Chapter 1
Introduction to Calculus

Review of Prerequisite Skills, pp. 2–3

1. a. $m = \frac{-7-5}{6-2}$
$= -3$

b. $m = \frac{4-(-4)}{-1-3}$
$= -2$

c. $m = \frac{4-0}{1-0}$
$= 4$

d. $m = \frac{4-0}{-1-0}$
$= -4$

e. $m = \frac{4-4.41}{-2-(-2.1)}$
$= -4.1$

f. $m = \frac{-\frac{1}{4}-\frac{1}{4}}{\frac{7}{4}-\frac{3}{4}}$
$= \frac{-\frac{2}{4}}{1}$
$= -\frac{1}{2}$

2. a. Substitute the given slope and y-intercept into $y = mx + b$.
$y = 4x - 2$

b. Substitute the given slope and y-intercept into $y = mx + b$.
$y = -2x + 5$

c. The slope of the line is
$m = \frac{12-6}{4-(-1)}$
$= \frac{6}{5}$
The equation of the line is in the form $y - y_1 = m(x - x_1)$. The point is $(-1, 6)$ and $m = \frac{6}{5}$.
The equation of the line is $y - 6 = \frac{6}{5}(x + 1)$ or $y = \frac{6}{5}(x + 1) + 6$.

d. $m = \frac{8-4}{-6-(-2)}$
$= -1$
$y - 4 = -1(x - (-2))$
$y - 4 = -x - 2$
$x + y - 2 = 0$

e. $x = -3$

f. $y = 5$

3. a. $f(2) = -6 + 5$
$= -1$

b. $f(2) = (8 - 2)(6 - 6)$
$= 0$

c. $f(2) = -3(4) + 2(2) - 1$
$= -9$

d. $f(2) = (10 + 2)^2$
$= 144$

4. a. $f(-10) = \frac{-10}{100+4}$
$= -\frac{5}{52}$

b. $f(-3) = \frac{-3}{9+4}$
$= -\frac{3}{13}$

c. $f(0) = \frac{0}{0+4}$
$= 0$

d. $f(10) = \frac{10}{100+4}$
$= \frac{5}{52}$

5. $f(x) = \begin{cases} \sqrt{3-x}, \text{ if } x < 0 \\ \sqrt{3+x}, \text{ if } x \geq 0 \end{cases}$

a. $f(-33) = 6$

b. $f(0) = \sqrt{3}$

c. $f(78) = 9$

d. $f(3) = \sqrt{6}$

6. $s(t) = \begin{cases} \frac{1}{t}, \text{ if } -3 < t < 0 \\ 5, \text{ if } t = 0 \\ t^3, \text{ if } t > 0 \end{cases}$

