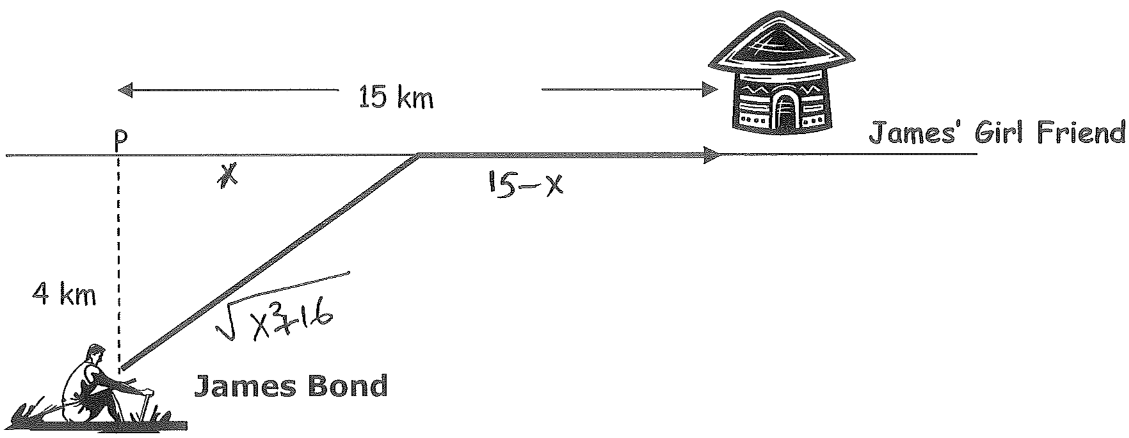
Day 4/5: 3.3 Optimization

James Bond's girl friend is being held hostage in a hut on the shore of a river. A bomb is set to explode in 3 hours. James is on a boat trying to reach the hut to save her.
The hut is 15 km down stream from point P.
James can jog at 7 km/h on shore and row at 4 km/h.
At what point of the shore should he land in order to save his girl friend in the shortest time?
Will James have enough time to save his girl friend? Can you help him?



$$\text{Time} = \frac{15-x}{7} + \frac{\sqrt{x^2+16}}{4}$$

$$(\text{Time})' = -\frac{1}{7} + \frac{1}{4}\left(\frac{1}{2}\right)(x^2+16)^{-\frac{1}{2}}(2x)$$

$$\frac{1}{7} = \frac{x}{4\sqrt{x^2+16}} \quad \text{(square both sides)}$$

$$\frac{1}{49} = \frac{x^2}{16(x^2+16)}$$

$$16x^2+256 = 49x^2$$

$$33x^2 = 256$$

$$x = \sqrt{\frac{256}{33}} \doteq 2.79 \text{ km.}$$

"Maybe I should get somebody else to help me with my homework."

$$T(2.79) = \frac{15-2.79}{7} + \sqrt{2.79^2+16}/4 = 2.96 < 3\text{hrs} !!$$

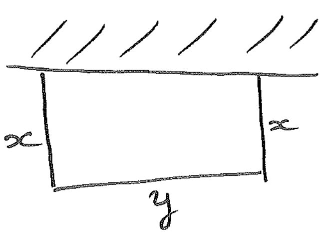
Solving Optimization Problems

Steps to Follow for Solving Optimization Problems

- Define given quantities and condition
- The quantity that I need to optimize is: ________________ .
- Draw diagram and identify given quantities and variables on it.
- Determine a function (formula) for that quantity. (This is the main formula that you will differentiate to find the derivative).
- When necessary, simplify the right side of the main formula to only one variable. You might need another equation on the side.
- Determine the domain of the function to be optimized, using information given in the problem.
- Follow the steps of finding absolute max/min to find the optimal solution.
- Write a concluding statement to answer the original question.

Area & Volume Problems

Example 1: A farmer has 300 m of fencing. He wants to enclose a rectangular portion of his field for the horses to roam. One side of the field is next to a road therefore it already has fencing. The farmer only needs to fence the other three sides. Find the dimensions of the field so that a maximum area is enclosed.



$2x+y=300$

$y=300-2x$

NOTE: $x>0$

$300-2x>0$

$x<150$

$\therefore 0<x<150$

$A=L\cdot w$

$=x\cdot y$

$=x(300-2x)$

$A(x)=300x-2x^2$

$A'=-4x+300$

set $A'(x)=0$ to find max: $\frac{-4x}{-4}=\frac{-300}{-4}$

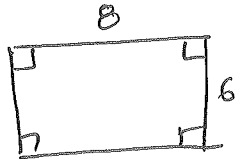
$x=75$ m

$y=300-2(75)$

$=150$ m.

$\therefore$ To maximize the area, the dimensions should be 75 m by 150 m.

Example 2: Cut congruent squares from the corners of a rectangular sheet. Fold to make a box (no lid) with maximum volume. Assume that the rectangular sheet has dimensions of 6 x 8 m.
How should the sheet be cut so that the box can have the maximum volume?



