Unit 2 Pre-Test Review – Factor Theorem and Inequalities
MHF4U

Section 1: 2.1 - Long and Synthetic Division / Remainder Theorem

1) What is the remainder when $x^4 - 4x^2 - 2x + 3$ is divided by $x + 1$? Do not divide. Support your answer with an explanation.

Let $P(x) = x^4 - 4x^2 - 2x + 3$

Rem $= P(-1) = (-1)^4 - 4(-1)^2 - 2(-1) + 3$

$= 1 - 4 + 2 + 3$

$= 2$ ∴ Remainder is 2 according to remainder theorem.

2) Is $x - 3$ a factor of the polynomial $3x^2 - 8x - 3$? Do not divide. Support your answer with an explanation.

$P(x) = 3x^2 - 8x - 3$

$P(3) = 3(3)^2 - 8(3) - 3$

$= 0$

∴ $x - 3$ is a factor of $P(x)$ according to factor theorem.

3) Divide $\frac{f(x)}{g(x)}$ and state the answer in quotient form. Use synthetic division where possible.

a) $f(x) = x^4 - 4x^2 - 2x + 3, g(x) = x - 2$

2	1	0	−4	−2	3
		2	4	0	−4
	1	2	0	−2	−1

∴ $\frac{f(x)}{g(x)} = x^3 + 2x^2 - 2 - \frac{1}{x-2}$

b) $f(x) = x^5 - x^4 + 2x^3 + 3x - 2, \quad g(x) = x^2 + 2$

$$
\begin{array}{r|l}
 & x^3 - x^2 + 2 \\
x^2 + 0x + 2 & x^5 - x^4 + 2x^3 + 3x - 2 \\
 & x^5 + 0x^4 + 2x^3 \\
\hline
 & -x^4 + 0x^3 + 0x^2 \\
 & -x^4 + 0x^2 - 2x^2 \\
\hline
 & 2x^2 + 3x - 2 \\
 & 2x^2 + 0x + 4 \\
\hline
 & 3x - 6
\end{array}
$$

∴ $\frac{f(x)}{g(x)} = x^3 - x^2 + 2 + \frac{3x-6}{x^2+2}$

4) Perform each division. Express the answer in quotient form and write the statement that could be used to check the division.

a) $x^3 + 9x^2 - 5x + 3$ divided by $x - 2$

$$\begin{array}{r|rrrr} 2 & 1 & 9 & -5 & 3 \\ & & 2 & 22 & 34 \\ \hline & 1 & 11 & 17 & 37 \end{array}$$

$\therefore\ x^3 + 9x^2 - 5x + 3 = (x^2 + 11x + 17)(x-2) + 37$

b) $12x^3 - 2x^2 + x - 11$ divided by $3x + 1$

$$\begin{array}{r|rrrr} -\frac{1}{3} & 12 & -2 & 1 & -11 \\ & & -4 & 2 & -1 \\ \hline & 12 & -6 & 3 & -12 \end{array}$$

$\underbrace{12 \quad -6 \quad 3}_{\text{divide by 3}}$

$\therefore\ 12x^3 - 2x^2 + x - 11 = (4x^2 - 2x + 1)(3x+1) - 12$

c) $-8x^4 - 4x + 10x^3 - x^2 + 15$ divided by $2x - 1$

$$\begin{array}{r|rrrrr} \frac{1}{2} & -8 & 10 & -1 & -4 & 15 \\ & & -4 & 3 & 1 & -1.5 \\ \hline & -8 & 6 & 2 & -3 & 13.5 \end{array}$$

$\therefore\ -8x^4 - 4x + 10x^3 - x^2 + 15 =$

$= (-4x^3 + 3x^2 + x - 1.5)(2x-1) + 13.5$

d) $x^3 + 4x^2 - 3$ divided by $x - 2$

$$\begin{array}{r|rrrr} 2 & 1 & 4 & 0 & -3 \\ & & 2 & 12 & 24 \\ \hline & 1 & 6 & 12 & 21 \end{array}$$

$\therefore\ (x^3 + 4x^2 - 3) = (x^2 + 6x + 12)(x-2) + 21$

5) Determine the value of k such that when $f(x) = x^4 + kx^3 - 3x - 5$ is divided by $x - 3$, the remainder is -10.

$f(3) = -10 \Rightarrow -10 = 3^4 + 27k - 15 - 5$

$-10 = 81 + 27k - 15 - 5$

$-10 = 61 + 27k$

$-71 = 27k$

$\boxed{k = \frac{-71}{27}}$

Section 2: 2.2 – Factor Theorem

6) Suppose the cubic polynomial $\underbrace{8x^3 + mx^2 + nx - 6}_{P(x)}$ has both $2x + 3$ and $x - 1$ as factors. Find m and n. Do not divide.

