Day 3: 4.3 Vertical and Horizontal Asymptotes

Vertical Asymptotes

Example 1: Find $\lim_{x\to 0^+} \frac{1}{x}$ and $\lim_{x\to 0^-} \frac{1}{x}$.

$= +\infty$ $\qquad$ $= -\infty$

Example 2: Find $\lim_{x\to 6}\left[2-\frac{5}{(x-6)^2}\right]$.

$$\lim_{x\to 6^+}\left[2-\frac{5}{(x-6)^2}\right] = 2 - \lim_{x\to 6^+}\frac{5}{(x-6)^2} = -\infty$$

$$\lim_{x\to 6^-}\left[2-\frac{5}{(x-6)^2}\right] = -\infty$$

$$\therefore \lim_{x\to 6}\left[2-\frac{5}{(x-6)^2}\right] = -\infty$$

To find vertical asymptotes of rational functions, we find the values of x where the **denominator** is zero and compute the limits of the function from the right and left.

Example 3 a) Find the vertical asymptotes of the function $y=\frac{x}{x^2-x-6}$.

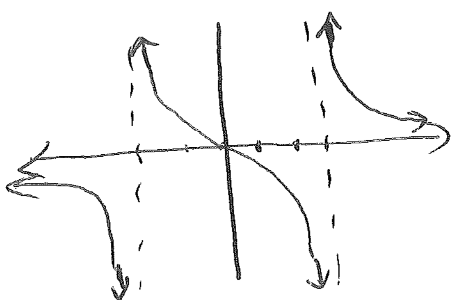
b) Sketch the graph near the asymptotes.

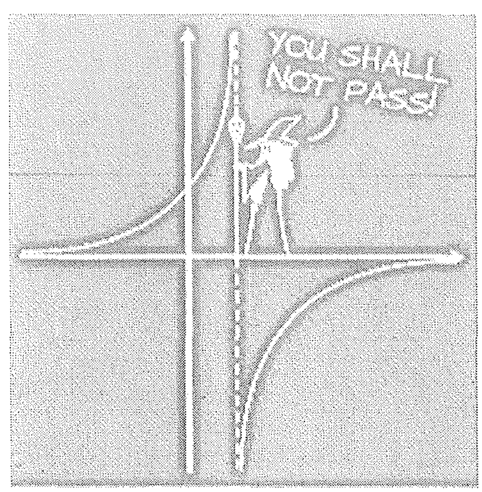
$$y = \frac{x}{(x-3)(x+2)}$$

VA: $x = 3$ $\qquad$ $x = -2$

$$\lim_{x\to 3^+}\frac{x}{(x-3)(x+2)} = +\infty \qquad \lim_{x\to -2^+}\frac{x}{(x-3)(x+2)} = +\infty$$

$$\lim_{x\to 3^-}\frac{x}{(x-3)(x+2)} = -\infty \qquad \lim_{x\to -2^-}\frac{x}{(x-3)(x+2)} = -\infty$$





Horizontal Asymptotes

The line $y = L$ is called a horizontal asymptote of the curve if either $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$.

To find H.A., we need to consider x approaches $+\infty$ and $-\infty$.

Example 1 Find $\lim_{x\to\infty} \frac{1}{x}$ and $\lim_{x\to-\infty} \frac{1}{x}$.

$= 0^+$ $\quad$ $= 0^-$

Example 2 Evaluate $\lim_{x\to\infty} \frac{4x^2 - x + 2}{6x^2 + 5x + 1}$.

$$= \lim_{x\to\infty} \frac{\cancel{x^2}\left(4 - \frac{1}{x} + \frac{2}{x^2}\right)}{\cancel{x^2}\left(6 + \frac{5}{x} + \frac{1}{x^2}\right)} = \frac{4}{6} = \frac{2}{3}$$