a. $s(-2) = -\frac{1}{2}$

b. $s(-1) = -1$

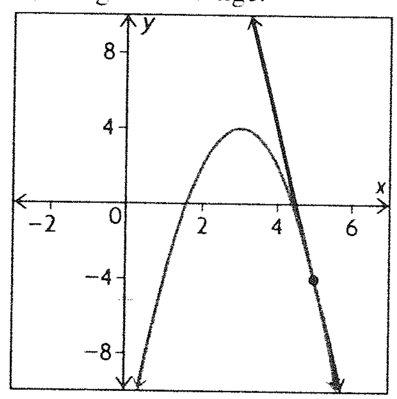
c. $s(0) = 5$
d. $s(1) = 1$
e. $s(100) = 100^3$ or 10^6
7. a. $(x - 6)(x + 2) = x^2 - 4x - 12$
b. $(5 - x)(3 + 4x) = 15 + 17x - 4x^2$
c. $x(5x - 3) - 2x(3x + 2) = 5x^2 - 3x - 6x^2 - 4x$
$= -x^2 - 7x$
d. $(x - 1)(x + 3) - (2x + 5)(x - 2)$
$= x^2 + 2x - 3 - (2x^2 + x - 10)$
$= -x^2 + x + 7$
e. $(a + 2)^3 = (a + 2)(a + 2)(a + 2)$
$= (a^2 + 4a + 4)(a + 2)$
$= a^3 + 6a^2 + 12a + 8$
f. $(9a - 5)^3 = (9a - 5)(9a - 5)(9a - 5)$
$= (81a^2 - 90a + 25)(9a - 5)$
$= 729a^3 - 1215a^2 + 675a - 125$
8. a. $x^3 - x = x(x^2 - 1)$
$= x(x + 1)(x - 1)$
b. $x^2 + x - 6 = (x + 3)(x - 2)$
c. $2x^2 - 7x + 6 = (2x - 3)(x - 2)$
d. $x^3 + 2x^2 + x = x(x^2 + 2x + 1)$
$= x(x + 1)(x + 1)$
e. $27x^3 - 64 = (3x - 4)(9x^2 + 12x + 16)$
f. $2x^3 - x^2 - 7x + 6$
$x = 1$ is a zero, so $x - 1$ is a factor. Synthetic or long division yields
$2x^3 - x^2 - 7x + 6 = (x - 1)(2x^2 + x - 6)$
$= (x - 1)(2x - 3)(x + 2)$
9. a. $\{x \in \mathbf{R} \mid x \geq -5\}$
b. $\{x \in \mathbf{R}\}$
c. $\{x \in \mathbf{R} \mid x \neq 1\}$
d. $\{x \in \mathbf{R} \mid x \neq 0\}$
e. $2x^2 - 5x - 3 = (2x + 1)(x - 3)$
$\left\{x \in \mathbf{R} \middle| x \neq -\frac{1}{2}, 3\right\}$
f. $\{x \in \mathbf{R} \mid x \neq -5, -2, 1\}$
10. a. $h(0) = 2$, $h(1) = 22.1$
average rate of change $= \dfrac{22.1 - 2}{1 - 0}$
$= 20.1$ m/s
b. $h(1) = 22.1$, $h(2) = 32.4$
average rate of change $= \dfrac{32.4 - 22.1}{2 - 1}$
$= 10.3$ m/s
11. a. The average rate of change during the second hour is the difference in the volume at $t = 120$ and $t = 60$ (since t is measured in minutes), divided by the difference in time.

$$\frac{V(120) - V(60)}{120 - 60} = \frac{0 - 1200}{60} = -20 \text{ L/min}$$

b. To estimate the instantaneous rate of change in volume after exactly 60 minutes, calculate the average rate of change in volume from minute 59 to minute 61.

$$\frac{V(61) - V(59)}{61 - 59} \doteq \frac{1186.56 - 1213.22}{2} = -13.33 \text{ L/min}$$

c. The instantaneous rate of change in volume is negative for $0 \leq t \leq 120$ because the volume of water in the hot tub is always decreasing during that time period, a negative change.
12. a., b.



The slope of the tangent line is -8.
c. The instantaneous rate of change in $f(x)$ when $x = 5$ is -8.

1.1 Radical Expressions: Rationalizating Denominators, p. 9

1. a. $2\sqrt{3} + 4$
b. $\sqrt{3} - \sqrt{2}$
c. $2\sqrt{3} + \sqrt{2}$
d. $3\sqrt{3} - \sqrt{2}$
e. $\sqrt{2} + \sqrt{5}$
f. $-\sqrt{5} - 2\sqrt{2}$
2. a. $\dfrac{\sqrt{3} + \sqrt{5}}{\sqrt{2}} \cdot \dfrac{\sqrt{2}}{\sqrt{2}}$
$= \dfrac{\sqrt{6} + \sqrt{10}}{2}$
b. $\dfrac{2\sqrt{3} - 3\sqrt{2}}{\sqrt{2}} \cdot \dfrac{\sqrt{2}}{\sqrt{2}}$
$= \dfrac{2\sqrt{6} - 6}{2}$
$= \sqrt{6} - 3$

c. $\dfrac{4\sqrt{3} + 3\sqrt{2}}{2\sqrt{3}} \cdot \dfrac{\sqrt{3}}{\sqrt{3}}$

$= \dfrac{12 + 3\sqrt{6}}{6}$

$= \dfrac{4 + \sqrt{6}}{2}$

d. $\dfrac{3\sqrt{5} - \sqrt{2}}{2\sqrt{2}} \cdot \dfrac{\sqrt{2}}{\sqrt{2}}$

$= \dfrac{3\sqrt{10} - 2}{4}$

3. a. $\dfrac{3}{\sqrt{5} - \sqrt{2}} \cdot \dfrac{\sqrt{5} + \sqrt{2}}{\sqrt{5} + \sqrt{2}}$

