# Day 7: 1.2 Rates of Change and Slope of a Curve

In some applications what is important is the effect of the change in one variable on a second variable.

Eg. How does a change in price affect sales?
How does a change in time affect position?
Independent vs. dependent variables

$ARoC = \frac{f(b)-f(a)}{b-a}$ if $a \le x \le b$

INVESTIGATION:

The volume of a sphere is related to its radius by the formula: $V = \frac{4}{3}\pi r^3$

A spherical balloon is being inflated. Find

a) the average rate of change of the volume with respect to the radius within the period of $2\text{ cm} \le r \le 6\text{ cm}$.

$$ARoC = \frac{\frac{4}{3}\pi(6)^3 - \frac{4}{3}\pi(2)^3}{6-2} \rightarrow \frac{\Delta V}{\Delta r}$$

$$\doteq \frac{871.27}{4}$$

$$\doteq 217.82\, cm^3/r$$

∴ The average rate of change is $217.82\ cm^3/r$.

[For 1 cm increase in radius volume increases by $217.82\ cm^3$]

b) the rate of change when the radius is 10 cm.

$10 \le r \le 10.001$

$$IROC = \frac{\frac{4}{3}\pi(10.001)^3 - \frac{4}{3}\pi(10)^3}{0.001}$$

$$= 1256.76\ cm^3/r$$

$$IRoC = \lim_{h\to 0} \frac{\frac{4}{3}\pi(10+h)^3 - \frac{4}{3}\pi(10)^3}{h}$$

$$= \lim_{h\to 0} \frac{\frac{4}{3}\pi[(10+h)^3 - 10^3]}{h}$$

$$= \lim_{h\to 0} \frac{\frac{4}{3}\pi[(10+h-10)((10+h)^2+(10+h)(10)+10^2)]}{\cancel{h}}$$

$$= \frac{4}{3}\pi(300) = 400\pi \doteq 1256.76\,cm^3/r$$

**Average Rate of Change between two points (slope of the secant)**

$$Ave\ RoC = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

**Instantaneous Rate of Change at a point x= a**

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Example 1: The total cost, in dollars, of manufacturing $x$ calculators is given by $C(x) = 10\sqrt{x} + 1000$
a. What is the total cost of manufacturing 100 calculators?

$$C(100) = 10\sqrt{100} + 1000$$
$$= 10(10) + 1000$$
$$= \$1100$$

b. What is the rate of change in the total cost with respect to the number of calculators, $x$, being produced when $x = 100$?

$$IROC = \lim_{h\to 0} \frac{C(100+h) - C(100)}{h}$$

$$= \lim_{h\to 0} \frac{10\sqrt{100+h} + 1000 - (10\sqrt{100} + 1000)}{h}$$ 1000 − 1000 cancels!

$$= \lim_{h\to 0} \frac{10\sqrt{100+h} - 100}{h} \cdot \frac{10\sqrt{100+h} + 100}{10\sqrt{100+h} + 100}$$ → Difference of squares!

$$= \lim_{h\to 0} \frac{100(\sqrt{100+h})^2 - (100)^2}{h(10\sqrt{100+h} + 100)}$$

$$= \lim_{h\to 0} \frac{100(100+h) - 100^2}{h(10\sqrt{100+h} + 100)}$$

$$= \lim_{h\to 0} \frac{\cancel{100^2} + 100h - \cancel{100^2}}{h(10\sqrt{100+h} + 100)} = \lim_{h\to 0} \frac{100\cancel{h}}{\cancel{h}(10\sqrt{100+h} + 100)}$$

$$= \frac{100}{10\sqrt{100+0} + 100} = \frac{100}{100+100} = \frac{100}{200} = \frac{1}{2}$$

∴ IROC at $x = 100$ is \$0.50/calculator.
(It means if the company were to produce one more calculator from $x = 100$ to 101, the cost will go up by \$0.50)

Example 2: A ball is dropped from the top of the CN Tower. $s(t) = -4.9t^2$

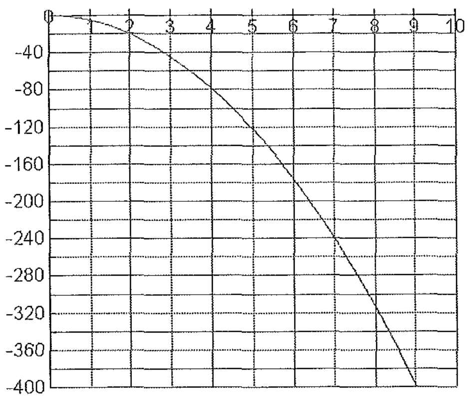
The position $s$ (in metres) of the ball is given by the equation: $s = -4.9t^2$ where $t$ is the time in seconds.