$V(x) = (x)(8-2x)(6-2x)$

$= x(48 - 16x - 12x + 4x^2)$

$= 4x^3 - 28x^2 + 48x$

$V'(x) = 0 \Rightarrow \quad 12x^2 - 56x + 48 = 0$

$3x^2 - 14x + 12 = 0$

$x = \dfrac{14 \pm \sqrt{14^2 - 4(3)(12)}}{6}$

$X_1 = \dfrac{14+\sqrt{52}}{6} \doteq 3.53$

$X_2 = \dfrac{14-\sqrt{52}}{6} \doteq 1.13$

$6 - 2x > 0$ $\quad$ $8 - 2x > 0$

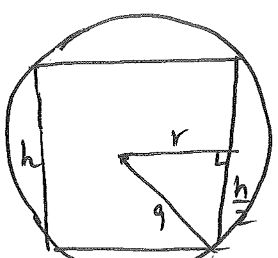
$x < 3$ $\quad$ $x < 4$

$x > 0$

$\therefore 0 < x < 3$

$\therefore$ 1.13 by 1.13 m, should be cut.

Example 3: Find the largest possible volume of a right circular cylinder that is inscribed in a sphere of radius 9 cm.



$V = \pi r^2 h$

$= \pi r^2 (2\sqrt{81 - r^2}) = 2\pi r^2 (81 - r^2)^{\frac{1}{2}}$

$V'(r) = 4\pi r (81 - r^2)^{\frac{1}{2}} + (2\pi r^2)(\frac{1}{2})(81 - r^2)^{-\frac{1}{2}}(-2r)$

$= 2\pi r (81 - r^2)^{-\frac{1}{2}} [2(81 - r^2) - r^2]$

$162 - 2r^2 - r^2 = 0$

$3r^2 = 162$

$r^2 = \dfrac{162}{3}$

$r^2 = \dfrac{(81)(2)}{3}$

$r = \dfrac{9\sqrt{2}}{\sqrt{3}}$ cm.

$r^2 + \left(\dfrac{h}{2}\right)^2 = 81$

$r^2 + \dfrac{h^2}{4} = 81$

$\dfrac{h^2}{4} = 81 - r^2$

$h^2 = 4(81 - r^2)$

$h = \sqrt{4(81 - r^2)}$

$= 2(81 - r^2)^{\frac{1}{2}}$

$h = 2\left(81 - \dfrac{162}{3}\right)^{\frac{1}{2}}$

$= 2(\sqrt{27})$

$= 2\sqrt{9}\sqrt{3}$

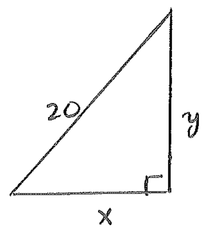
$= 6\sqrt{3}$ cm

$\therefore V = \pi \left(\dfrac{162}{3}\right)(6\sqrt{3})$

$= \dfrac{972\pi\sqrt{3}}{3}$ cm^3

$\doteq 1763$ cm^3

Example 4: Find the maximum perimeter of a right-angled triangle with hypotenuse equal to 20cm



$x^2+y^2=400$

$y^2=400-x^2$

$y=\sqrt{400-x^2}$

$P(x)=x+y+20$

$P(x)=x+\sqrt{400-x^2}+20$

$P'(x)=1+\frac{1}{\cancel{2}}(400-x^2)^{-\frac{1}{2}}(-\cancel{2}x)$ set $P'(x)=0$

$1=\frac{x}{\sqrt{400-x^2}} \Rightarrow \sqrt{400-x^2}=x$

$400-x^2=x^2$

$2x^2=400$

$x^2=200$

$x=\sqrt{200}=10\sqrt{2}$ cm

$y=\sqrt{400-x^2}=10\sqrt{2}$ cm

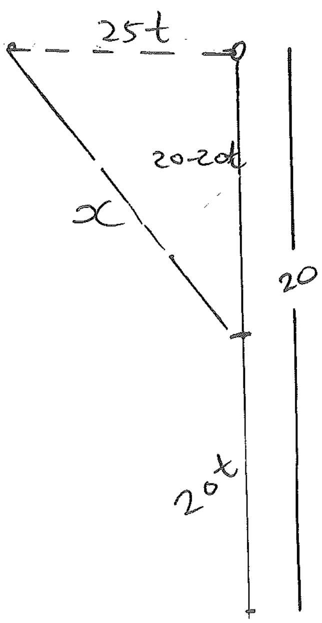
$\therefore P=x+y+20$

$=10\sqrt{2}+10\sqrt{2}+10$

$=20\sqrt{2}+10$ cm

Shortest Distance Problem

Example 5: A boat leaves a dock at noon and heads west at a speed of 25 km/h. Another boat heads north at 20 km/h and reaches the same dock at 1:00 p.m. When were the boats closest to each other?



$0 \le t < 1$

$x^2=(25t)^2+(20-20t)^2$

$x=\sqrt{625t^2+400-800t+400t^2}$

$=\sqrt{1025t^2-800t+400}$

$\frac{dx}{dt}=\frac{1}{2}(1025t^2-800t+400)^{-\frac{1}{2}}(2050t-800)$

set $\frac{dx}{dt}=0$

$2050t-800=0$

$t=\frac{800}{2050}$

≈ 0.39 hrs $\Rightarrow$ 23 mins 25 seconds