \dfrac{x^2 - 6x + 3}{(\quad)}$

$x^2 - 6x + 3 = 0$

$x = \dfrac{6 \pm \sqrt{36 - 12}}{2} = \dfrac{6 \pm \sqrt{24}}{2} = \dfrac{6 \pm 2\sqrt{6}}{2} = 3 \pm \sqrt{6}$

(0.55)　　　　5.45
$3-\sqrt{6}$　　3　　$3+\sqrt{6}$

y'　　+　　−　　−　　+

intervals of inc: $x \in (-\infty, 3-\sqrt{6}) \cup (3+\sqrt{6}, \infty)$

intervals of dec: $x \in (3-\sqrt{6}, 3) \cup (3, 3+\sqrt{6})$

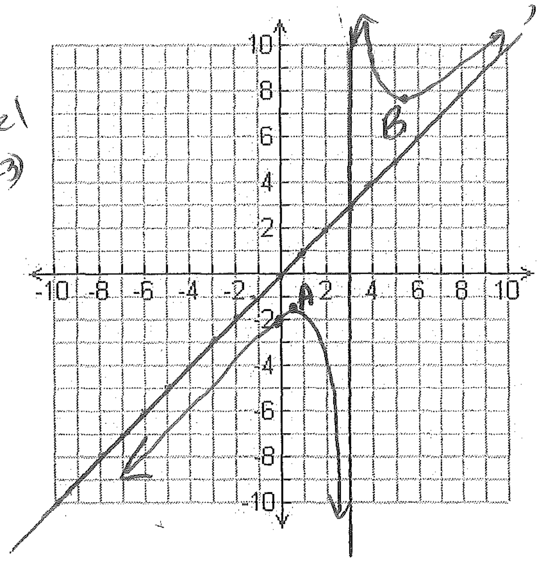
A Local min $(0.55, -1.90)$

B Local max $(5.45, 7.90)$

5) Concavity, Points of Inflection $y' = \dfrac{x^2-6x+3}{(x-3)^2}$

$$y'' = \frac{(2x-6)(x-3)^2 - (x^2-6x+3)(2)(x-3)}{(x-\cancel{3})^4}$$

factor and cancel $(x-3)$



$$= \frac{(2x-6)(x-3) - (x^2-6x+3)(2)}{(x-3)^3}$$

$$= \frac{2x^2-6x-6x+18-2x^2+12x-6}{(x-3)^3} = \frac{12}{(x-3)^3}$$

6) Any special value?

y''　　−　　3　　+

7) Chart & Sketch　　y　CD　　CU

x	$(-\infty, 3-\sqrt{6})$	$(3-\sqrt{6}, 3)$	$(3, 3+\sqrt{6})$	$(3+\sqrt{6}, \infty)$
$f'(x)$	+	−	−	+
$f''(x)$	−	−	+	+
$f(x)$	inc, CD	dec, CD	dec, CU	inc, CU