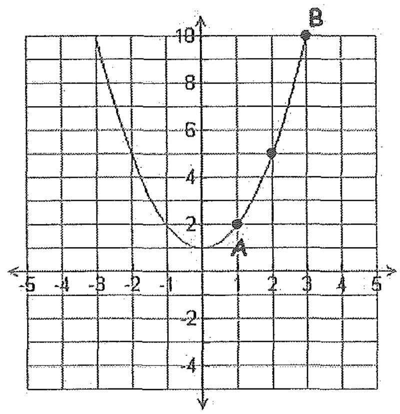
Day 3: 1.3 Limit of a Function

In mathematics, the function describes how two quantities are related. Calculus is concerned with HOW a change in one quantity is related or dependent upon a change in the other.

Example 1
What happens to the value of a function, *f(x)*, as *x* gets closer and closer to a particular value, x = a ?

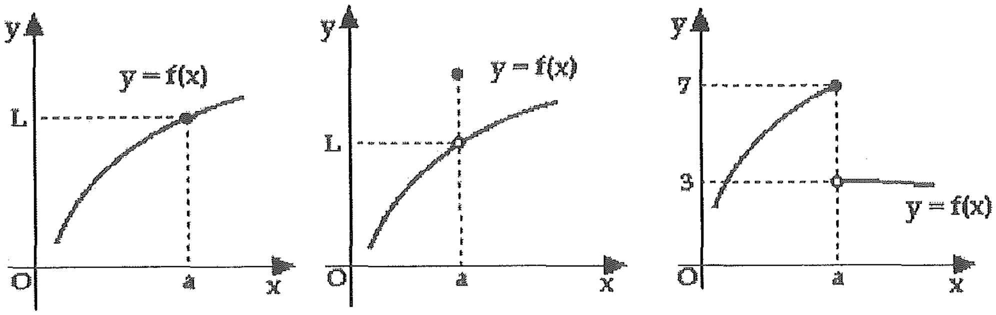


Given the graph of $f(x) = x^2 + 1$ examine what the value of *f(x)* approaches (from both sides of the function) when:

x = 1	from the left = 2	from the right = 2
x = 2	from the left = 5	from the right = 5
x = 3	from the left = 10	from the right = 10

Example 2
For each of the graphs below, examine what the value of *f(a)* approaches (from both sides of the function).

From the left = L	From the left = L	From the left = 7
From the right = L	From the right = L	From the right = 3



(a) (b) (c)

The Limit of a Function

The notation: $\lim_{x \to a} f(x) = L$

What it means: the limit of $f(x)$ as x approaches a is L (as x approaches a from either side).

Important!
We are looking for the limit of f(x) as x approaches a, which is not necessarily the value of f(a). A good example of when the limit does not equal f(a) is example 2 b.

In 2b, $L \neq f(a)$

One-Sided Limits

Left-hand Limit: $\lim_{x \to a^-} f(x)$ denotes the limit approaching a from the left side

Right-hand Limit: $\lim_{x \to a^+} f(x)$ denotes the limit approaching a from the right side

Two-sided limits (just called the limit)

$\lim_{x \to a} f(x)$ denotes the limit approaching a from both sides

The Existence of a limit

A function *f(x)* has a limit as x approaches a ***if and only if*** the right-hand and left-hand limits at a exist and are equal.

$$\lim_{x \to a} f(x) = L \Leftrightarrow \lim_{x \to a^+} f(x) = L \; and \; \lim_{x \to a^-} f(x) = L$$

(iff = if and only if, in symbols it is represented by $\Leftrightarrow$)

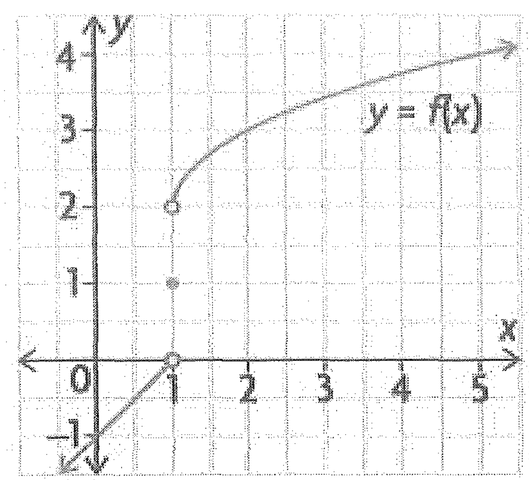
What is the difference between if and iff?

A pie is on the table. Sophie will eat the pie if it is a blueberry pie. (If the pie is pumpkin will Sophie eat it? Maybe, we don't know).

A pie is on the table. Sophie will eat the pie iff it is a blueberry pie (If the pie is pumpkin will Sophie eat it? Nope).