$\therefore y = 3$ is the HA

Example 3

Determine the value of each of the following:

a. $\lim_{x\to+\infty} \frac{2x - 3}{x + 1}$ $= 2$

b. $\lim_{x\to-\infty} \frac{x}{x^2 + 1}$ $= 0^-$

c. $\lim_{x\to+\infty} \frac{2x^2 + 3}{3x^2 - x + 4}$ $= \frac{2}{3}$

Example 4 Find the HA & VA's of the function and sketch its graph near the asymptotes.

a) $y = \frac{x+1}{x-2}$

(b) $y = \frac{x}{x^2 - x - 6}$.

see previous page.

a) VA: $x = 2$

$x \to 2^+$ $y \to \infty$

$x \to 2^-$ $y \to -\infty$

HA: $y = 1$

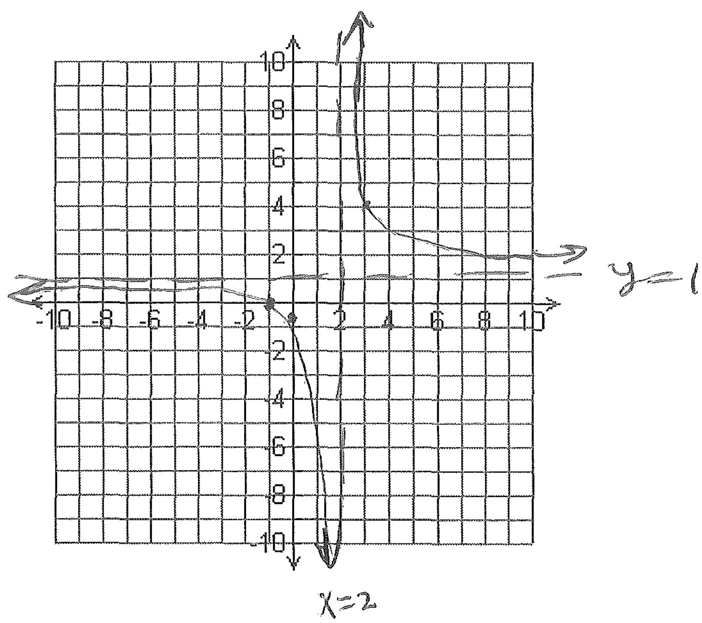
$x \to \infty$ $y \to 1^+$

$x \to -\infty$ $y \to 1^-$

x-int: $x + 1 = 0$

$x = -1$

y-int: $y = -\frac{1}{2}$



Vertical Asymptote, Point of Discontinuity & X-intercept Of A Rational Function

Find all 'special' values of x which make the denominator or numerator zero.

Example 1 $\quad y = \dfrac{x+1}{x-2}$

- *when* $x = 2$, $y = \dfrac{3}{0}$ *(undefined). There is a vertical asymptote at* $x = 2$.
- *when* $x = -1$, $y = \dfrac{0}{-3} = 0$. *There is an x-intercept at* $(-1, 0)$.

Example 2 $\quad y = \dfrac{(x+3)(x-5)}{(x-2)(x+3)}$

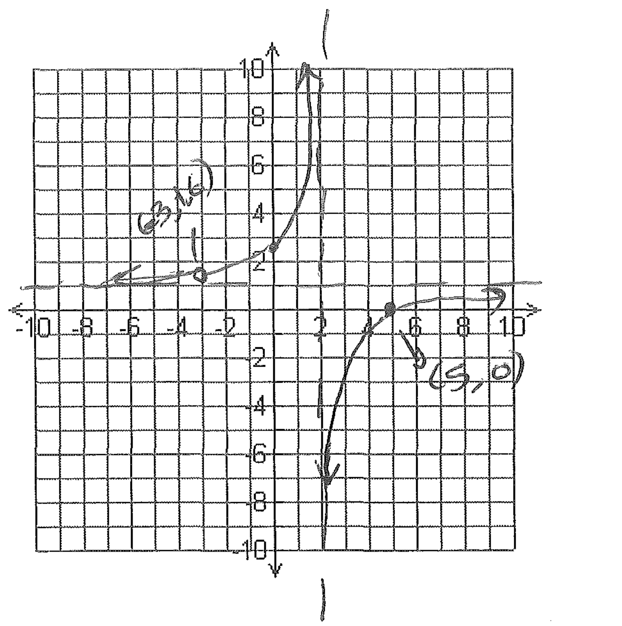
- *when* $x = 2$, $y = \dfrac{-3}{0}$ *(undefined). There is a* ***vertical asymptote*** *at* $x=2$
- *when* $x = -3$, $y = \dfrac{0}{0}$ *(undefined). There is a* ***point of discontinuity*** *('hole') on the curve.*

