Part 2 – Exponential Decay

The exponential function can be used as a model to solve problems involving exponential decay.

$$\boldsymbol{A = a(1 - r)^n}$$

where: A = final amount / number
a = initial amount / number
r = decay rate in decimal
n = number of decay periods

Ex1. A new car costs $24 000. It loses 18% of its value each year after it is purchased.

a) Find an expression to represent the value of the car after *x* years.

$a = 24000$
$r = 18\% = 0.18/\text{year}$
$A = ?$

$A = 24000(1 - 0.18)^x$
$A = 24000(0.82)^x$

b) Determine the value of the car after 30 months.

$n = 30$ months
Convert months to years
$\frac{30}{12} = \frac{5}{2}$

$A = 24000(0.82)^{5/2}$
$\doteq 14613.22$

For most accurate results
24000 [×] 0.82 ^ (5 [A b/c] 2)

∴ The car's value will be approximately $14,613.22.

Ex2. A used-car dealer sells a five-year-old car for $4 200. What was the original value of the car if the depreciation is 15% a year?

$r = 0.15/\text{year}$
$A = 4200$
$n = 5$
$a = ?$

$A = a(1 - r)^n$
$4200 = a(1 - 0.15)^5$
$\frac{4200}{(0.85)^5} = \frac{a(0.85)^5}{(0.85)^5}$

$9465.74 \doteq a$

Calculator
4200 [÷] (0.85) [^] 5

∴ It was approximately $9465.74.

HALF LIFE

The "half-life" of a radioactive material (an isotope) is the time it takes for a sample to decay to half of the original amount. In general, radioactive materials decay according to the following equation:

$$A_L = A_0\left(\frac{1}{2}\right)^{t/h}$$

where A_L is the amount of isotope left

A_0 is the original amount of isotope

t is the elapsed time

h is the half-life of the isotope

For example, suppose you have a radioactive isotope that has a mass of 64 mg,

after one half-life, 32 mg is left

after two half-lives, 16 mg is left

after three half-lives, 8 mg is left

after four half-lives, 4 mg is left

Would the sample ever reach a mass of 0 mg? No

Ex1. The half-life of ruthenium-106 is 1 year. If an original sample of ruthenium-106 had an original mass of 128 mg, and there are 2 mg left, what is the elapsed time?

$h = 1$ year

$A_0 = 128$ mg

$A_L = 2$ mg

$t = ?$

$A_L = A_0\left(\frac{1}{2}\right)^{t/h}$

$\frac{2}{128} = \frac{128}{128}\left(\frac{1}{2}\right)^{t/1}$

$\frac{1}{64} = \left(\frac{1}{2}\right)^{t}$

$\left(\frac{1}{2}\right)^6 = \left(\frac{1}{2}\right)^t$

if the bases are equal, then $t = 6$

∴ 6 years passed.

Ex2. A radioactive isotope, iodine-131, is used to determine whether a person has a thyroid deficiency. The iodine-131 is injected into the blood stream. A healthy thyroid gland absorbs all of the iodine. The half-life of iodine-131 is 8.2 days. After how long would 25% of the iodine-131 remain in the thyroid gland of a healthy person?

$h = 8.2$ days

$t = ?$

$A_0 = 100\%$

$A_L = 25\%$

$A_L = A_0\left(\frac{1}{2}\right)^{t/h}$

$\frac{25\%}{25\%} = \frac{100\%}{100\%}\left(\frac{1}{2}\right)^{t/8.2}$

$\frac{1}{4} = \left(\frac{1}{2}\right)^{t/8.2}$

if $\left(\frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^{t/8.2}$

then $8.2 \times 2 = \frac{t}{8.2} \times 8.2$

$t = 16.4$

∴ 16.4 days have elapsed.

PRACTICE

1. To determine whether a pancreas is functioning normally, a tracer dye is injected. A normally functioning pancreas secretes 4% of the dye each minute. A doctor injects 0.50 g of the dye.
    a. If the pancreas is functioning normally, how much dye should remain after 20 minutes?
    b. The doctor determines that the actual level after 20 minutes is 0.35g. Is the pancreas functioning normally?

2. An element is decaying at the rate of 12%/h. Initially we have 100g.
    a. How much remains after 10 h? (round to the nearest gram)
    b. How much remains after 30 h? (round to the nearest gram)
    c. When will there be 40 g left?

3. A research assistant made 160 mg of radioactive sodium (Na^{24}) and found that there were only 20 mg left 45 h later.
    a. What is the half-life of Na^{24} ?
    b. If the laboratory requires 100 mg of Na^{24} 12 h from now, how much Na^{24} should the research assistant make now?
    c. How much of the original 20 mg would be left in 12 h?

4. On the day her daughter is born, an excited mother wants to give her new daughter a season's ticket to watch the Toronto Marlies. A season's ticket costs $900 when the daughter is born, but the mother decides to wait until her daughter is 6 years old before buying the ticket. If inflation is assumed to be 3% per year, how much money will the mother need in 6 years to buy the season's ticket?

5. Two different strains of cold virus were isolated and put in cultures to grow. Virus A doubles every 4.8 h while Virus B triples every 8 h. If each culture has 1000 viruses to start, which has more after 24 h?

6. A certain radioactive material has a half-life of 35 years. If 100 g is present now, how many grams will be present in 350 years?

7. In 30 hours, a sample of plutonium decays to $\frac{1}{256}$ of its original amount. What is the half-life of the substance?

