Day 7: Implicit Differentiation

Consider the following 2 equations describing familiar curves:

$y = x^2 + 3$ parabola

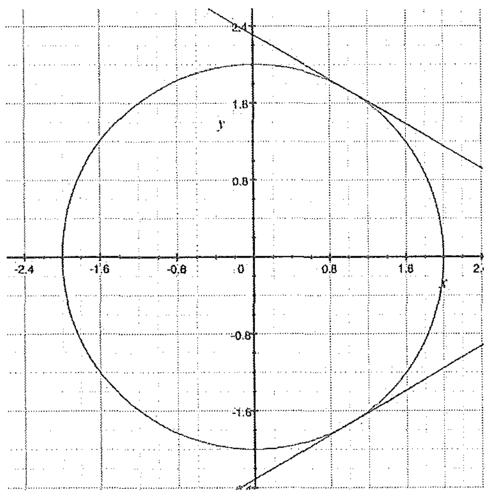
$x^2 + y^2 = 4$ circle

The first equation defines y as a function of x ***explicitly***, since for each x, the equation gives an explicit formula $y = f(x)$ for finding the corresponding value of y.

The second equation does not define a function, since it fails the vertical line test. However you can solve for y and find at least 2 functions ($y = -\sqrt{4-x^2}$ and $y = \sqrt{4-x^2}$) that are defined ***implicitly*** by the equation $x^2 + y^2 = 4$. If we wanted to find the derivative at a point $x = c$ then we would have to calculate $f'(c)$ for each implicitly defined function. In the case of the circle, there are 2 slopes for every $x = c$ where the derivative exists (in this example x=2 would give a vertical slope, so no derivative there).

Example at $x = 1$

Bottom Half	Top Half
$y = -\sqrt{4-x^2}$	$y = \sqrt{4-x^2}$
$y' = \dfrac{x}{\sqrt{4-x^2}}$	$y' = \dfrac{-x}{\sqrt{4-x^2}}$
$y'(1) = \dfrac{1}{\sqrt{3}}$	$y'(1) = \dfrac{-1}{\sqrt{3}}$



Find the equation of each tangent line:

eqn 1 $y - y_1 = m(x - x_1)$ $\left(m = \frac{1}{\sqrt{3}}, x = 1, y = -\sqrt{3}\right)$

$y + \sqrt{3} = \frac{1}{\sqrt{3}}(x - 1)$

$y = \frac{1}{\sqrt{3}}x - \frac{1}{\sqrt{3}} - \sqrt{3}$

eqn ② $y = \frac{1}{\sqrt{3}}x - \frac{1}{\sqrt{3}} + \sqrt{3}$

Here's how we use implicit differentiation on our circle example:
$x^2 + y^2 = 4$

$2x + 2yy' = 0$

$y' = \frac{-2x}{2y} = \frac{-x}{y}$

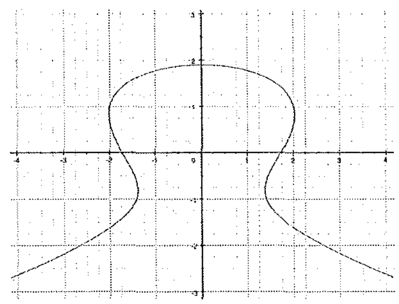
Check that the points $(1, \sqrt{3})$ and $(1, -\sqrt{3})$ gives the same results as when we did explicit differentiation.

$y' = \frac{-1}{\sqrt{3}}$ when $x=1$, $y=\sqrt{3}$

$y' = \frac{-1}{-\sqrt{3}} = \frac{1}{\sqrt{3}}$ when $x=1$ $y=-\sqrt{3}$

Example

Find $\frac{dy}{dx}$ for $x^2 + y^3 - 2y = 3$. Then, find the slope of the tangent line at the point (2,1).