Example 3:

$$f(x) = \begin{cases} x - 1, \text{if } x < 1 \\ 1, \text{if } x = 1 \\ 2 + \sqrt{x - 1}, \text{if } x > 1 \end{cases}$$



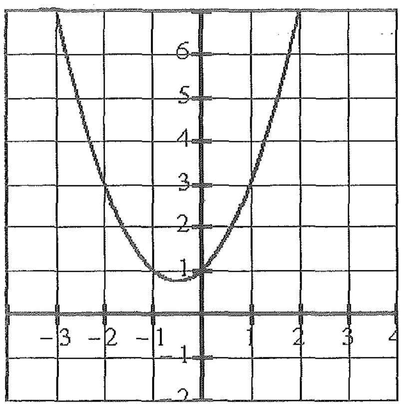
Determine if the limit exits:

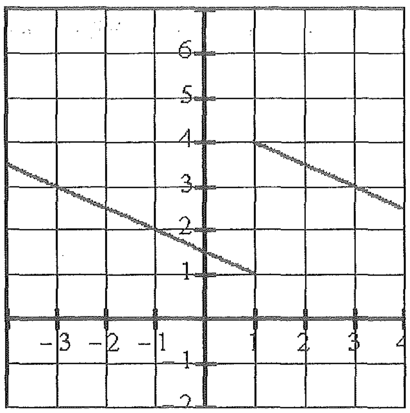
$\lim_{x \to 1} f(x) = d.n.e$

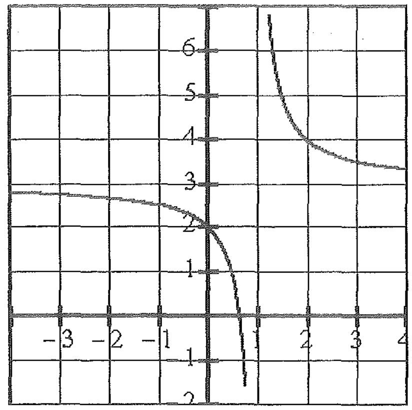
$\lim_{x \to 1^+} f(x) = 2$ $\lim_{x \to 1^-} f(x) = 0$

$\lim_{x \to 1} f(x)$ d.n.e since $\lim_{x \to 1^+} f(x) \neq \lim_{x \to 1^-} f(x)$

Example 4: Determine if the limit exists:







$\lim_{x \to 1} f(x) = 3$

$\lim_{x \to 1} f(x) =$ d.n.e

$\lim_{x \to 1^+} f(x) \neq \lim_{x \to 1^-} f(x)$

$\lim_{x \to 1} f(x) =$ d.n.e

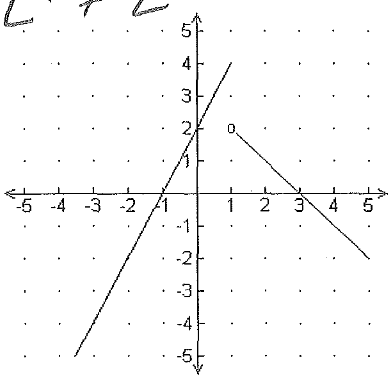
$L^+ = \infty$

$L^- = -\infty$

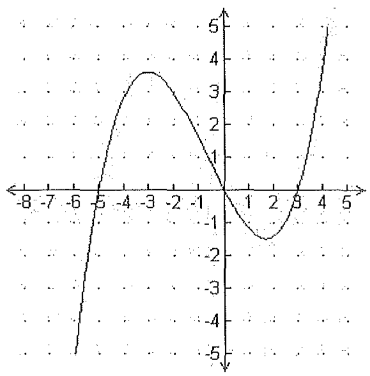
Example 5: Determine the following.

a) $\lim_{x \to 1} f(x)$ = d.n.e

$L^+ \neq L^-$



b) $\lim_{x \to 0} f(x) = 0$



Properties of Limits

For any real number a, suppose that f and g both have limits that exist at $x = a$.

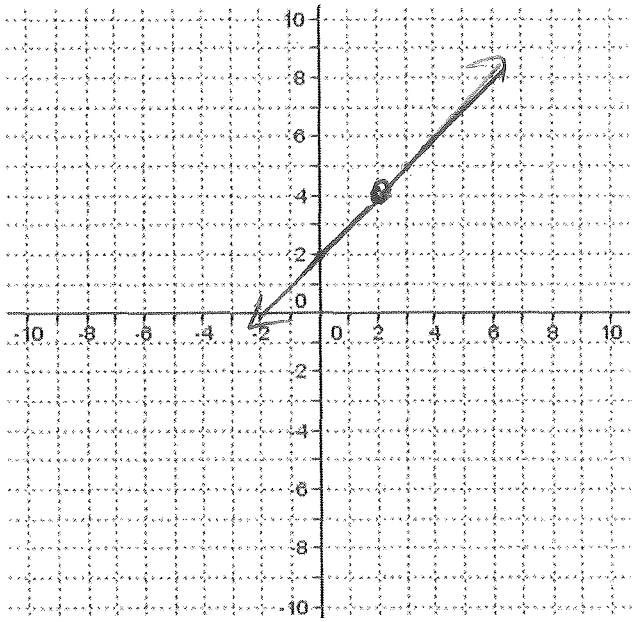
1. $\lim_{x \to a} k = k$, for any constant k
2. $\lim_{x \to a} x = a$
3. $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$
4. $\lim_{x \to a} [cf(x)] = c[\lim_{x \to a} f(x)]$, for any constant c
5. $\lim_{x \to a} [f(x)g(x)] = [\lim_{x \to a} f(x)][\lim_{x \to a} g(x)]$
6. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, provided that $\lim_{x \to a} g(x) \neq 0$
7. $\lim_{x \to a} [f(x)]^n = [\lim_{x \to a} f(x)]^n$, for any rational number n