1. To determine whether a pancreas is functioning normally, a tracer dye is injected. A normally functioning pancreas secretes 4% of the dye each minute. A doctor injects 0.50 g of the dye.
   a. If the pancreas is functioning normally, how much dye should remain after 20 minutes?
   b. The doctor determines that the actual level after 20 minutes is 0.35g. Is the pancreas functioning normally?

a)

$r = 0.04/\text{min}$

$a = 0.50g$

$$A = a(1-r)^n$$
$$= 0.50(1-0.04)^{20}$$
$$= 0.50(0.96)^{20}$$
$$= 0.22$$

$\therefore$ There is 0.22g left.

b) No

2. An element is decaying at the rate of 12%/h. Initially we have 100g.
   a. How much remains after 10 h? (round to the nearest gram)
   b. How much remains after 30 h? (round to the nearest gram)
   c. When will there be 40 g left?

a)

$a = 100g$

$r = 0.12/h$

$$A = a(1-r)^n$$
$$= 100(1-0.12)^{10}$$
$$= 100(0.88)^{10}$$
$$\doteq 28$$

$\therefore$ There's approximately 28g remaining.

b) $A = 100(0.88)^{30}$

$\doteq 2$

$\therefore$ There's approximately 2g remaining.

c) $40 = 100(0.88)^n$

try numbers for n.

When $n = 5$ 53g remaining

$n = 6$ 46g

$n = 7$ 41g

$n = 7.1$ 40g

$\therefore$ It's after approximately 7.1 h we have 40g left.

3. A research assistant made 160 mg of radioactive sodium (Na^{24}) and found that there were only 20 mg left 45 h later.

a. What is the half-life of Na^{24}?

b. If the laboratory requires 100 mg of Na^{24} 12 h from now, how much Na^{24} should the research assistant make now?

c. How much of the original 20 mg would be left in 12 h?

a)

$A_0 = 160mg$

$A_L = 20mg$

$t = 45h$

$h = ?$

$A_L = A_0\left(\frac{1}{2}\right)^{t/h}$

$\frac{20}{160} = \frac{160\left(\frac{1}{2}\right)^{45/h}}{160}$

$\frac{1}{8} = \left(\frac{1}{2}\right)^{45/h}$

$\left(\frac{1}{2}\right)^3 = \left(\frac{1}{2}\right)^{45/h}$

$3 = \frac{45}{h}$

$3h = 45$

$\boxed{h = 15}$

∴ It's 15 hours

b) $A_0 = ?$ mg

$A_L = 100$

$t = 12h$

$h = 15h$

$A_L = A_0\left(\frac{1}{2}\right)^{t/h}$

$100 = A_0\left(\frac{1}{2}\right)^{12/15}$

$\frac{100}{(0.5)^{0.8}} = \frac{A_0(0.5)^{0.8}}{(0.5)^{0.8}}$

$174 \doteq A_0$

calculator

100 [÷] [(] 0.5 [)] [^] 0.8

∴ The assistant needs to prepare 174mg.

c) $A_0 = 20mg$

$A_L = ?$

$t = 12$

$h = 15$

$A_L = 20\left(\frac{1}{2}\right)^{12/15}$

$A_L \doteq 11.5$

calc

20 [×] (0.5) [^] (12 [÷] 15)

∴ There'll be approximately 11.5 mg left.

4. On the day her daughter is born, an excited mother wants to give her new daughter a season's ticket to watch the Toronto Marlies. A season's ticket costs $900 when the daughter is born, but the mother decides to wait until her daughter is 6 years old before buying the ticket. If inflation is assumed to be 3% per year, how much money will the mother need in 6 years to buy the season's ticket?

$a = 900$

$n = 6$

$A = ?$

$r = 0.03$

$$A = a(1+r)^n$$
$$= 900(1+0.03)^6$$
$$= 900(1.03)^6$$
$$\doteq 1074.65$$

∴ The ticket will be $1074.65

5. Two different strains of cold virus were isolated and put in cultures to grow. Virus A doubles every 4.8 h while Virus B triples every 8 h. If each culture has 1000 viruses to start, which has more after 24 h?

Virus A	Virus B
$A = a(2)^{t/\text{double}}$	$A = a(3)^{t/\text{triple}}$
$= 1000(2)^{24/4.8}$	$= 1000(3)^{24/8}$
Virus A $= 32{,}000$	Virus B $= 27{,}000$

∴ Virus A is more by 5000.

6. A certain radioactive material has a half-life of 35 years. If 100 g is present now, how many grams will be present in 350 years?

$h = 35$

$A_0 = 100$

$t = 350$

$A_L = ?$

$$A_L = 100\left(\frac{1}{2}\right)^{350/35}$$
$$= 100\left(\frac{1}{2}\right)^{10}$$
$$\doteq 0.0976$$

∴ There'll be 0.0976g left.

7. In 30 hours, a sample of plutonium decays to $\frac{1}{256}$ of its original amount. What is the half-life of the substance?

$h = ?$

$t = 30$

$A_0 = 256$ (if we had 256g)

$A_L = 1$ (we'd have 1g left)

$$A_L = A_0\left(\frac{1}{2}\right)^{t/h}$$

$$\frac{1}{256} = \frac{256\left(\frac{1}{2}\right)^{30/h}}{256}$$

$$\frac{1}{256} = \left(\frac{1}{2}\right)^{30/h}$$

$$\left(\frac{1}{2}\right)^{8} = \left(\frac{1}{2}\right)^{30/h}$$

256	2
128	2
64	2
32	2
16	2
8	2
4	2
2	2
1	

$\} \; 2^8$

$$8 = \frac{30}{h}$$

$$8h = 30$$

$$h = 3.75$$

$\therefore$ The half life is 3.75h.