Day 2: 5.1 Derivatives of Exponential Functions y = e^x

INVESTIGATION

Let try to find the derivative of an exponential function y = b^x

By definition $y' = \lim_{h\to 0}\frac{b^{x+h}-b^x}{h} = \lim_{h\to 0}\frac{b^x(b^h-1)}{h} = (b^x)\lim_{h\to 0}\frac{b^h-1}{h}$

Any value of b such that y' = b^x? that means $\lim_{h\to 0}\frac{b^h-1}{h} = 1$

The value is "e", a special irrational number, like the number π. It is called **the natural number, or** Euler's number in honour of the Swiss mathematician Leonhard Euler (pronounced "oiler"), who lived from 1707 to 1783. We use a rational approximation for *e* of about 2.718.

The value of e is calculated as:

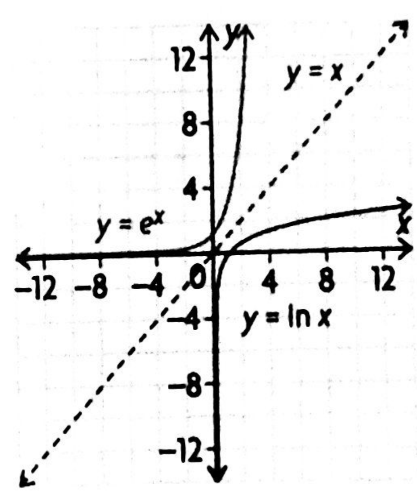
$$e = \lim_{x\to 0}(1+x)^{\frac{1}{x}} \qquad e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \qquad e = \sum_{n=0}^{\infty}\frac{1}{n!} = \frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots$$

These are called the Fundamental Limit, a famous limit in Calculus.

The function $y = e^x$
Since $y = e^x$ is an exponential function, it has the same properties as other exponential functions you have studied.

Recall that the logarithm function is the inverse of the exponential function.
For example, $y = log_2x$ is the inverse of $y = 2^x$

The function $y = e^x$ also has an inverse, $y = \log_e x$. Their graphs are reflections in the line $y = x$. The function $y = \log_e x$ can be written as $y = \ln x$ and is called the <u>natural logarithm function</u>.



Locate the e^x and the ln(x) button on your calculator

Properties of $y = e^x$ and $y = \ln x$

$y = e^x$	$y = \ln x$
• The domain is $\{x \in \mathbf{R}\}$.	• The domain is $\{x \in \mathbf{R} \mid x > 0\}$.
• The range is $\{y \in \mathbf{R} \mid y > 0\}$.	• The range is $\{y \in \mathbf{R}\}$.
• The function passes through (0, 1).	• The function passes through (1, 0).
• $e^{\ln x} = x, x > 0$.	• $\ln e^x = x, x \in \mathbf{R}$.
• The line $y = 0$ is the horizontal asymptote.	• The line $x = 0$ is the vertical asymptote.

Derivative of $f(x) = e^x$

$$\boxed{f(x) = e^x, \qquad f'(x) = e^x.}$$

Note: $y = e^x$ is the one and the only one function that its derivative is itself!

Derivative of composite functions involving e^x

$$\boxed{\text{if } f(x) = e^{g(x)} \text{ then } f'(x) = e^{g(x)}g'(x)}$$

Example1: Differentiate

a) $y = -3e^x$

$y' = -3e^x$

b) $f(x) = e^{x^2-2x+1}$

$f'(x) = (2x-2)e^{x^2-2x+1}$

c) $g(x) = \frac{e^x}{x^2}$

$g'(x) = \frac{e^x x^2 - e^x(2x)}{x^4}$

$= \frac{xe^x(x-2)}{x^4} = \frac{e^x(x-2)}{x^3}$

Example 2: Differentiate then determine $f'(1)$ of $f(x) = e^{x^2} + 3e^{-x}$

$f'(x) = e^{x^2} \cdot 2x + (3)e^{-x}(-1)$

$f'(1) = e^{1^2} \cdot 2(1) + 3e^{-1}(-1)$

$= 2e - 3e^{-1}$

$= 2e - \frac{3}{e}$

Example 3: Find the equation of the tangent line to the curve $y = 1 + xe^{2x}$ at x = 0

$y' = 0 + 1e^{2x} + xe^{2x}(2)$

$y'(0) = m = e^0 + 0(e^0)(2) = 1, \quad y(0) = 1$

$\therefore \; y - y_1 = m(x - x_1)$

$y - 1 = 1(x - 0) \Rightarrow y = x + 1 \quad$ OR $\quad x - y + 1 = 0$

Example 4: Find the point(s) at which the tangent to the curve y = xe^x is horizontal.

$y' = 1e^x + xe^x$ ($y' = 0$ if tangent is horizontal)

$e^x(1 + x) = 0 \Rightarrow e^x = 0 \Rightarrow$ NO sol^n

OR

$x + 1 = 0 \Rightarrow x = -1$

$\therefore \left(-1, \frac{-1}{e}\right)$ is the point

Example 5: Determine the equation of the tangent line to the graph of

$y = \frac{e^x}{x^2} \quad x \neq 0 \quad$ at x = 2

$$y' = \frac{e^x(x)^2 - e^x(2x)}{x^4}$$

$$y'(2) = \frac{e^2(2)^2 - e^2(2(2))}{2^4}$$

$$= \frac{4e^2 - 4e^2}{16}$$

$$= 0$$

$\therefore m = 0$ (horizontal tangent.)

$$\left.\begin{array}{l} y = \frac{e^2}{4} \\ m = 0 \\ x = 2 \end{array}\right\} \begin{array}{l} y - y_1 = m(x - x_1) \\ y - \frac{e^2}{4} = 0(x - 2) \end{array}$$

$\boxed{\therefore y = \frac{e^2}{4}}$ is the equation of tangent.