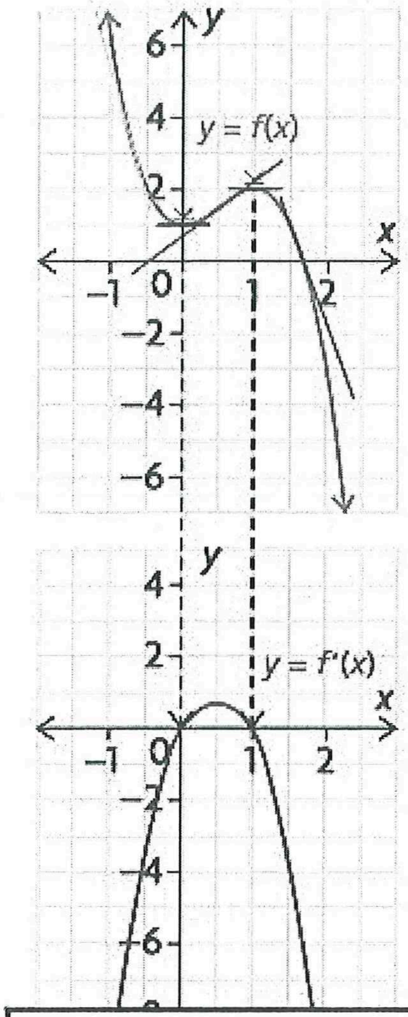
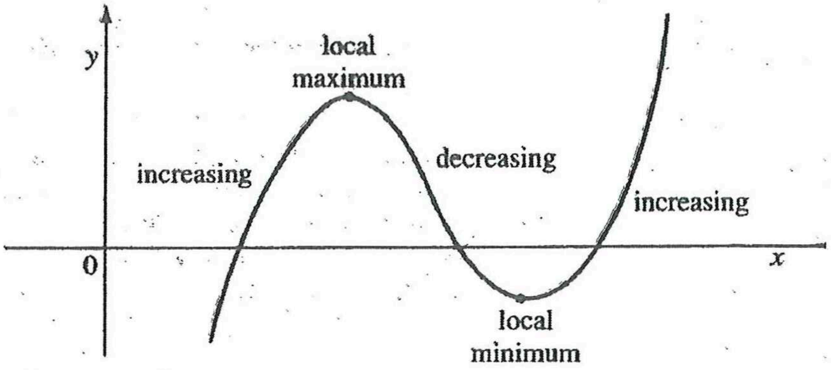
Day 2: 4.2 Critical Points, Local Maxima, Local Minima





Observe the graph of $f(x) = -2x^3 + 3x^2 + 1$ *and* $f'(x) = -6x(x-1)$

What do you notice about the two graphs?

the derivative is 0 when $x=0$ $x=1$

on the f(x) graph, at $x=0$ LOCAL MIN

$x=1$ LOCAL MAX

CRITICAL NUMBERS

A **critical number** of a function is a number c such that either $f'(c) = 0$ or $f'(c)$ is undefined.

'c' is part of the domain

For (i) Rational functions of the form $y = \frac{f(x)}{g(x)}$:

We need to find the *x*'s that make the "top" zero and the *x*'s that make the "bottom" zero.
In other words, we need to solve the two equations of $f(x) = 0$ and $g(x) = 0$

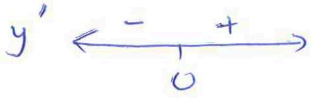
(ii) For Non-Rational functions of $y = f(x)$:
the job is to solve the equation of $f'(x) = 0$

Examples: Find all the critical numbers for the following functions:

1. $y = x^2 - 5$.

$y' = 2x$ $\quad$ $y' = 0$ if $x = 0$

when $x = 0$ $\quad$ $y = -5$



$\therefore$ $(0, -5)$ is a local min.

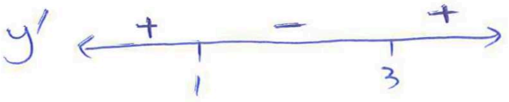
2. $y = 6x^3 - 36x^2 + 54x$.

$y' = 18x^2 - 72x + 54$

$18(x^2 - 4x + 3) = 0$

$18(x-3)(x-1) = 0$

$x = 3 \qquad x = 1$



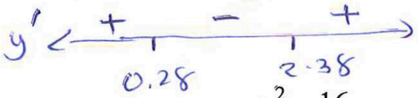
$\therefore$ $(1, 24)$ is a local max

$(3, 0)$ is a local min.

