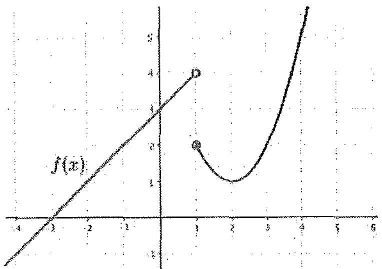
Day 5: Continuity

A function is continuous over a given interval if you can draw the function without lifting your pencil over that interval. No Break, No Jumps!!! Otherwise, it is discontinuous.

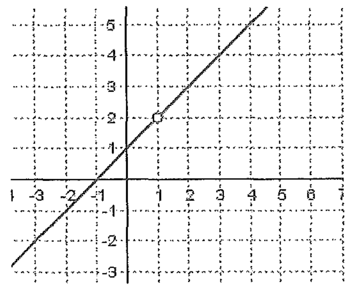
Example 1: State whether each is continuous or not, and justify your answers.

a)



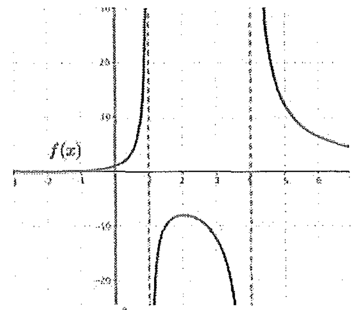
at $x=1$
jump discontinuity

b)



at $x=1$
point/removeable discontinuity

c)



at $x=1,4$
infinite discontinuity

A function is *continuous* at *a* if

1. *The value a is in the domain of f.*
2. $\lim_{x\to a} f(x)$ *exists*
3. $\lim_{x\to a} f(x) = f(a)$

A function is *continuous* if it is continuous at *a*, for all values of *a* in the domain.

A function is discontinuous *at a when* f(a) *is not defined;*

$\lim_{x\to a} f(x)$ *does not exists or* $\lim_{x\to a} f(x) \neq f(a)$

Example 2: Test the continuity of each function at $x = 2$. If it is not continuous there, state why.

a) $f(x) = x^3 - x$

continuous (polynomials continuous everywhere)

b) $f(x) = \frac{1}{(x-2)^2}$

discontinuous at $x=2$
(infinite discontinuity)

c) $f(x) = \frac{x^2 - x - 2}{x - 2}$

$= \frac{\cancel{(x-2)}(x+1)}{\cancel{(x-2)}}$

hole at $x=2$

$\therefore$ discontinuous at $x=2$

d) $g(x) = \begin{cases} 4 - x^2, & x < 2 \\ 3, & x \geq 2 \end{cases}$

$\lim_{x\to 2^-} 4 - x^2 = 0$

$\lim_{x\to 2^+} 3 = 3$

$\therefore$ jump at $x=2$
discontinuous at $x=2$

Functions that are continuous (over their domains)

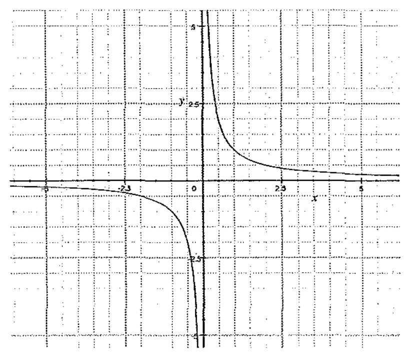
- *Polynomial functions*
- *Rational functions (Continuous over its domain (however there are discontinuities, just not in the domain)*
- *Absolute value function* $y = |x|$
- *Exponential functions*
- *Logarithmic functions*
- *Trigonometric functions*
- *Radical functions (*$y = \sqrt[n]{x}$*, if n is a positive integer greater than 1)**

In the previous section we discussed the limit of $f(x)$ as $x \to c$, where c is a *finite* value. However one might be interested in how the function behaves at infinite points. There are 2 different types of limits involving infinity.

Finite Limits
In this type we take the limit as $x \to \infty$ or $x \to -\infty$ and we get back a finite value. Two easy examples are:

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

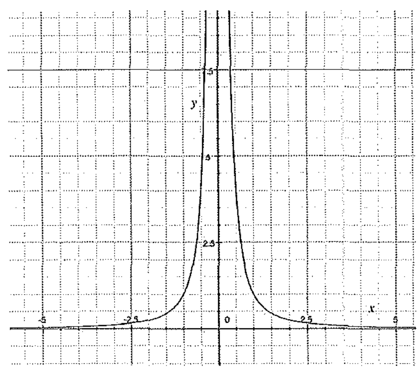
$$\lim_{x \to -\infty} \frac{1}{x} = 0$$



Infinite Limits
In this type we take the limit as $x \to c$ and we get back an infinite limit. Three easy examples are:

$$\lim_{x \to 0^+} \frac{1}{x^2} = \infty \qquad \lim_{x \to 0^-} \frac{1}{x^2} = \infty$$

$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$



In the examples we can see that as x approaches 0, the limit approaches infinity. A vertical asymptote is actually defined in terms of infinite limits.

