Day 7: Related Rates

Oil spilled from a tanker spreads in a circle whose area increases at a constant rate of 6 km^2/h. How fast is the radius of the spill increasing when the area is 9π km^2? Knowing the rate of increase of the radius is important in planning the containment operation.

In this lesson, you will encounter some interesting problems that will help you understand the applications of derivatives and how they can be used to describe and predict the phenomena of change. In many practical applications, several quantities vary in relation to each other. The rates at which they vary are also related to one another. With Calculus, we can describe and calculate such rates.

In a related rates problem, we are given the rate of change of one or more quantities and we are asked to find the rate of change of a related quantity. To do this, we must find an equation that relates the given quantities.

Example 1: The radius of a spherical balloon is increasing at a rate of 0.2 m/min. What is the rate of change of the volume of the balloon when:

a) The radius is 1.6 m.

b) The surface area is 144 m^2.

Solution: a) Step 1: How are the radius of the balloon and the volume of the balloon related?

$$V = \frac{4}{3}\pi r^3$$

Step 2: Differentiate both sides of the equation with respect to time.

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Step 3: Substitute $\frac{dr}{dt} = 0.2$ and $r = 1.6$ into $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$.

$$\frac{dV}{dt} = 4\pi(1.6)^2(0.2)$$

$$\frac{dV}{dt} = 2.048\pi$$

$$\frac{dV}{dt} \cong 6.434$$

Step 4: Express your answer in a statement.

Therefore, the rate of change of the volume is approximately 6.434 m^3/min when the radius is 1.6 m.

b) The formula for the surface area of a sphere is $A = 4\pi r^2$. Therefore in the differential equation $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$, we can substitute 144 for $4\pi r^2$ and $\frac{dr}{dt} = 0.2$.

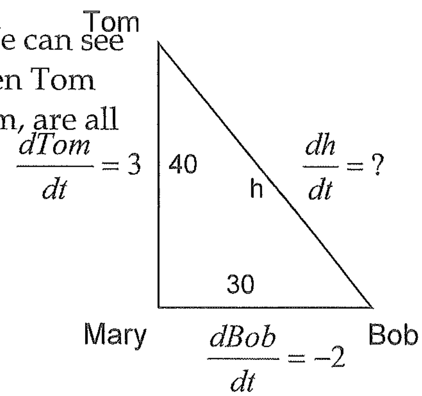
$$\frac{dV}{dt} = 144(0.2)$$

$$\frac{dV}{dt} = 28.8$$

Therefore, the rate of change of the volume is 28.8 m^3/min when the surface area is 144 m^2.

Example 2: Tom, Mary, and Bob are on a football field. Tom walks away from Mary (who is standing still), heading due north, at 3 m/s. Bob, who is due east of Mary, walks towards her at 2 m/s. How fast is the distance between Bob and Tom changing when Tom is 40 m away from Mary and Bob is 30 m away from Mary?

Solution: In this problem, a diagram is necessary. We can see from the diagram that the distances between Tom and Mary, Bob and Mary, and Bob and Tom, are all related by Pythagorean Theorem.



$$Tom^2 + Bob^2 = h^2$$

$$2Tom\frac{dTom}{dt} + 2Bob\frac{dBob}{dt} = 2h\frac{dh}{dt}$$

$$2(40)(3) + 2(30)(-2) = 2(50)\frac{dh}{dt}$$

$$240 - 120 = 100\frac{dh}{dt}$$

$$120 = 100\frac{dh}{dt}$$

$$1.2 = \frac{dh}{dt}$$

When Tom = 40 and Bob = 30,

$$h^2 = 40^2 + 30^2$$

$$h^2 = 1600 + 900$$

$$h^2 = 2500$$

$$h = 50$$

Therefore, the distance between Bob and Tom is changing at 1.2 m/s when Tom is 40 m away from Mary and Bob is 30 m away from Mary.

Review

Steps:

- *Gather information*
- *Create the function*
- *Take derivative, let it equal to zero, solve for x*
- *Find the value of the function when the derivative is zero, check out end point values as well.*

Summary:

- The derivative of the derivative function is called the second derivative.
- If the position of an object, $s(t)$, is a function of time, t, then the first derivative of this function represents the velocity of the object at time t. $v(t) = s'(t) = \frac{ds}{dt}$
- Acceleration, $a(t)$, is the instantaneous rate of change of velocity with respect to time. Acceleration is the first derivative of the velocity function and the second derivative of the position function. $a(t) = v'(t) = s''(t)$, or $a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$
- The maximum and minimum values of a function on an interval are also called extreme values, or absolute extrema.
- The maximum value of a function that has a derivative at all points in an interval occurs at a "peak" ($f'(c) = 0$) or at an endpoint of the interval, $[a, b]$.
- The minimum value occurs at a "valley" ($f'(c) = 0$) or at an endpoint of the interval, $[a, b]$.
- In an optimization problem, you must determine the maximum or minimum value of a quentity.
- An optimization problem can be solved using a mathematical model that is developed using information given in the problem. The numerical solution represents the extreme value of the model.
- Profit, cost, and revenue are quantities whose rates of change are measured in terms of the number of units produced of sold.
- Economic situations usually involve minimizing costs or maximizing profits.

Cone question:

$$V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$$

$$V = \frac{\pi}{12}h^3$$

$$\frac{dV}{dt} = \frac{3\pi}{12}h^2 \cdot \frac{dh}{dt}$$

$$\pi = \frac{3\pi(8)^2}{12}\left(\frac{dh}{dt}\right)$$

$$\frac{dh}{dt} \doteq \frac{1}{16} \text{ m/min.}$$

$$\frac{5}{10} = \frac{r}{h}$$

$$\frac{1}{2} = \frac{r}{h}$$

$$r = \frac{h}{2}$$

Given $\frac{dV}{dt} = \pi$ m^3/min

$h = 8$