3. $y = x^3 - 4x^2 + 2x - 5$

$y' = 3x^2 - 8x + 2 = 0$ $\quad$ USE QF

$\left.\begin{matrix} x = 2.38 \\ y = -9.42 \end{matrix}\right\}$ local min $\qquad$ $\left.\begin{matrix} x = 0.28 \\ y = -4.47 \end{matrix}\right\}$ local max



4. $y = \dfrac{x^2 - 16}{x^2 + 6x + 8} = \dfrac{(x-4)\cancel{(x+4)}}{(x+2)\cancel{(x+4)}}$

$$y' = \frac{(1)(x+2) - (1)(x-4)}{(x+2)^2} = \frac{6}{(x+2)^2}$$

No solution for $y' = 0$

Note: $x \neq -2$ or -4.

5. $y = \frac{5x}{x^2+1}$

$$y' = \frac{5(x^2+1) - 5\cancel{x}(2x)}{(x^2+1)^2}$$

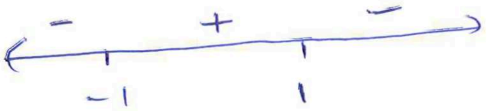
$$= \frac{5x^2 + 5 - 10x^2}{(x^2+1)^2}$$

$$= \frac{-5x^2 + 5}{(x^2+1)^2}$$

$$= \frac{-5(x^2-1)}{(x^2+1)^2}$$

$y' = 0 \Rightarrow -5(x^2-1) = 0$

$x = \pm 1$



$(-1, -\frac{5}{2})$ Local min.

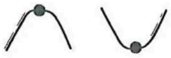
$(1, \frac{5}{2})$ Local max

Graphical Meaning At The Critical Numbers

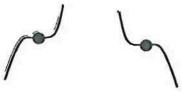
Case 1:

$f'(c) = 0$ means the slope of the graph at $x = c$ is zero. A zero slope can be one of the following situations.

- a local maximum or local minimum.



- a point of inflection

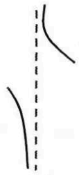


Case 2:

$f'(c) =$ **undefined** means the slope at $x = c$ is undefined. It can be one of the following situations.

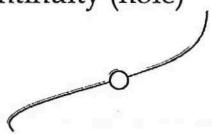
- the graph is **discontinuous** at $x = c$.

e.g Vertical Asymptote



or

Point of Discontinuity (hole)

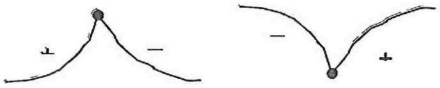


These have been discovered when you analyze $y = f(x)$.

- the graph has a **vertical tangent** or slope at $x = c$.

POINTS WITH VERTICAL SLOPE

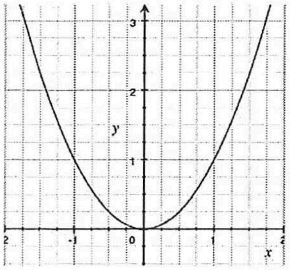
- the graph has a **cusp** at $x = c$.



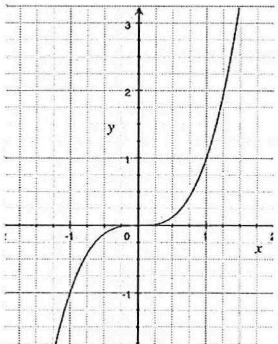
CUSPS

Examples:

$f(x) = x^2$
$f'(x) = 2x$

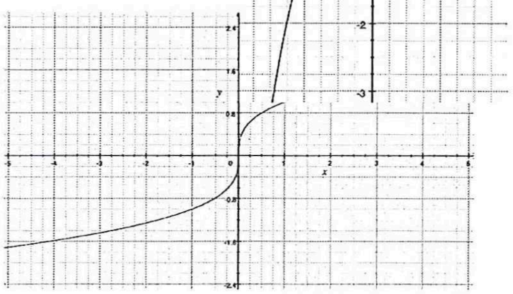


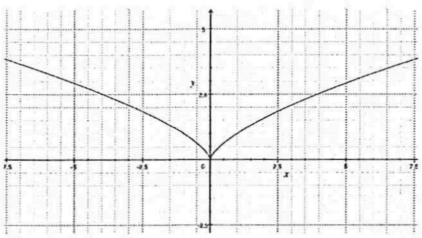
$f(x) = x^3$
$f'(x) = 3x^2$



A vertical **tangent**

$f(x) = \sqrt[3]{x}$





A cusp $f(x) = x^{\frac{2}{3}}$

FIRST DERIVATIVE TEST

Let c be a critical number of a continuous function *f*.

1. If f'*(x)* changes from positive to negative at c, then f has a local maximum at c.
2. If f'(*x*) changes from negative to positive at c, then f has local minimum at c.
3. If f'(*x*) does not change sign at c, then f has no maximum or minimum at c.