$P(-\frac{3}{2}) = 0$

$8(-\frac{3}{2})^3 + m(-\frac{3}{2})^2 + n(-\frac{3}{2}) - 6 = 0$

$8(\frac{-27}{8}) + \frac{9m}{4} - \frac{3n}{2} - 6 = 0$

$\frac{9m}{4} - \frac{3n}{2} = +33 \quad \times 4$

$\therefore \boxed{9m - 6n = 132}$ ①

$P(1) = 0 \longrightarrow 8(1)^3 + m(1)^2 + n(1) - 6 = 0$

$8 + m + n - 6 = 0$

$\boxed{m + n = -2}$ ②

$9m - 6n = 132$ ①

$6m + 6n = -12$ ② $\times 6$

$+$ $15m = 120$

$\boxed{m = 8}$ sub in $m + n = -2$

$8 + n = -2$

$\boxed{n = -10}$

7) Factor each of the following

a) $x^3 - 4x^2 + x + 6 = P(x)$

$P(2) = 8 - 4(4) + 2 + 6 = 0 \Rightarrow x - 2$ factor

$$\begin{array}{r|rrrr} 2 & 1 & -4 & 1 & 6 \\ & \downarrow & 2 & -4 & -6 \\ \hline & 1 & -2 & -3 & 0 \end{array}$$

$\therefore P(x) = (x-2)(x^2 - 2x - 3)$

$= (x-2)(x-3)(x+1)$

b) $3x^3 - 5x^2 - 26x - 8 = g(x)$

$g(-2) = 0 \Rightarrow x + 2$ is a factor

$$\begin{array}{r|rrrr} -2 & 3 & -5 & -26 & -8 \\ & \downarrow & -6 & 22 & 8 \\ \hline & 3 & -11 & -4 & 0 \end{array}$$

$\therefore g(x) = (x+2)(3x^2 - 11x - 4)$

$= (x+2)(3x+1)(x-4)$

c) $-4x^3 - 4x^2 + 16x + 16$

$= -4x^2(x+1) + 16(x+1)$

$= (-4x^2+16)(x+1)$

$= -4(x^2-4)(x+1)$

$= -4(x-2)(x+2)(x+1)$

d) $x^3 - 64$

$a^3-b^3 = (a-b)(a^2+ab+b^2)$

$= (x-4)(x^2+4x+16)$

or Let $P(x) = x^3-64$

$P(4)=0 \Rightarrow$ $x-4$ is a factor

$$\begin{array}{c|cccc} 4 & 1 & 0 & 0 & -64 \\ & \downarrow & 4 & 16 & 64 \\ \hline & 1 & 4 & 16 & 0 \end{array}$$

$\therefore P(x) = (x-4)(x^2+4x+16)$

Section 3: 2.3&2.6 – Factoring to Solve Equations and Inequalities

8) Determine the real roots of each equation.

a) $(5x^2 + 20)(3x^2 - 48) = 0$

$5(x^2+4)(3)(x^2-16) = 0$

$15(x^2+4)(x-4)(x+4) = 0$

$\downarrow$

Sum of squares

No real roots.

$x = \{\pm 4\}$

b) $(2x^2 - x - 13)(x^2 + 1) = 0$

$\downarrow$

No real roots

$\downarrow$ USE QF $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$a=2 \quad b=-1 \quad c=-13$

$x = \dfrac{1 \pm \sqrt{1-4(2)(13)}}{4}$

$= \dfrac{1 \pm \sqrt{105}}{4}$ (exact answers)

9) Solve the following polynomial equations.

a) $2x^3 + 1 = x^2 + 2x$

$2x^3 - x^2 - 2x + 1 = 0$

$x^2(2x - 1) - 1(2x - 1) = 0$

$(x^2 - 1)(2x - 1) = 0$

$(x - 1)(x + 1)(2x - 1) = 0$

$x = \{\pm 1, \frac{1}{2}\}$

b) $x^3 + 6x^2 + 11x + 6 = 0$

Let $P(x) = x^3 + 6x^2 + 11x + 6$

$P(-1) = 0 \Rightarrow x + 1$ is a factor

$$\begin{array}{r|rrrr} -1 & 1 & 6 & 11 & 6 \\ & \downarrow & -1 & -5 & -6 \\ \hline & 1 & 5 & 6 & 0 \end{array}$$

$\therefore P(x) = (x+1)(x^2 + 5x + 6)$

$= (x+1)(x+2)(x+3)$

$P(x) = 0 \Rightarrow x = \{-3, -2, -1\}$

c) $x^5 - 4x^3 - x^2 + 4 = 0$

$x^3(x^2 - 4) - (x^2 - 4) = 0$

$(x^3 - 1)(x^2 - 4) = 0$

$(x - 1)(x^2 + x + 1)(x - 2)(x + 2) = 0$

$\therefore x = \{\pm 2, 1\}$

d) $3x^3 + 2x^2 - 11x - 10 = 0$

$P(x) = 3x^3 + 2x^2 - 11x - 10$

$P(2) = 0 \Rightarrow (x - 2)$ is a factor

$$\begin{array}{r|rrrr} 2 & 3 & 2 & -11 & -10 \\ & \downarrow & 6 & 16 & 10 \\ \hline & 3 & 8 & 5 & 0 \end{array}$$

