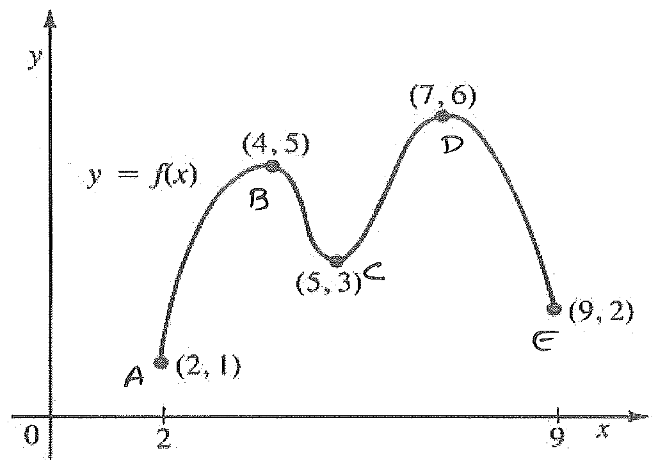
Day 3: 3.2 Maximum and Minimum on an Interval (Extreme Value)

We have already investigated some applications of derivatives, but now that we know the differentiation rules we are in a better position to pursue the applications of differentiation in greater depth. Here we will learn how derivatives affect the shape of a graph of a function and how they help us locate maximum and minimum values of a function. Many practical problems require us to minimize a cost or maximize an area or somehow find the best possible outcome of a situation.

Some of the most important applications of differential calculus are ***optimization problems*** in which we are required to find the optimal (best) way of doing something. These problems can be reduced to finding the maximum or minimum values of a function. But first we need to explain exactly what we mean by maximum and minimum values.

Warm up: Look at the graph of f(x):



a) Domain $\{x \in \mathbb{R} \mid 2 \leq x \leq 9\}$

b) Range $\{y \in \mathbb{R} \mid 1 \leq y \leq 6\}$

c) Maximum value $f(7) = 6$

d) Minimum value $f(2) = 1$

e) Peak/valley

B: local max

C: local min

D: global max (absolute max)

A: global min (absolute min)

NOTE: Endpoints can not be considered for local max/min BUT must be considered for global/absolute extrema.

Definitions

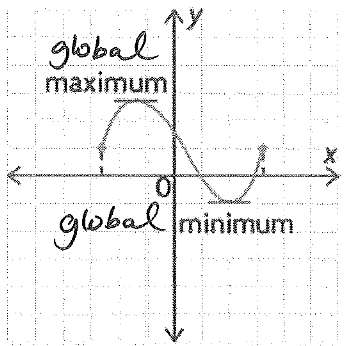
Let f be a function with domain D.
Then $f(c)$ is:

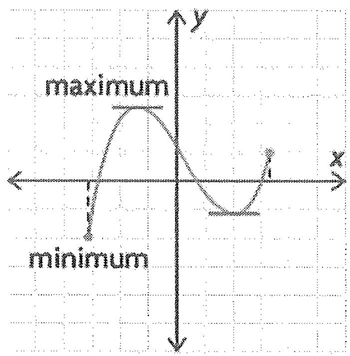
a) ***absolute maximum value*** *on* D *iff* $f(x) \le f(c)$ *for all* $x \in D$
b) ***absolute minimum value*** *on* D *iff* $f(x) \ge f(c)$ *for all* $x \in D$

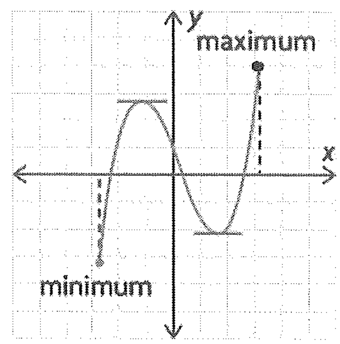
Let c be an interior point of the domain D of the function.
Then $f(c)$ is:

a) ***local maximum value*** *at c iff* $f(x) \le f(c)$ *for all* x *in some open interval containing* c.
b) ***local minimum value*** *at c iff* $f(x) \ge f(c)$ *for all* x *in some open interval containing* c.