Ex 1: Find the local maximum and minimum values of $f(x) = x^3 - 3x + 1$

$f' = 3x^2 - 3$

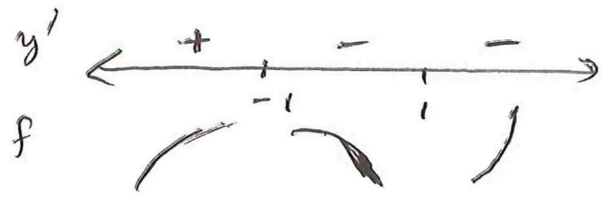
$3x^2 - 3 = 0$

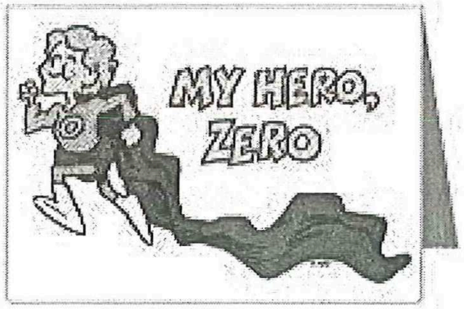
$3(x^2 - 1) = 0$

$3(x-1)(x+1) = 0$

$x = 1$ $\quad$ $x = -1$

NOTE: Before $x = -1$ the y' is +ve which means function is increasing then decreasing which implies $f(-1)$ is a local max.





∴ $(-1, 3)$ is a local max

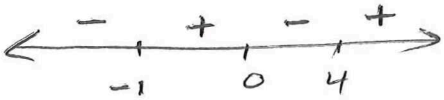
$(1, -1)$ is a local min

Ex 2: Find the local maximum and minimum of $g(x)=x^4 - 4x^3 - 8x^2 - 1$ and use this information to sketch the graph of g(x).

$g'(x)=0 \Rightarrow \quad 4x^3-12x^2-16x=0$

$4x(x^2-3x-4)=0$

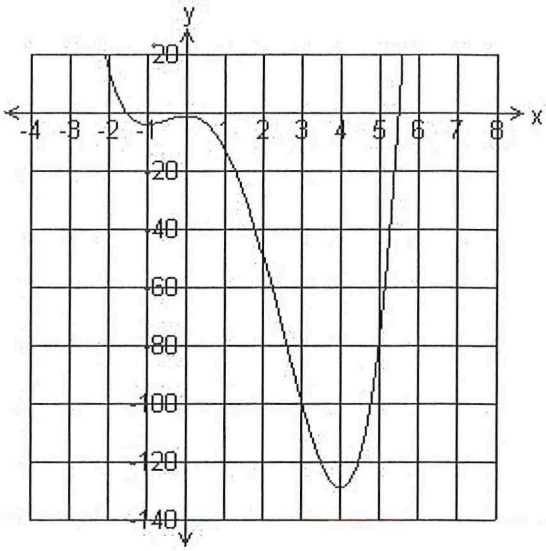
$4x(x-4)(x+1)=0$



$(-1,-4)$ Local min

$(0,-1)$ local max

$(4,-129)$ local min



Ex 3: Find the critical numbers, interval of increase and decrease, and local max and min of

f(x)= 2x – 3x^{2/3}

$f'(x)= 2 - \cancel{3}\left(\frac{2}{\cancel{3}}\right)x^{-\frac{1}{3}} = 2 - \frac{2}{3\sqrt{x}}$

$2 = \frac{2}{\sqrt[3]{x}} \Rightarrow \quad 2\,{}^3\sqrt{x}=2$

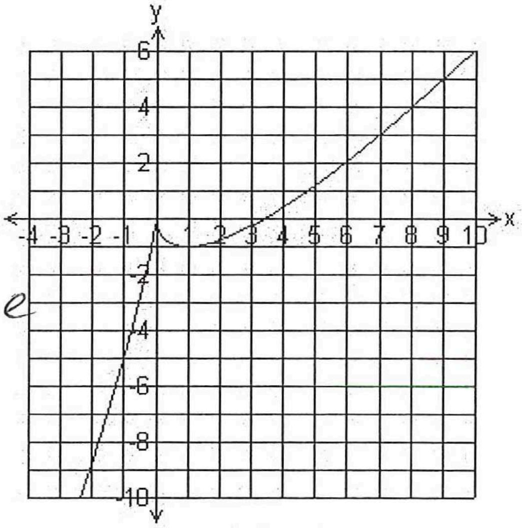
$3\sqrt{x}=1$

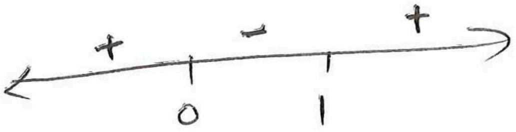
$x=1^3=1$

$x=0$ critical number since $f'(x)$ d.n.e

$x=1$ is also a critical number

since $f'(x)=0$





∴ Intervals of increase: $x \in (-\infty,0) \cup (1,\infty)$

decrease: $x \in (0,1)$

∴ at $(1,-1)$ there is a local min.