$P(x) = (x - 2)(3x^2 + 8x + 5)$

$= (x - 2)(3x + 5)(x + 1)$

$P(x) = 0 \Rightarrow x = \{-\frac{5}{3}, -1, 2\}$

10) Solve the following polynomial inequalities. (Refer to #9 where you factored the polynomials)

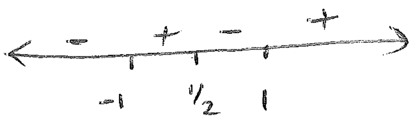
a) $2x^3 + 1 < x^2 + 2x$

$2x^3 - x^2 - 2x + 1 < 0$

$x^2(2x-1) - (2x-1) < 0$

$(2x-1)(x^2-1) < 0$

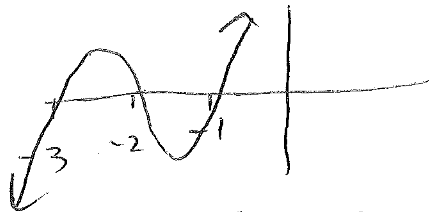
$(2x-1)(x-1)(x+1) < 0$



$x \in (-\infty, -1) \cup (\frac{1}{2}, 1)$

b) $x^3 + 6x^2 + 11x + 6 > 0$

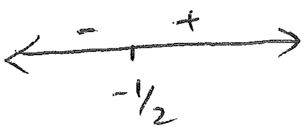
$(x+1)(x+2)(x+3) > 0$



$x \in (-3, -2) \cup (-1, \infty)$

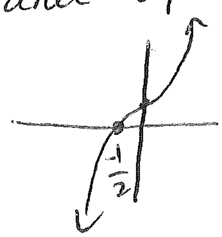
11) Where is the polynomial $y = 8x^3 + 1$ positive? Justify your solution.

$y = (2x)^3 + 1^3 = (2x+1)(4x^2 - 2x + 1)$



$\therefore x \in (-\frac{1}{2}, \infty)$

Transformations to cubic: horizontally compressed by a factor of $\frac{1}{2}$ and 1 unit up.



12) Solve $6x^3 + 13x^2 - 41x + 12 \le 0$ using a sign chart.

Let $P(x) = 6x^3 + 13x^2 - 41x + 12$

$P(1) \neq 0 \quad P(-1) \neq 0 \quad P(2) \neq 0 \quad P(-2) \neq 0$

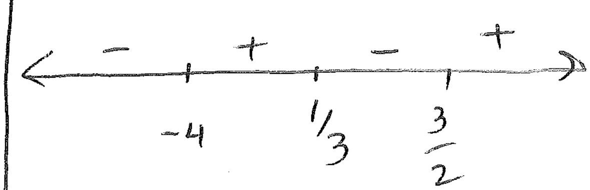
$P(-4) = 0 \Rightarrow (x+4)$ is a factor

$$\begin{array}{r|rrrr} -4 & 6 & 13 & -41 & 12 \\ & \downarrow & -24 & 44 & -12 \\ \hline & 6 & -11 & 3 & 0 \end{array}$$

$P(x) = (x+4)(6x^2 - 11x + 3)$

$= (x+4)(3x-1)(2x-3)$

$\therefore (x+4)(3x-1)(2x-3) \le 0$



$\therefore x \in (-\infty, -4] \cup [\frac{1}{3}, \frac{3}{2}]$

Section 4: 2.4 – Families of Polynomials

13) Find the equation for the family of quartic polynomials that have real roots of 3 (order 2) and $2 \pm \sqrt{2}$.

$y = a(x-3)^2(x-2+\sqrt{2})(x-2-\sqrt{2})$

$= a(x-3)^2((x-2)^2-2)$

$= a(x-3)^2(x^2-4x+4-2)$

$\therefore a(x-3)^2(x^2-4x+2)=0$ where $a \neq 0$

14) A family of cubic polynomials has roots of -2, -3 and -5. Find the member of this family that passes through the point (2,-35). What is this polynomials *y*-intercept?

