Review for Unit 1

Evaluate the following limits. (write DNE if the limit does not exist)

a) $\lim_{x\to 3} 4x^2 - 2x$

$= 4(3)^2 - 2(3)$

$= 36 - 6$

$= 30$

b) $\lim_{x\to 2} 4$

$= 4$

c) $\lim_{x\to 3} \frac{x^2-2x-3}{x-3}$

$= \lim_{x\to 3} \frac{\cancel{(x-3)}(x+1)}{\cancel{(x-3)}}$

$= 4$

d) $\lim_{x\to 0} \frac{\sqrt{x+9}-3}{x} \cdot \frac{\sqrt{x+9}+3}{\sqrt{x+9}+3}$

$= \lim_{x\to 0} \frac{\cancel{x}+\cancel{9}-\cancel{9}}{\cancel{x}(\sqrt{x+9}+3)}$

$= \frac{1}{\sqrt{9}+3} = 1/6$

e) $\lim_{x\to 0} \frac{\sqrt{x+4}-2}{3x} \cdot \frac{\sqrt{x+4}+2}{\sqrt{x+4}+2}$

$= \lim_{x\to 0} \frac{\cancel{x}+\cancel{4}-\cancel{4}}{3\cancel{x}(\sqrt{x+4}+2)}$

$= \frac{1}{3(4)} = \frac{1}{12}$

f) $\lim_{x\to 0} \frac{(x+27)^{\frac{1}{3}}-3}{x}$

$u = (x+27)^{\frac{1}{3}}$

$x=0 \quad u=3$

$= \lim_{u\to 3} \frac{u-3}{u^3-27}$

$= \lim_{u\to 3} \frac{\cancel{u-3}}{\cancel{(u-3)}(u^2+3u+9)}$

$= 1/27$

g) $\lim_{x\to 3} \frac{|x-3|}{x-3} = d.n.e$

$\lim_{x\to 3^+} \frac{|x-3|}{x-3} = \lim_{x\to 3^+} \frac{(x-3)}{(x-3)} = 1$

$\lim_{x\to 3^-} \frac{|x-3|}{x-3} = \lim_{x\to 3^-} \frac{-\cancel{(x-3)}}{\cancel{x-3}} = -1$

h) $\lim_{x\to 0} \frac{|x|}{x} = d.n.e$

$L^+ = 1$

$L^- = -1$

see (g)

i) $\lim_{x\to -1} \frac{3x^3-2x+1}{4x^2+4x-6}$

$= \frac{3(-1)^3 - 2(-1)+1}{4(-1)^2 + 4(-1) - 6}$

$= \frac{0}{-6} = 0.$

j) $\lim_{x\to\infty} \frac{2x^2-7x+3}{5x^3-3x+1}$

$= \lim_{x\to\infty} \frac{\frac{2x^2}{x^3} - \frac{7x}{x^3} + \frac{3}{x^3}}{\frac{5x^3}{x^3} - \frac{3x}{x^3} + \frac{1}{x^3}}$

$= \frac{0}{5} = 0$

k) $\lim_{t\to 4} \frac{t^2-16}{2-\sqrt{t}} \cdot \frac{2+\sqrt{t}}{2+\sqrt{t}}$

$= \lim_{t\to 4} \frac{(t-4)(t+4)(2+\sqrt{t})}{4-t}$

$= \lim_{t\to 4} \frac{\cancel{(t-4)}(t+4)(2+\sqrt{t})}{-\cancel{(t-4)}} = -32$

l) $\lim_{x\to 1} \frac{x^4-5x^2+1}{x+2}$

$= \frac{1^4 - 5(1)^2 + 1}{1+2}$

$= \frac{-3}{3} = -1$

m) $\lim_{x\to 2} \frac{x^3-8}{x^2-3x+2}$

$= \lim_{x\to 2} \frac{\cancel{(x-2)}(x^2+2x+4)}{\cancel{(x-2)}(x+1)}$

$= \frac{12}{3} = 4$

n) $\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} \cdot \frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}$

$= \lim_{x\to 0} \frac{\cancel{x}+\cancel{1}-\cancel{1}}{\cancel{x}(\sqrt{x+1}+1)}$