To find the location of this 'hole', we can sub in x = –3 after reducing (cross out) the common factor (x+3) from the numerator and the denominator.

The 'hole' is located at $\left(-3, \dfrac{8}{5}\right) = (-3, 1.6)$.

- *when* $x = 5$, $y = \dfrac{0}{3} = 0$. *There is an x-intercept at* $(5, 0)$.
- *The chart to determine whether the graph is above or below the x-axis.*

	-3	2	5	
x	$(-\infty, -3)$	$(-3, 2)$	$(2, 5)$	$(5, \infty)$
y	+	+	–	+



Use limits to describe the behaviour of the graph near the vertical asymptote.

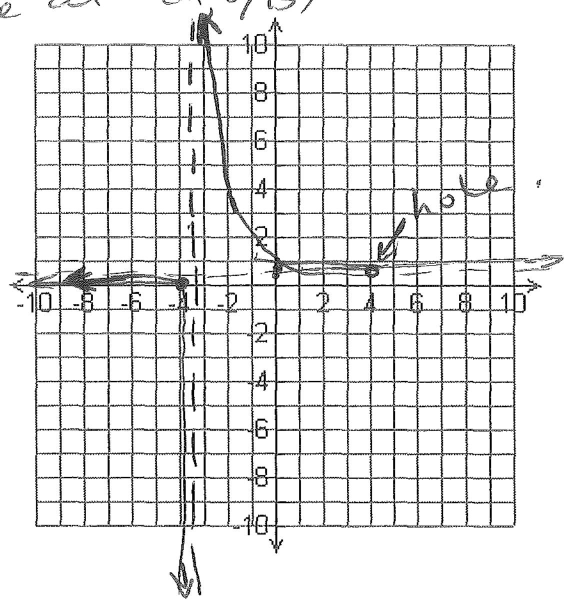
$\lim_{x \to 2^-} f(x) = +\infty \qquad \lim_{x \to 2^+} f(x) = -\infty$

SUMMARY

- $y = \frac{0}{\#} = 0$ x-intercept
- $y = \frac{0}{0}$ = undefined a hole'
- $y = \frac{\#}{0}$ = undefined vertical asymptote

Sketch the rational function $f(x) = \dfrac{x^2-16}{2x^2-x-28} = \dfrac{(x-4)(x+4)}{(2x+7)(x-4)}$

$f(x) = \dfrac{x+4}{2x+7}$, $x \neq 4$ $y \neq \dfrac{8}{15}$ $\Rightarrow$ hole at $(4, 8/15)$



x-int: $x = -4$ $(-4, 0)$

y-int: $y = 4/7$ or $(0, 4/7)$

hole $(4, 8/15)$

VA: $x = -7/2$

$\lim\limits_{x \to -\frac{7}{2}^+} f(x) = \infty$ $\lim\limits_{x \to -\frac{7}{2}^-} f(x) = -\infty$

HA: $y = \frac{1}{2}$

$\lim\limits_{x \to \infty} f(x) = \frac{1}{2}^+$ $\lim\limits_{x \to -\infty} f(x) = \frac{1}{2}^-$