family: $y = a(x+2)(x+3)(x+5)$

member: sub $x=2$ $y=-35$ solve for a

$-35 = a(4)(5)(7)$

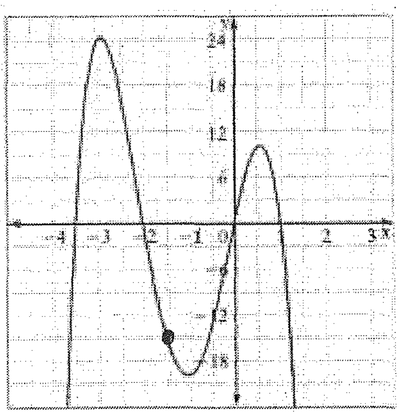
$-35 = 170a$

$a = -1/4$ $\therefore y = -\frac{1}{4}(x+2)(x+3)(x+5)$

y-int: sub $x=0$ $y = \frac{-15}{2}$

15) Find an equation for each of the following functions

a)



$y = a(x)(x-1)(x+2)(2x+7)$

Sub $x=-1.5$ $y=-15$

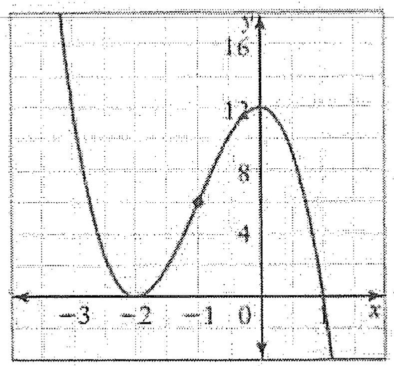
$-15 = a(-1.5)(-2.5)(0.5)(4)$

$-15 = 7.5a$

$a = -2$

$\therefore y = -2(x)(x-1)(x+2)(2x+7)$

b)



$y = a(x+2)^2(x-1)$

sub $y=12$ $x=0$

$12 = a(0+2)^2(0-1)$

$12 = -4a$

$a = -3$

$\therefore y = -3(x+2)^2(x-1)$

ANSWER KEY

1) $P(-1) = 2$ =remainder . This was found using remainder theorem.

2) $P(3) = 0$, so $x-3$ is a factor because remainder is 0 (Factor Theorem)

3)a) $\frac{x^4-4x^2-2x+3}{x-2} = x^3 + 2x^2 - 2 - \frac{1}{x-2}$ **b)** $\frac{x^5-x^4+2x^3+3x-2}{x^2+2} = x^3 - x^2 + 2 + \frac{3x-6}{x^2+2}$

4)a) $\frac{x^3+9x^2-5x+3}{x-2} = x^2 + 11x + 17 + \frac{37}{x-2}$; $x^3 + 9x^2 - 5x + 3 = (x-2)(x^2 + 11x + 17) + 37$

b) $\frac{12x^3-2x^2+x-11}{3x+1} = 4x^2 - 2x + 1 - \frac{12}{3x+1}$; $12x^3 - 2x^2 + x - 11 = (3x+1)(4x^2 - 2x + 1) - 12$

c) $\frac{-8x^4-4x+10x^3-x^2+15}{2x-1} = -4x^3 + 3x^2 + x - \frac{3}{2} + \frac{27}{2(2x-1)}$; $-8x^4 - 4x + 10x^3 - x^2 + 15 = (2x-1)\left(-4x^3 + 3x^2 + x - \frac{3}{2}\right) + \frac{27}{2}$

d) $\frac{x^3+4x^2-3}{x-2} = x^2 + 6x + 12 + \frac{21}{x-2}$; $x^3 + 4x^2 - 3 = (x-2)(x^2 + 6x + 12) + 21$

5) $k = -\frac{71}{27}$

6) $m = 8$, $n = -10$

7)a) $(x+1)(x-3)(x-2)$ **b)** $(x+2)(3x+1)(x-4)$ **c)** $-4(x+1)(x+2)(x-2)$ **d)** $(x-4)(x^2 + 4x + 16)$

8)a) (-4, 0) and (4, 0) **b)** $\left(\frac{1-\sqrt{105}}{4}, 0\right)$ and $\left(\frac{1+\sqrt{105}}{4}, 0\right)$

9) a) $x = -1, 1, \frac{1}{2}$ **b)** $x = -1, -2, -3$ **c)** $x = 1, -2, 2$ **d)** $x = -1, -\frac{5}{3}, 2$

10)a) $x \in (-\infty, -1) \cup (0.5, 1)$ **b)** $x \in (-3, -2) \cup (-1, \infty)$

11) $x \in \left(-\frac{1}{2}, \infty\right)$

12) $x \in (-\infty, -4] \cup \left[\frac{1}{3}, \frac{3}{2}\right]$

13) $P(x) = k(x-3)^2(x^2 - 4x + 2)$

14) $f(x) = -\frac{1}{4}(x+2)(x+3)(x+5)$, y-int is $\left(0, -\frac{15}{2}\right)$

15)a) $P(x) = -2x(x-1)(x+2)(2x+7)$ **b)** $P(x) = -3(x+2)^2(x-1)$