$= \frac{1}{2}$

o) $\lim_{x\to 4} \frac{\frac{1}{\sqrt{x}} - \frac{1}{2}}{x-4} = \lim_{x\to 4} \frac{2-\sqrt{x}}{(2\sqrt{x})(x-4)} \cdot \frac{2+\sqrt{x}}{2+\sqrt{x}}$

$= \lim_{x\to 4} \frac{4-x}{(2\sqrt{x})(x-4)(2+\sqrt{x})}$

$= \lim_{x\to 4} \frac{-\cancel{(x-4)}}{(2\sqrt{x})\cancel{(x-4)}(2+\sqrt{x})} = \frac{-1}{16}$

2. A toy rocket is launched in the air so that its height, s, at time t is measured by $s(t) = -5t^2 + 30t + 2$. What is the instantaneous velocity at 4 seconds?

$$V(4) = \lim_{h\to 0} \frac{s(4+h) - s(4)}{h}$$

$$= \lim_{h\to 0} \frac{-5h^2 - 10h + \cancel{42} - \cancel{42}}{h}$$

$$= \lim_{h\to 0} -5h - 10 = -10 \text{ m/s.}$$

$$s(4+h) = -5(4+h)^2 + 30(4+h) + 2$$
$$= -5(16 + 8h + h^2) + 120 + 30h + 2$$
$$= -80 - 40h - 5h^2 + 120 + 30h + 2$$
$$= -5h^2 - 10h + 42$$
$$s(4) = -5(4)^2 + 30(4) + 2$$
$$= 42$$

3. Find the slope of the tangent to $y = 2x^2 - 4x$ at the point $(2, 0)$. Then find the equation of the tangent line in standard form.

$$m = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h\to 0} \frac{2h^2 + 4h - 0}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(2h+4)}{\cancel{h}} = 4$$

$\therefore m = 4 \quad x = 2 \quad y = 0$

$$f(2+h) = 2(2+h)^2 - 4(2+h)$$
$$= 2(4 + 4h + h^2) - 8 - 4h$$
$$= 8 + 8h + 2h^2 - 8 - 4h$$
$$= 2h^2 + 4h$$
$$f(2) = 2(2)^2 - 4(2) = 0$$

$$y - y_1 = m(x - x_1)$$
$$y - 0 = 4(x - 2)$$
$$y = 4x - 8$$
$$4x - y - 8 = 0$$

4. Find the slope of the tangent of the curve $f(x) = \frac{7x-1}{x}$ at $P(1, 6)$

$$m = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \left(\frac{7(1+h) - 1}{1+h} - \frac{7(1) - 1}{1}\right) \div h$$

$$= \lim_{h\to 0} \left(\frac{7h+6}{1+h} - \frac{6}{1}\right) \div h = \lim_{h\to 0} \frac{7h + 6 - 6(1+h)}{(1+h)(1)(h)}$$

$$= \lim_{h\to 0} \frac{7h + \cancel{6} - \cancel{6} - 6h}{(1+h)(1)(h)} = \lim_{h\to 0} \frac{\cancel{h}}{(1+h)(1)(\cancel{h})} = 1$$

5. Find the slope of the tangent to the curve $f(x) = \sqrt{2x+1}$ at $x = 4$.

$$m = \lim_{h\to 0} \frac{\sqrt{2(4+h)+1} - \sqrt{2(4)+1}}{h} = \lim_{h\to 0} \frac{\sqrt{2h+9} - 3}{h} \cdot \frac{\sqrt{2h+9} + 3}{\sqrt{2h+9} + 3}$$

$$= \lim_{h\to 0} \frac{2\cancel{h} + \cancel{9} - \cancel{9}}{\cancel{h}(\sqrt{2h+9} + 3)} = \frac{2}{6} = \frac{1}{3}.$$

6. Find the value of c that makes $f(x)$ continuous.

$$f(x) = \begin{cases} x^2 - c^2, & x < 4 \\ cx + 20, & x \geq 4 \end{cases}$$

if $x<4$: $f(x) = x^2 - c^2$ parabola which is continuous on its domain.

if $x>4$: $f(x) = cx+20$ Linear which is continuous

at $x=4$: $\lim_{x\to 4^+} f = \lim_{x \to 4^-} f \Rightarrow 4^2 - c^2 = 4c+20$

$c^2 + 4c + 4 = 0$

$(c+2)^2 = 0 \Rightarrow c = -2$

7. Determine the horizontal asymptotes of $f(x) = \frac{4x^2-1}{x^2+1}$

$$\lim_{x\to\infty} \frac{4x^2-1}{x^2+1}$$

$\therefore$ $y=4$ is the HA.