$= \dfrac{3(\sqrt{5} + \sqrt{2})}{3}$

$= \sqrt{5} + \sqrt{2}$

b. $\dfrac{2\sqrt{5}}{2\sqrt{5} + 3\sqrt{2}} \cdot \dfrac{2\sqrt{5} - 3\sqrt{2}}{2\sqrt{5} - 3\sqrt{2}}$

$= \dfrac{20 - 6\sqrt{10}}{20 - 18}$

$= 10 - 3\sqrt{10}$

c. $\dfrac{\sqrt{3} - \sqrt{2}}{\sqrt{3} + \sqrt{2}} \cdot \dfrac{\sqrt{3} - \sqrt{2}}{\sqrt{3} - \sqrt{2}}$

$= \dfrac{3 + 2\sqrt{6} + 2}{3 - 2}$

$= 5 + 2\sqrt{6}$

d. $\dfrac{2\sqrt{5} - 8}{2\sqrt{5} + 3} \cdot \dfrac{2\sqrt{5} - 3}{2\sqrt{5} - 3}$

$= \dfrac{20 - 22\sqrt{5} + 24}{20 - 9}$

$= \dfrac{44 - 22\sqrt{5}}{11}$

$= 4 - 2\sqrt{5}$

e. $\dfrac{2\sqrt{3} - \sqrt{2}}{5\sqrt{2} + \sqrt{3}} \cdot \dfrac{5\sqrt{2} - \sqrt{3}}{5\sqrt{2} - \sqrt{3}}$

$= \dfrac{10\sqrt{6} - 6 - 10 + \sqrt{6}}{50 - 3}$

$= \dfrac{11\sqrt{6} - 16}{47}$

f. $\dfrac{3\sqrt{3} - 2\sqrt{2}}{3\sqrt{3} + 2\sqrt{2}} \cdot \dfrac{3\sqrt{3} - 2\sqrt{2}}{3\sqrt{3} - 2\sqrt{2}}$

$= \dfrac{27 - 12\sqrt{6} + 8}{27 - 8}$

$= \dfrac{35 - 12\sqrt{6}}{19}$

4. a. $\dfrac{\sqrt{5} - 1}{4} \cdot \dfrac{\sqrt{5} + 1}{\sqrt{5} + 1}$

$= \dfrac{5 - 1}{4(\sqrt{5} + 1)}$

$= \dfrac{1}{\sqrt{5} + 1}$

b. $\dfrac{2 - 3\sqrt{2}}{2} \cdot \dfrac{2 + 3\sqrt{2}}{2 + 3\sqrt{2}}$

$= \dfrac{4 - 18}{2(2 + 3\sqrt{2})}$

$= \dfrac{-7}{2 + 3\sqrt{2}}$

c. $\dfrac{\sqrt{5} + 2}{2\sqrt{5} - 1} \cdot \dfrac{\sqrt{5} - 2}{\sqrt{5} - 2}$

$= \dfrac{5 - 4}{10 - 5\sqrt{5} + 2}$

$= \dfrac{1}{12 - 5\sqrt{5}}$

5. a. $\dfrac{8\sqrt{2}}{\sqrt{20} - \sqrt{18}} \cdot \dfrac{\sqrt{20} + \sqrt{18}}{\sqrt{20} + \sqrt{18}}$

$= \dfrac{8\sqrt{40} + 8\sqrt{36}}{20 - 18}$

$= \dfrac{16\sqrt{10} + 48}{2}$

$= 8\sqrt{10} + 24$

b. $\dfrac{8\sqrt{2}}{2\sqrt{5} - 3\sqrt{2}} \cdot \dfrac{2\sqrt{5} + 3\sqrt{2}}{2\sqrt{5} + 3\sqrt{2}}$

$= \dfrac{16\sqrt{10} + 48}{20 - 18}$

$= \dfrac{16\sqrt{10} + 48}{2}$

$= 8\sqrt{10} + 24$

c. The expressions in the two parts are equivalent. The radicals in the denominator of part a. have been simplified in part b.