Examples using Finite Limits (use graphing calculator to check)

$$\lim_{x\to\infty} \frac{6x-13}{2x+5}$$

$$= \lim_{x\to\infty} \frac{\frac{6x}{x} - \frac{13}{x}\to 0}{\frac{2x}{x} + \frac{5}{x}\to 0} = \frac{6}{2} = 3$$

$$\lim_{x\to-\infty} \frac{3x-10}{4x^3+5}$$

$$= \lim_{x\to-\infty} \frac{\frac{3x}{x^3}\to 0 - \frac{10}{x^3}\to 0}{\frac{4x^3}{x^3} + \frac{5}{x^3}\to 0} = \frac{0}{4} = 0$$

$$\lim_{x\to\infty} \frac{1-x^2}{10x+7}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}\to 0 - \frac{x^2}{x}}{\frac{10x}{x} + \frac{7}{x}\to 0}$$

$$= \lim_{x\to\infty} \frac{-x}{10} = -\infty$$

$$\lim_{x\to-\infty} \frac{2x+1}{\sqrt{x^2+3}}$$

$x \to \infty$ $\quad x = \sqrt{x^2}$

$x \to -\infty$ $\quad x = -\sqrt{x^2}$

$$= \lim_{x\to-\infty} \frac{\frac{2x}{x} + \frac{1}{x}}{\sqrt{x^2+3} \div x} = \lim_{x\to-\infty} \frac{2 + \frac{1}{x}}{-\sqrt{\frac{x^2}{x^2} + \frac{3}{x^2}}}$$

$x = -\sqrt{x^2}$

$\therefore x \to -\infty$

$$= \lim_{x\to-\infty} \frac{2 + \frac{1}{x}\to 0}{-\sqrt{1 + \frac{3}{x^2}\to 0}} = -2$$

Examples using Infinite Limits (use graphing calculator to check)

$$\lim_{x\to2^+} \frac{3x-1}{x-2}$$

Sub in 2.001

$= +\infty$

$$\lim_{x\to2^-} \frac{3x-1}{x-2}$$

sub in 1.999

$= -\infty$

$$\lim_{x\to3^-} \frac{x^2}{x^2-9}$$

check 2.999

$$= \lim_{x\to3^-} \frac{x^2}{(x-3)(x+3)}$$

$= -\infty$ since $x^2 > 0$

$x - 3 < 0$

$x + 3 > 0$

$$\lim_{x\to5^-} \frac{\sqrt{25-x^2}}{x-5}$$

$\to$ check $x = 4.999$

$= -\infty$

$$\lim_{x\to-3} \frac{1}{x+3}$$

$= -\infty$

Finding Asymptotes

Find the horizontal and vertical asymptotes of the function $f(x)=\dfrac{3x+5}{x-2}$

To find the horizontal asymptotes, examine the $\lim\limits_{x\to\infty} f(x)$ and the $\lim\limits_{x\to-\infty} f(x)$

$$\lim_{x\to\infty}\frac{3x+5}{x-2}$$

$$=\lim_{x\to\infty}\frac{\frac{3x}{x}+\cancel{\frac{5}{x}}\to 0}{\frac{x}{x}-\cancel{\frac{2}{x}}\to 0}=\frac{3}{1}=3$$

$$\lim_{x\to-\infty}\frac{\frac{3x}{x}+\frac{5}{x}}{\frac{x}{x}-\frac{2}{x}}=3$$

$y=3$ is the HA

To find the vertical asymptotes, look for x value that the denominator is zero and find the limit at that point

$x=2$ (VA)

$$\lim_{x\to2^+}\frac{3x+5}{x-2}=+\infty$$

$$\lim_{x\to2^-}\frac{3x+5}{x-2}=-\infty$$

NOTE: $x\to VA^+, VA^-$ helps when graphing functions. (known as behavior around VA)

Practice

Find the horizontal and vertical asymptotes of the function $f(x)=\dfrac{x}{x^2-4}$. Graph to verify.

$$\lim_{x\to\infty}\frac{x}{x^2-4}=\lim_{x\to\infty}\frac{\frac{x}{x^2}}{\frac{x^2}{x^2}-\frac{4}{x^2}}=0 \qquad \lim_{x\to-\infty}f(x)=0 \qquad HA: y=0$$

VA: $x=\pm2$

$$\lim_{x\to2^+}\frac{x}{(x-2)(x+2)}=\infty \qquad \lim_{x\to2^-}\frac{x}{(x-2)(x+2)}=-\infty$$

$$\lim_{x\to2^+}\frac{x}{(x-2)(x+2)}=\infty \qquad \lim_{x\to-2^-}\frac{x}{(x-2)(x+2)}=-\infty$$

x-int: 0 y-int: 0

at $x=0$: It is point of inflection.

Point where concavity changes.