$$= \lim_{x\to\infty} \frac{\frac{4x^2}{x^2} - \frac{1}{x^2}}{\frac{x^2}{x^2} + \frac{1}{x^2}} = 4$$

8. Determine any horizontal asymptotes for $f(x) = 2^{-x} + \frac{x-1}{x+2}$

$$\lim_{x\to\infty} 2^{-x} + \frac{x-1}{x+2} = \lim_{x\to\infty} \frac{1}{2^x} + \lim_{x\to\infty} \frac{x-1}{x+2}$$

$= 0 + 1 = 1 \Rightarrow y = 1$ is the HA.

9. Thinking Questions:

a) For the piecewise function $f(x) = \begin{cases} ax + b, \ x < 0 \\ x^2 + 3, 0 \leq x \leq 3 \\ -a(x-5)^2, x > 3 \end{cases}$

if the limits as x approaches both 0 and 3 exist, find the values of a and b.

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) \Rightarrow \lim_{x\to 0^+} x^2+3 = \lim_{x\to 0^-} ax+b \Rightarrow b=3.$

$\lim_{x\to 3^+} -a(x-5)^2 = \lim_{x\to 3^-} (x^2+3) \Rightarrow -a(-2)^2 = 3^2+3$

$-4a = 12$

$\boxed{a = -3}$

b) Given the function $f(x) = \frac{x^3+ax^2+bx+5}{x^3+2x^2-x-2}$ determine whether there are numbers a and b such that both $\lim_{x\to 1} f(x)$ exists and $\lim_{x\to -1} f(x)$ exists. If so, evaluate both limits.

Let $P(x) = x^3 + ax^2 + bx + 5$ $\quad Q(x) = x^3 + 2x^2 - x - 2$

$= x^2(x+2) - 1(x+2)$

$= (x^2-1)(x+2)$

$= (x-1)(x+1)(x+2)$

If $\lim_{x\to 1} f(x)$, $\lim_{x\to -1} f(x)$ exist,

$x+1$ and $x-1$ are factors of $P(x)$

$P(1)=0 \Rightarrow 1+a+b+5=0 \Rightarrow a+b=-6$

$P(-1)=0 \Rightarrow -1+a-b+5=0 \Rightarrow a-b=-4$

$2a=-10 \Rightarrow a=-5$

$b=-1$

c) Find a and b if $\lim_{x\to 0} \frac{\sqrt{ax+b}-4}{x} = 3$

$\lim_{x\to 0} \frac{\sqrt{ax+b}-4}{x} \cdot \frac{\sqrt{ax+b}+4}{\sqrt{ax+b}+4} = 3$

$\lim_{x\to 0} \frac{ax+b-16}{(\sqrt{ax+b}+4)(x)} = 3$

$\to x$ must cancel : $b-16=0$

$b=16$

$\therefore \lim_{x\to 0} \frac{a}{\sqrt{ax+b}+4} = 3$

$\frac{a}{\sqrt{b}+4} = 3 \Rightarrow \frac{a}{4+4} = 3$

$\boxed{a=24}$

d) Evaluate $\lim_{x\to\infty} f(x)$, where $f(x) = \sqrt{x^2+1} - x$.

$\lim_{x\to\infty} \frac{(\sqrt{x^2+1}-x)\cdot(\sqrt{x^2+1}+x)}{\sqrt{x^2+1}+x} = \lim_{x\to\infty} \frac{\cancel{x^2}+1-\cancel{x^2}}{\sqrt{x^2+1}+x}$

$= 0$

e) Evaluate $\lim_{x\to 2} \frac{\sqrt{6-x}-2}{\sqrt{3-x}-1} \cdot \frac{\sqrt{6-x}+2}{\sqrt{6-x}+2} \cdot \frac{\sqrt{3-x}+1}{\sqrt{3-x}+1}$

$= \lim_{x\to 2} \frac{(6-\cancel{x}-4)(\sqrt{3-x}+1)}{(3-\cancel{x}-1)(\sqrt{6-x}+2)} = \frac{\sqrt{1}+1}{\sqrt{4}+2} = \frac{2}{4}$

$= \frac{1}{2}$