$2x + 3y^2y' - 2y' = 0$

$3y^2y' - 2y' = -2x$

$y'(3y^2 - 2) = -2x$

$y' = \frac{-2x}{3y^2 - 2}$

when $x=2$ $y=1$ $\quad y' = \frac{-2}{3-2} = \frac{-2}{1} = -2$

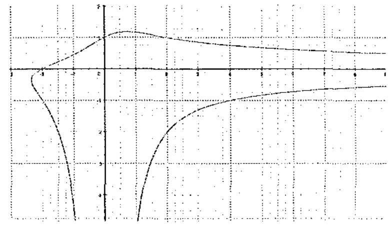
$\therefore y - y_1 = m(x - x_1)$

$y - 1 = -2(x-2)$

$\boxed{y = -2x+5}$ OR $2x + y - 5 = 0$

Example

Find $\frac{dy}{dx}$ for $x^2y^2 - 2x = 4 - 4y$. Then, find the slope of the tangent line at the point (2,-2).



$$(2x)(y^2) + x^2 2yy' - 2 = -4y'$$

$$x^2 2yy' + 4y' = -2xy^2 + 2$$

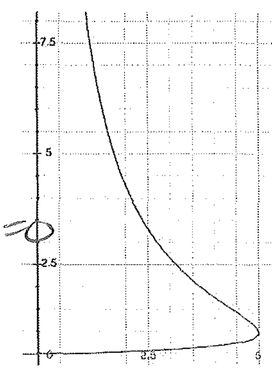
$$y'(2x^2y + 4) = -2xy^2 + 2$$

$$y' = \frac{-2xy^2 + 2}{2x^2y + 4}$$

Example

Hey, remember van der Waal's equation $\left(P + \frac{n^2a}{V^2}\right)(V - nb) = nRT$.

Find $\frac{dV}{dP}$ if n=1, a=5, b=0.03, R=0.97, and T=10. How is volume changing w.r.t. pressure when the pressure is 1 and the volume is 5? Those crazy chemists have some funky curves.



$$\left(P + \frac{n^2a}{V^2}\right)(V - nb) = nRT$$

$$\left(1 + (n^2a)(-2)V^{-3} \cdot \frac{dV}{dP}\right)(V - nb) + \left(P + \frac{n^2a}{V^2}\right)\left(\frac{dV}{dP}\right) = 0$$

$$\left(1 + \frac{(5)(-2)}{5^3} \cdot \frac{dV}{dP}\right)(5 - 1(0.03)) + \left(1 + \frac{5}{5^2}\right)\frac{dV}{dP} = 0$$

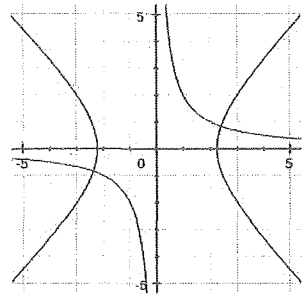
$$\left(1 - \frac{2}{25}V'\right)(4.97) + (1.2)V' = 0$$

$$4.97 - \frac{9.94}{25}V' + 1.2V' = 0$$

$$0.8024V' = -4.97$$

$$V' = -5.39$$

Example (90°, right, perpendicular, normal, orthogonal)
Show that any curve of the form $xy = c$ for some constant c intersects any curve of the form $x^2 - y^2 = k$ for some constant k, at right angles (that is the tangent lines to the curve at the intersection point are perpendicular). In this case, we say that the families of curves are ***orthogonal***. The graph shown is when c=2 and k=5



$xy = c$

$1(y) + xy' = 0$

$y' = \frac{-y}{x}$

$x^2 - y^2 = k$

$2x - 2yy' = 0$

$-2yy' = -2x$

$y' = \frac{-2x}{-2y}$

$y' = \frac{x}{y}$

To show
If two curves intersect at 90°, we need to show their slopes multiply to −1 OR they are reciprocal of each other.

$\left(\frac{-y}{x}\right)\left(\frac{x}{y}\right) = -1$, $x \neq 0$ $y \neq 0$

Hence, $xy = c$ and $x^2 - y^2 = k$ curves are orthogonal.