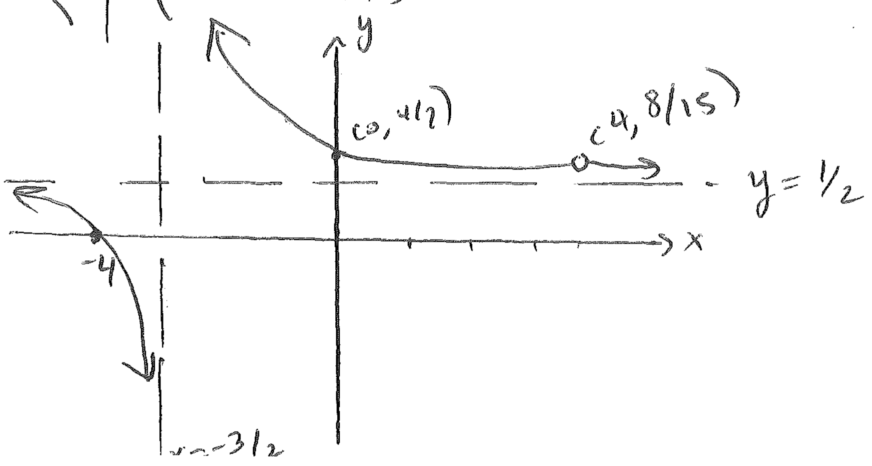
Intervals of inc/dec

$f' = \dfrac{1(2x+7) - (x+4)(2)}{(2x+7)^2} = \dfrac{\cancel{2x} + 7 - \cancel{2x} - 8}{(2x+7)^2} = -\dfrac{1}{(2x+7)^2}$

	$(-\infty, -\frac{7}{2})$	$(-\frac{7}{2}, 4)$	$(4, \infty)$
y'	$-$	$-$	$-$
y	↘	↘	↘

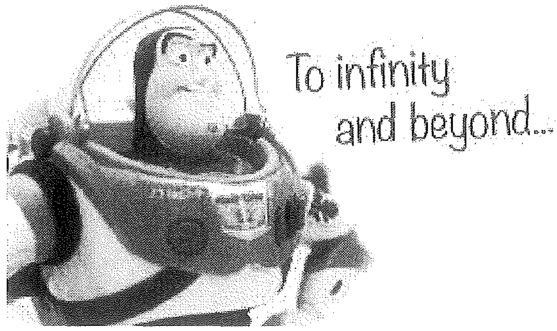
the f(x) is decreasing on its domain.

graph →



Oblique Asymptotes

Oblique asymptotes are straight lines (not parallel to any axes) that the curve can cross locally but that the curve approaches infinitely closely at the extremes (same behaviour as the horizontal asymptote). They occur with rational functions in which the degree of the numerator polynomial ix exactly one degree more than the degree of the denominator polynomial. When this happens, there is NO horizontal asymptote, although vertical asymptotes are still possible.



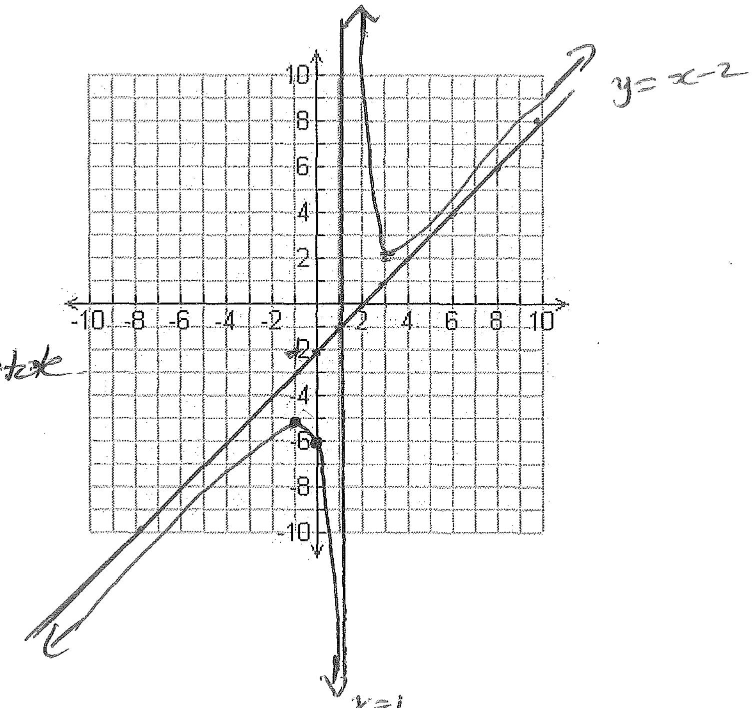
Find the Oblique Asymptote for each of the following:

Ex 1: $f(x) = \dfrac{x^2 - 3x + 6}{x - 1} = (x-2) + \dfrac{4}{x-1}$

$$\begin{array}{r|rrr} 1 & 1 & -3 & 6 \\ & & 1 & -2 \\ \hline & 1 & -2 & 4 \end{array}$$

$$\begin{array}{r} x - 2 \\ x-1 \overline{) x^2 - 3x + 6} \\ \underline{x^2 - x} \\ -2x + 6 \\ \underline{-2x + 2} \\ 4 \end{array}$$

Using synthetic/long division,
$y = x - 2$ is the oblique asymptote



$x = 1$ is the VA

$\lim_{x \to 1^+} f(x) = +\infty$ $\lim_{x \to 1^-} f(x) = -\infty$

x-int: $x^2 - 3x + 6 = 0$

$x = \dfrac{3 \pm \sqrt{9 - 4(1)(6)}}{2(1)}$

No x-int (No sol^n)

NOTE: We learned how to graph f(x) in MHF4U but now we can locate max/min.

y-int: -6

$f'(x) = \dfrac{(2x-3)(x-1) - (x^2 - 3x + 6)(1)}{(x-1)^2}$

$= \dfrac{2x^2 - 5x + 3 - x^2 + 3x - 6}{(x-1)^2}$

$= \dfrac{x^2 - 2x - 3}{(x-1)^2} = \dfrac{(x+1)(x-3)}{(x-1)^2}$

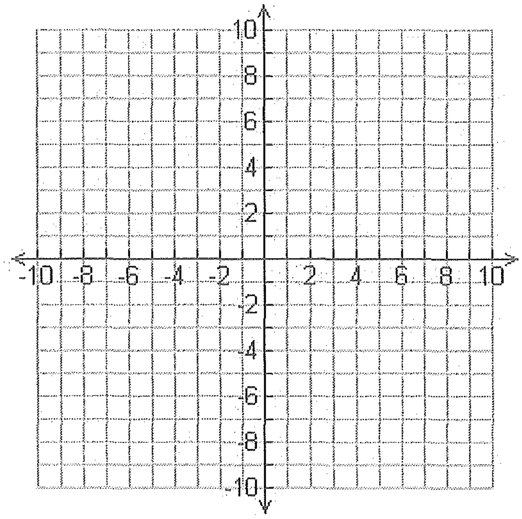
	-1	1	3	
y' +	−	−	+	

$(-1, -5)$ local max

$(3, 3)$ local min

Ex 2: $f(x) = \dfrac{2x^3 - 3x^2 + x - 3}{x^2 + 1}$

$$\begin{array}{r|l} & \quad 2x - 3 \\ \hline x^2 + 0x + 1 & 2x^3 - 3x^2 + x - 3 \\ & 2x^3 + 0x^2 + 2x \\ \hline & \quad -3x^2 - x - 3 \\ & \quad -3x^2 + 0x - 3 \\ \hline & \qquad -x + 0 \end{array}$$



$\therefore$ $y = 2x - 3$ is the equation of the oblique asymptote.

Go back to page 2 for the graph.

Special Cases

$$f(x) = \frac{\mathbf{4x^2 + 5x - 1}}{\mathbf{2x^2 + 3}}$$

NOTE: This function has no VA. It has a horizontal asymptote at $y = 2$. To find the local max/min, we can find the x where

$f'(x) = 0$