(a) Find the average velocity of the ball over the time period of $3 \le t \le 7$.

$$AROC = \frac{s(7) - s(3)}{7-3}$$

$$= \frac{-4.9(7)^2 - (-4.9(3)^2)}{4} = \frac{-196}{4} = -49\ m/s$$

(b) Find the velocity of the ball at $t = 4$.




$$V(4) = \lim_{h \to 0} \frac{s(4+h) - s(4)}{h}$$

$$= \lim_{h \to 0} \frac{-4.9(4+h)^2 - (-4.9(4)^2)}{h}$$

$$= \lim_{h \to 0} \frac{-4.9\left[(4+h)^2 - 4^2\right]}{h}$$ → You can expand and simplify or factor

$$= \lim_{h \to 0} \frac{-4.9\left[(4+h+4)(\cancel{4}+h-\cancel{4})\right]}{\cancel{h}}$$

$$= -4.9\left[(4+0+4)(1)\right]$$

$$= -4.9(8)$$

$$= -39.2\ m/s$$

The velocity of an object with position function s(t) at a time t = a is

$$v(a) = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{h \to 0} \frac{s(a+h) - s(a)}{h}$$

Example 3:

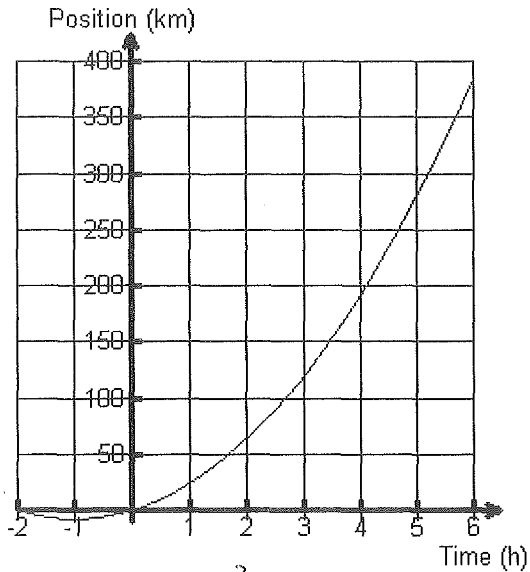
The position function of a moving object is $s(t) = 8t^2 + 16t$ where $s$ is in km and t is in hours. $0 \le t \le 5$.

a) Find the average velocity during the time interval of t=2 to t=5.

$$IROC = \frac{s(5) - s(0)}{5 - 0} = \frac{[8(5)^2 + 16(5)] - [8(0)^2 + 16(0)]}{5}$$

$$= \frac{8(25) + 80}{5} = \frac{280}{5} = 56 \text{ km/hour}$$

b) Find the instantaneous velocity at $t = 2$.




$$v(2) = \lim_{h \to 0} \frac{s(2+h) - s(2)}{h}$$ → see below the graph

$$= \lim_{h \to 0} \frac{8h^2 + 48h + \cancel{64} - \cancel{64}}{h}$$

$$= \lim_{h \to 0} \frac{(8h + 48)(\cancel{h})}{\cancel{h}}$$

$$= 8(0) + 48$$

$$= 48 \text{ km/hour.}$$

∴ Velocity at $t = 2$ is 48 km/hour.

$$s(2+h) = 8(2+h)^2 + 16(2+h)$$

$$= 8(4 + 4h + h^2) + 32 + 16h$$

$$= 32 + 32h + 8h^2 + 32 + 16h$$

$$= 8h^2 + 48h + 64$$

$$s(2) = 8(2)^2 + 16(2)$$

$$= 32 + 32$$

$$= 64$$