6. a. $\dfrac{2\sqrt{2}}{2\sqrt{3}-\sqrt{8}} \cdot \dfrac{2\sqrt{3}+\sqrt{8}}{2\sqrt{3}+\sqrt{8}}$

$= \dfrac{4\sqrt{6}+8}{6-8}$

$= -2\sqrt{3}-4$

b. $\dfrac{2\sqrt{6}}{2\sqrt{27}-\sqrt{8}} \cdot \dfrac{2\sqrt{27}+\sqrt{8}}{2\sqrt{27}+\sqrt{8}}$

$= \dfrac{4\sqrt{162}+2\sqrt{48}}{54-8}$

$= \dfrac{36\sqrt{2}+8\sqrt{3}}{46}$

$= \dfrac{18\sqrt{2}+4\sqrt{3}}{23}$

c. $\dfrac{2\sqrt{2}}{\sqrt{16}-\sqrt{12}}$

$= \dfrac{2\sqrt{2}}{4-2\sqrt{3}} \cdot \dfrac{4+2\sqrt{3}}{4+2\sqrt{3}}$

$= \dfrac{8\sqrt{2}+4\sqrt{6}}{16-12}$

$= 2\sqrt{2}+\sqrt{6}$

d. $\dfrac{3\sqrt{2}+2\sqrt{3}}{\sqrt{12}-\sqrt{8}} \cdot \dfrac{\sqrt{12}+\sqrt{8}}{\sqrt{12}+\sqrt{8}}$

$= \dfrac{3\sqrt{24}+12+12+2\sqrt{24}}{12-8}$

$= \dfrac{24+15\sqrt{3}}{4}$

e. $\dfrac{3\sqrt{5}}{4\sqrt{3}-5\sqrt{2}} \cdot \dfrac{4\sqrt{3}+5\sqrt{2}}{4\sqrt{3}+5\sqrt{2}}$

$= \dfrac{12\sqrt{15}+15\sqrt{10}}{48-50}$

$= -\dfrac{12\sqrt{15}+15\sqrt{10}}{2}$

f. $\dfrac{\sqrt{18}+\sqrt{12}}{\sqrt{18}-\sqrt{12}} \cdot \dfrac{\sqrt{18}+\sqrt{12}}{\sqrt{18}+\sqrt{12}}$

$= \dfrac{18+2\sqrt{216}+12}{18-12}$

$= \dfrac{30+12\sqrt{6}}{6}$

$= 5+2\sqrt{6}$

7. a. $\dfrac{\sqrt{a}-2}{a-4} \cdot \dfrac{\sqrt{a}+2}{\sqrt{a}+2}$

$= \dfrac{a-4}{(a-4)(\sqrt{a}-2)}$

$= \dfrac{1}{\sqrt{a}-2}$

b. $\dfrac{\sqrt{x+4}-2}{x} \cdot \dfrac{\sqrt{x+4}+2}{\sqrt{x+4}+2}$

$= \dfrac{x+4-4}{x(\sqrt{x+4}+2)}$

$= \dfrac{x}{x(\sqrt{x+4}+2)}$

$= \dfrac{1}{\sqrt{x+4}-2}$

c. $\dfrac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \dfrac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$

