Day 5/6: 2.5 The Chain Rule
The Derivative of Composite Functions

<u>Using the Chain Rule (*"proper"* way using Leibniz notation)</u>

> **Rule 8 The Chain Rule**
> If f is differentiable at the point $u = g(x)$, and g is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and
> $$(f \circ g)'(x) = f'(g(x))g'(x)$$
>
> In Leibniz notation, if $y = f(u)$ and $u = g(x)$, then $\dfrac{dy}{dx} = \dfrac{dy}{du} \square \dfrac{du}{dx}$
>
> where $\dfrac{dy}{du}$ is evaluated at $u = g(x)$

$y(x) = (2 - x^3)^{50}$ $\quad \dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$

$y(u) = u^{50}, \quad u = 2 - x^3$

$y'(x) = y'(u) \cdot u'$

$= 50u^{49} \cdot (-3x^2)$

$= 50(2 - x^3)^{49}(-3x^2)$

$= -150x^2(2 - x^3)^{49}$

or $\quad y = [g(x)]^n$

$y' = n[g(x)]^{n-1} \cdot g'(x)$

$y = (2 - x^3)^{50}$

$y' = 50(2 - x^3)^{49}(-3x^2)$

$= -150x^2(2 - x^3)^{49}$

$x = -2$

$u = 1$

Ex: Use the chain rule, in Leibniz notation, to find $\frac{dy}{dx}$ at the given value of x:

$y = u\left(u^2+3\right)^3, u = (x+3)^2, x = -2$ $\qquad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$y' = \left[1(u^2+3)^3 + u(3)(u^2+3)^2 \cdot 2u\right](2)(x+3)$

$= \left[(1+3)^3 + 3(1+3)^2 \cdot 2\right]\left[(2)(-2+3)\right]$

$= [64 + 54][2]$

$= (118)(2) = 236$

Ex: Find the derivatives of the following:

$g(x) = \frac{1}{(3-4x)^2} = (3-4x)^{-2}$

$g'(x) = -2(3-4x)^{-3}(-4)$

$= \frac{8}{(3-4x)^3}$

$f(x) = \overset{p}{\left(x^3-5\right)^4}\overset{q}{\left(2-3x^2\right)^2}$

$f'(x) = \underbrace{4(x^3-5)^3(3x^2)}_{p'}(q)$

$+ p\left((2)(2-3x^2)^1(-6x)\right)$

We can also use the quotient rule

$f'(x) = 6x^2(x^3-5)^3(2-3x^2)^2 + (-12x)(x^3-5)^4(2-3x^2)$

$= 6x(x^3-5)^3(2-3x^2)\left[x(2-3x^2) - 2(x^3-5)\right]$

$= 6x(x^3-5)^3(2-3x^2)(2x-3x^3-2x^3+10)$

$= 6x(x^3-5)^3(2-3x^2)(-5x^3+2x+10)$

Find the derivative and simplify

a. $f(x) = (2x-5)^3(3x^2+4)^5$

$$f'(x) = 3(2x-5)^2(2)(3x^2+4)^5 + (2x-5)^3(5)(3x^2+4)^4(6x)$$

$$= 6(2x-5)^2(3x^2+4)^5 + 30x(2x-5)^3(3x^2+4)^4$$

$$= 6(2x-5)^2(3x^2+4)^4[3x^2+4+5x(2x-5)]$$

$$= 6(2x-5)^2(3x^2+4)^4(13x^2-25x+4)$$

↳ does not factor.

b. $g(x) = 8x^3(4x^2+2x-3)^5$

$$g'(x) = (24x^2)(4x^2+2x-3)^5 + (8x^3)(5)(4x^2+2x-3)^4(8x+2)$$

$$= 8x^2(4x^2+2x-3)^4[3(4x^2+2x-3)+5x(8x+2)]$$

$$= 8x^2(4x^2+2x-3)^4(52x^2+16x-9)$$

expand and Simplify

c. $y = (5+x)^2(4-7x^3)^6$

$$y' = 2(5+x)(4-7x^3)^6 + (5+x)^2(6)(4-7x^3)^5(-21x^2)$$

$$= 2(5+x)(4-7x^3)^5[4-7x^3+(3)(-21x^2)(5+x)]$$

$$= 2(5+x)(4-7x^3)^5(4-7x^3-315x^2-63x^3)$$

$$= 2(5+x)(4-7x^3)^5(-70x^3-315x^2+4)$$

d. $h(x) = \frac{6x-1}{(3x+5)^4}$

$$h'(x) = \frac{6(3x+5)^4 - (6x-1)(4)(3x+5)^3(3)}{(3x+5)^8}$$

$$= \frac{6(3x+5)^3[3x+5-2(6x-1)]}{(3x+5)^8}$$

$$= \frac{6[-9x+7]}{(3x+5)^3}$$

e. $y = \frac{(2x^2-5)^3}{(x+8)^2}$

$$y' = \frac{3(2x^2-5)^2(2)(x+8)^2 - (2x^2-5)^3(2)(x+8)}{(x+8)^4}$$

$$= \frac{2(2x^2-5)^2(x+8)\left[3(x+8) - (2x^2-5)\right]}{(x+8)^4}$$

$$= \frac{2(2x^2-5)^2(-2x^2+3x+29)}{(x+8)^3}$$

f. $p(x) = \frac{-3x^4}{\sqrt{4x-9}} = -3x^4(4x-9)^{-1/2}$

$$p'(x) = -12x^3(4x-9)^{-\frac{1}{2}} - 3x^4\left(-\frac{1}{2}\right)(4x-9)^{-\frac{3}{2}}(4)$$

$$= -12x^3(4x-9)^{-\frac{1}{2}} + 6x^4(4x-9)^{-3/2}$$

$$= -6x^3(4x-9)^{-3/2}\left[2(4x-9)-x\right] = \frac{-6x^3(7x-18)}{(4x-9)^{3/2}}$$

g. $r(x) = \left(\frac{2x+5}{6-x^2}\right)^4$

$$r'(x) = 4\left(\frac{2x+5}{6-x^2}\right)^3\left(\frac{2x+5}{6-x^2}\right)'$$ use the quotient rule to find the last part

$$= 4\left(\frac{2x+5}{6-x^2}\right)^3\left[\frac{2(6-x^2)-(2x+5)(-2x)}{(6-x^2)^2}\right] = \frac{4(2x+5)^3(2x^2+10x+12)}{(6-x^2)^3}$$

$$= \frac{8(2x+5)^3(x^2+5x+6)}{(6-x^2)^3}$$

$$= \frac{8(2x+5)^3(x+2)(x+3)}{(6-x^2)^3}$$

h. $t(x) = \left[\frac{1}{(4x+x^2)^3}\right]^3 = (4x+x^2)^{-9}$

$$t'(x) = -9(4x+x^2)^{-10}(4+2x)$$

$$= \frac{-18(x+2)}{(4x+x^2)^{10}}$$