*local extrema are also called relative extrema







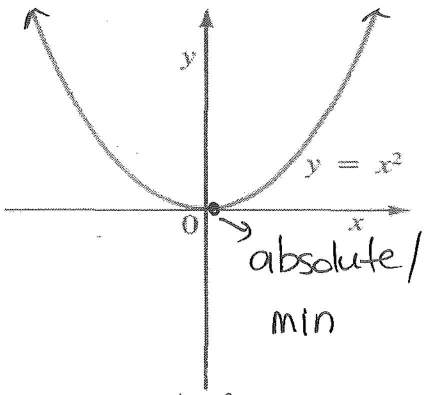
Algorithm for Finding Maximum or Minimum (Extreme) Values

If a function $f(x)$ has a derivative at every point in the interval $a \le x \le b$, calculate $f(x)$ at

- all points in the interval $a \le x \le b$, where $f'(x) = 0$
- the end points $x = a$ and $x = b$ *(i.e. $f(b)$, $f(a)$)*

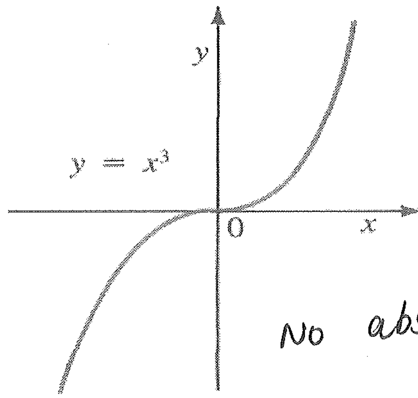
The maximum value (global/absolute max) of $f(x)$ on the interval $a \le x \le b$ is the largest of these values, and the minimum value (global/absolute min) of $f(x)$ on the interval is the smallest of these values.

Example 1: Find the maximum and minimum of the following functions (extreme values or absolute maximum/minimum values)

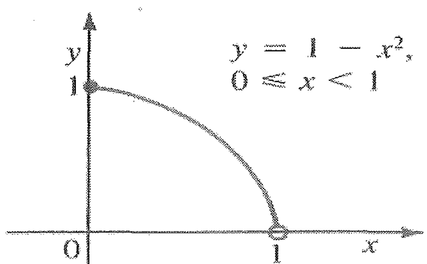


absolute/global min

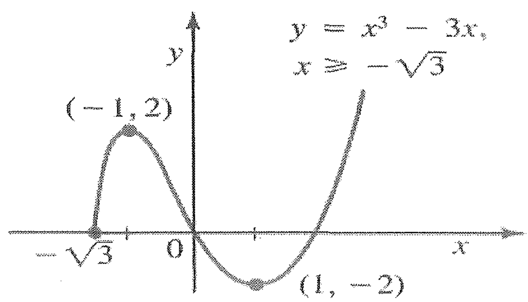
NO global max



No absolute max/min.



absolute max = 1



Example 2: Find the extreme values of $y = 2x^2 - 8x + 9$.

We know from gr 10: 'y' is a parabola opening up. hence, it will have global min

when $y' = 0$: $4x - 8 = 0$

$x = 2$

$\therefore f(2) = 2(2)^2 - 8(2) + 9$

$= 1$ is the global min

NOTE: We can also complete the square !!

$y = 2(x-2)^2 + 1$

Example 3 : $f(x) = x^3 - 3x^2$, $-2 \le x \le 1$

STEP 1: find $f'(x) = 0$ solve for $x \in [-2, 1]$.

$3x^2 - 6x = 0$

$3x(x-2) = 0$

$x = 0, 2$ $\quad x = 0 \in [-2, 1]$ $\quad x = 2 \notin [-2, 1]$

STEP2: find $f(0) = 0 \Rightarrow$ global max

$f(-2) = -8 - 12 = -20 \Rightarrow$ global min.

$f(1) = -2$

Example 4: $f(x) = \frac{1}{x^2 - 2x + 2}$ $\quad 0 \le x \le 2$

① $f'(x) = 0 \Rightarrow -1(x^2 - 2x + 2)^{-2}(2x - 2) = 0$

$\Rightarrow 2x - 2 = 0$

$x = 1 \in [0, 2]$

② $f(0) = 1/2$

$f(1) = 1 \Rightarrow$ absolute max

$f(2) = 1/2$

absolute min $= f(0) = f(2) = \frac{1}{2}$.