Example 6: Use a table of values to graph:

$$f(x) = \frac{x^2-4}{x-2} = \frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}} = x+2$$

hole at (2,4)

x	y
-3	-1
-2	0
-1	1
0	2
1	3
2	4
3	5



But what if we really needed to know the value of $f(x)$ as $x \to 2$

	x approaches 2 from the left					x approaches 2 from the right			
x	1	1.5	1.9	1.999	2	2.001	2.1	2.5	3
$\frac{x^2-4}{x-2}$	3	3.5	3.9	3.999		4.001	4.1	4.5	5
	$f(x)$ approaches 4 from 2^-					$f(x)$ approaches 4 from 2^+			

Example 7: Sketch the graph of any function that satisfies the given conditions.

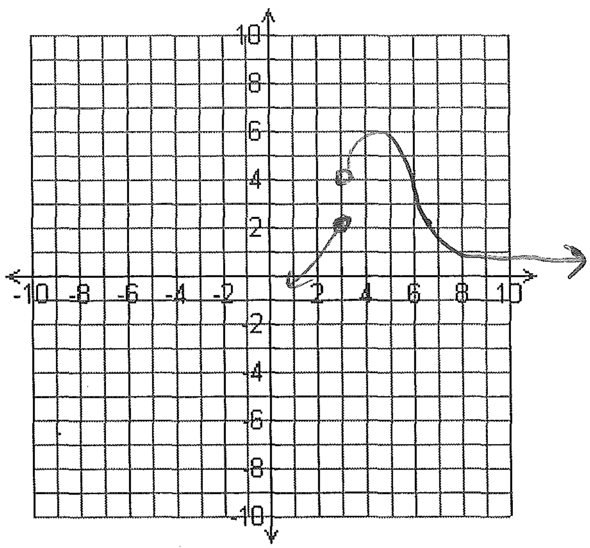
i) $f(3) = 2$

ii) $\lim_{x \to 3^-} f(x) = 2$

iii) $\lim_{x \to 3^+} f(x) = 4$

iv) $\lim_{x \to \infty} f(x) = 0$

↳ HA is $y=0$



For Limit Properties, see the table on page 35 of the textbook.

Example 8: Determine (direct substitution).

a) $\lim_{x\to -1} 5 = 5$

b) $\lim_{x\to -\frac{5}{2}} (-x+2) = -(\frac{-5}{2})+2 = \frac{5}{2}+2 = \frac{9}{2}$

c) $\lim_{x\to 3} -\sqrt{x+3} = -\sqrt{3+3}$

$= -\sqrt{6}$

d) $\lim_{x\to 5}\sqrt{3x+1} = \sqrt{3(5)+1}$

$= \sqrt{16} = 4$

e) $\lim_{x\to 2}(x^3 - x^2 - 4)$

$= 2^3 - 2^2 - 4$

$= 8 - 8$

$= 0$

f) $\lim_{x\to 4} x^2 + 2x - 1$

$= 4^2 + 2(4) - 1$

$= 24 - 1$

$= 23$

g) $\lim_{x\to 2}\frac{x^2+2x+4}{x+2}$

$= \frac{2^2+2(2)+4}{2+2} = \frac{12}{4} = 3$

h) $\lim_{x\to 1} -\frac{x-4}{x^2-6x+8}$

$= \frac{-(1-4)}{1^2-6(1)+8} = \frac{3}{3} = 1$

i) $\lim_{x\to \pi}\sin(x)$

$= \sin(\pi)$

$= 0$

j) $\lim_{x\to -\frac{1}{3}}\sqrt{3x+1}$ = d.n.e (not defined).

$\lim_{x\to -\frac{1}{3}^+}\sqrt{3x+1} = 0$ $\quad$ $\lim_{x\to -\frac{1}{3}^-}\sqrt{3x+1}$ = d.n.e

Example 9: Let $g(x) = Ax + B$, where A and B are constants. If $\lim_{x\to 1} g(x) = -2$ and $\lim_{x\to -1} g(x) = 4$, find the values of A and B.

$\lim_{x\to 1}(Ax+B) = -2 \Rightarrow A+B = -2$

$\lim_{x\to -1}(Ax+B) = 4 \Rightarrow -A+B = 4$

System of equations solve by substitution or elimination

$2B = 2$

$B = 1$

$A = -3$