Day 2: 2.2 The Derivatives of Polynomial Functions

So this is the lesson where most calculus students get upset because they learn that they were doing derivatives the long way. Anyway, it is important to understand where the derivative comes from, how it is defined in terms of limits, and when it is defined. Now that you completely understand the theory we can now turn our attention to the practice of calculating derivatives. So here we go.

Rule 1 Derivative of a Constant function

If f is the function with the constant value c, then $f'(c) = 0$ or $\frac{df}{dx} = \frac{d}{dx}(c) = 0$

Proof of Rule 1

This should make complete sense since the function is a horizontal straight line, hence its slope should be zero.

Example of Rule 1

What is the derivative of $f(x) = 5$?

$f'(x) = 0$

$f(x) = 3455624$?

$F'(x) = 0$

Rule 2 Power Rule for Positive Integer Powers of x

If n is a positive integer, then $\frac{d}{dx}(x^n) = nx^{n-1}$

Proof of Rule 2 (see page 77 in the textbook) $f(x) = x^n$

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{[(x+h) - x]}\,[(x+h)^{n-1} + (x+h)^{n-2}x + (x+h)^{n-3}x^2 + \ldots + x^{n-1}]}{\cancel{h}}$$

$$= (x)^{n-1} + x^{n-2}x + x^{n-3}x^2 + \ldots + x^{n-1}$$

$$= \underbrace{(x^{n-1}) + (x^{n-1}) + \ldots + (x^{n-1})}_{n \text{ times}}$$

$$= nx^{n-1}$$

Example of Rule 2

What is the derivative of $f(x)=x^5$?

$f'(x) = 5x^4$

$y=\dfrac{t^{107}}{t^{17}} = t^{90}$

$y' = 90t^{89}$

$y=\sqrt[3]{x^2} = x^{\frac{2}{3}}$

$y' = \frac{2}{3}x^{\frac{2}{3}-1}$

$= \frac{2}{3}x^{-\frac{1}{3}}$

$= \dfrac{2}{3\sqrt[3]{x}}$

Rule 3 The Constant Multiple Rule

If g(x) is a differentiable function of x and k is a constant, then $\frac{d}{dx}(kg(x)) = k\frac{d}{dx}[g(x)]$

Or if $f(x) = kg(x)$, then $f'(x) = kg'(x)$

Proof of Rule 3 (see page 78 in the textbook)

$f'(x) = \lim_{h \to 0} \dfrac{k(g(x+h)) - kg(x)}{h}$ factor k'

$= k \underbrace{\lim_{h \to 0} \dfrac{g(x+h) - g(x)}{h}}_{\text{definition of } g'(x)}$

$= kg'(x)$

Example of Rule 3

What is the derivative of $f(x)=5x^5$?

$f'(x) = 5(5x^4)$

$= 25x^4$

$y=\pi x^{\pi}$

$y' = \pi(\pi x^{\pi-1})$

$= \pi^2 x^{\pi-1}$

Rule 4 The Sum and Difference Rule

If u and v are differentiable functions of x, then their sum and difference are differentiable at every point where u and v are differentiable. At such points

$$\frac{d}{dx}(f(x)+g(x)) = \frac{df}{dx} + \frac{dg}{dx} = f'(x) + g'(x)$$

Proof of Rule 4 (see page 79 in the textbook) let $p(x) = f(x) + g(x)$

$$p'(x) = \lim_{h \to 0} \frac{p(x+h) - p(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= f'(x) + g'(x)$$

Example of Rule 4

What is the derivative of $f(x) = x^5 + x^4$?

$$f'(x) = 5x^4 + 4x^3$$

$y = t^7 - 4t^4 + 5t^3 - 20t^2 + 7t + 9$

$$y' = 7t^6 - 16t^3 + 15t^2 - 40t + 7$$

$y = 2x^6 + 3\sqrt{x}$

$$y' = 12x^5 + \frac{3}{2\sqrt{x}}$$

$$(\sqrt{x})' = \frac{1}{2\sqrt{x}}$$

$$\hookrightarrow \left(x^{\frac{1}{2}}\right)' = \frac{1}{2} x^{-\frac{1}{2}}$$

$f(x) = \dfrac{4x^2 - 3x + 2\sqrt{x}}{x}$

$$F(x) = \frac{4x^2}{x} - \frac{3x}{x} + \frac{2\sqrt{x}}{x}$$

$$= 4x - 3 + 2x^{-\frac{1}{2}}$$

$$f'(x) = 4 + 2\left(-\frac{1}{2} x^{-\frac{3}{2}}\right)$$

$$= 4 - x^{-\frac{3}{2}}$$

$$= 4 - \frac{1}{\sqrt{x^3}}$$