Day 1: Introduction to Derivatives

In the previous sections, we looked at rates of change and were able to determine the slope of a tangent line as the limit of the slopes of secant lines. This amazing result allowed us to calculate instantaneous rates of change of a function at any point. The instantaneous rate of a function is called the ***derivative*** and the study of derivatives is called ***differential calculus***.

> Definition: **Derivative**
> The derivative of a function f w.r.t. (with respect to) the variable x is **the function f'** whose value at x is:
>
> $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$ provided the limit exits
>
> [$f'(x)$ is pronounced "f prime of x"]

Does this definition look familiar? It does since it is the same formula we used in the previous chapter to find the slope of the tangent line; which is what the derivative is.

Notation: The derivative of a function $y = f(x)$ can be denoted by the following symbols:

$$y' \quad f'(x) \quad D_x y \quad \frac{dy}{dx} \quad \frac{d}{dx}f(x) \quad (\textit{Leibniz notation})$$

Derivative of f(x) at x = 3 can be written as f'(3)

Ex1: Find the derivative function of y = $x^2 - 3x + 5$. Use the derivative to find the slope of the tangent at x=3 and then find the equation of the tangent.

$$f(x+h) = (x+h)^2 - 3(x+h) + 5$$
$$= x^2 + 2xh + h^2 - 3x - 3h + 5$$

$$y' = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 3x - 3h + 5 - (x^2 - 3x + 5)}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2 - 3h}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2x + h - 3)}{\cancel{h}}$$ sub $h=0$ once h cancels!

$$= 2x - 3$$

when $x=3$, $y'(3) = 2(3) - 3 = 3 = m$ $\quad y(3) = 3^2 - 3(3) + 5 = 5$

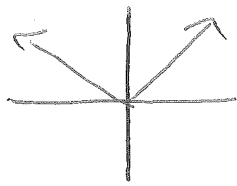
$y - y_1 = m(x - x_1) \Rightarrow y - 5 = 3(x - 3)$

$$\boxed{y = 3x - 4}$$

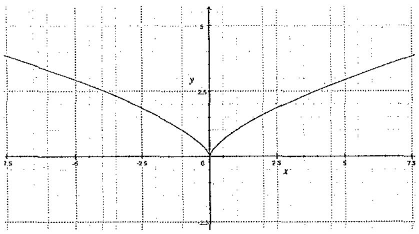
Conditions for the Derivative to Exist

- *The domain of f' is the set of all points for which the domain of f for which the limit exists. It may be smaller than the domain of f.*
- *If $f'(x)$ exists, we say f has a **derivative (is differentiable)** at x.*
- *A function that is differentiable at every point of its domain is a **differentiable function.***

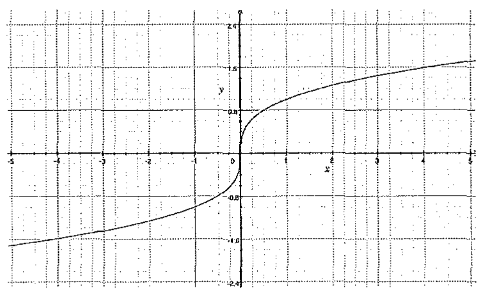
4 examples of how $f'(a)$ does not exist



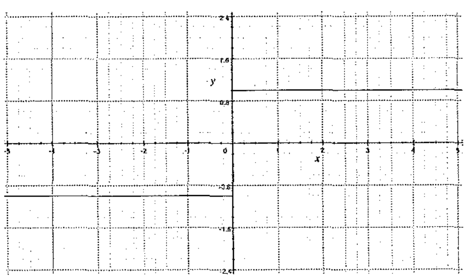
1. ***A corner****, where 1 sided derivatives differ, example* $f(x) = |x|$



2. ***A cusp****, where the slopes of the secants approach* ∞ *from one side and* $-\infty$ *from the other side, example* $f(x) = x^{\frac{2}{3}}$



3. ***A vertical tangent****, where the slopes of the tangent approach* ∞ *or* $-\infty$ *from both sides, example* $f(x) = \sqrt[3]{x}$



4. ***A discontinuity****, where one or both 1-sided limits FAIL to exist, example*

$$f(x) = \begin{cases} -1, x < 0 \\ 1, x \geq 0 \end{cases}$$

What do you notice about the above examples?

① and ② slope from the left does not equal to slope from the right.

③ vertical tangent

④ discontinuous (Jump)

Using the difference of quotient to find the derivatives $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

1. $f(x) = x$

$$f'(x) = \lim_{h \to 0} \frac{\cancel{x}+h-\cancel{x}}{h}$$

$$= \lim_{h \to 0} \frac{h}{h}$$

$$= 1$$

2. $f(x) = x^2$

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{(x+h+x)(\cancel{x}+\cancel{h}-\cancel{x})}{\cancel{h}}$$

$$= 2x$$

3. $f(x) = x^3$

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{(\cancel{x}+\cancel{h}-\cancel{x})((x+h)^2 + (x+h)(x) + x^2)}{\cancel{h}}$$

$$= (x+0)^2 + (x+0)(x) + x^2$$

$$= 3x^2$$

4. $f(x) = x^4$

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^4 - x^4}{h}$$

$$= \lim_{h \to 0} \frac{[(x+h)^2 - x^2][(x+h)^2 + x^2]}{h}$$

$$= \lim_{h \to 0} \frac{(x+h+x)(\cancel{x}+\cancel{h}-\cancel{x})[(x+h)^2 + x^2]}{\cancel{h}}$$

$$= (2x)(x^2 + x^2) = (2x)(2x^2)$$

$$= 4x^3$$

5. $f(x) = \sqrt{x}$

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h \to 0} \frac{\cancel{x}+\cancel{h}-\cancel{x}}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$

6. $f(x) = \frac{1}{x}$

$$f'(x) = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$= \lim_{h \to 0} \frac{x - (x+h)}{(x+h)(x)} \div h$$

$$= \lim_{h \to 0} \frac{\cancel{x} - \cancel{x} - \cancel{h}}{(x+h)(x)(\cancel{h})}$$

$$= \frac{-1}{x^2}$$

Summary:

y	x	x^2	x^3	x^4	$\sqrt{x}$	$1/x$
y'	1	$2x$	$3x^2$	$4x^3$	$1/2\sqrt{x}$	$-1/x^2$