$= \dfrac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}$

$= \dfrac{h}{h(\sqrt{x+h}+\sqrt{x})}$

$= \dfrac{1}{\sqrt{x+h}+\sqrt{x}}$

1.2 The Slope of a Tangent, pp. 18–21

1. a. $m = \dfrac{-8-7}{-3-2}$

$= 3$

b. $m = \dfrac{-\frac{7}{2}-\frac{3}{2}}{\frac{7}{2}-\frac{1}{2}}$

$= \dfrac{-\frac{10}{2}}{\frac{6}{2}}$

$= -\dfrac{5}{3}$

c. $m = \dfrac{-1-(-2.6)}{1.5-6.3}$

$= -\dfrac{1}{3}$

2. a. The slope of the given line is 3, so the slope of a line perpendicular to the given line is $-\frac{1}{3}$.

b. $13x - 7y - 11 = 0$

$-7y = -13x - 11$

$y = \dfrac{13}{7}x + \dfrac{11}{7}$

The slope of the given line is $\frac{13}{7}$, so the slope of a line perpendicular to the given line is $-\frac{7}{13}$.

3. a. $m = \dfrac{-\frac{5}{3} - (-4)}{\frac{5}{3} - (-4)}$

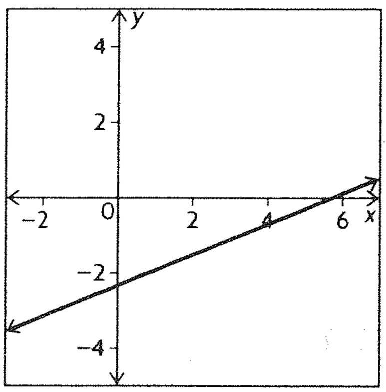
$= \dfrac{\frac{7}{3}}{\frac{17}{3}}$

$= \dfrac{7}{17}$

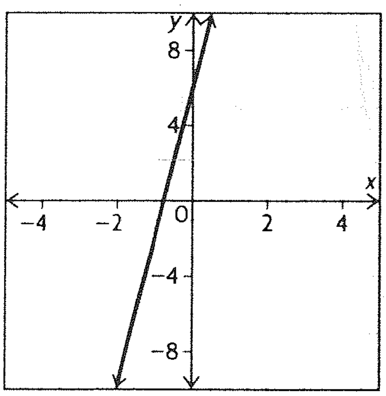
$y - (-4) = \dfrac{7}{17}(x - (-4))$

$17y + 68 = 7x + 28$

$7x - 17y - 40 = 0$



b. The slope and y-intercept are given.
$y = 8x + 6$



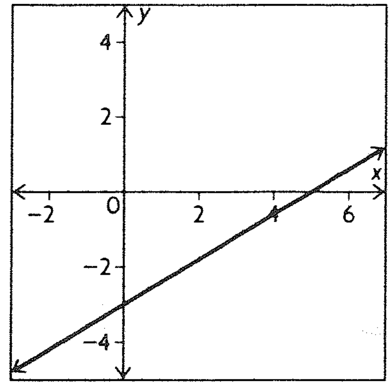
c. $(0, -3), (5, 0)$

$m = \dfrac{0 - (-3)}{5 - 0}$

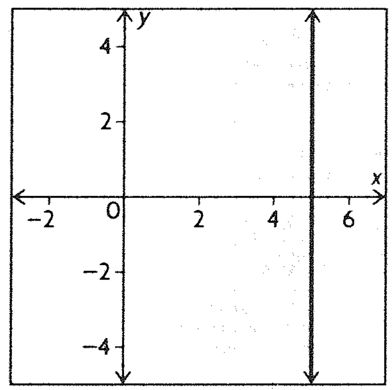
$= \dfrac{3}{5}$

$y - 0 = \dfrac{3}{5}(x - 5)$

$3x - 5y - 15 = 0$



d. The line is a vertical line because both points have the same x-coordinate.
$x = 5$



4. a. $\dfrac{(5 + h)^3 - 125}{h}$

$= \dfrac{(5 + h - 5)((5 + h)^2 + 5(5 + h) + 25)}{h}$

$= \dfrac{h(75 + 15h + h^2)}{h}$

$= 75 + 15h + h^2$

b. $\dfrac{(3 + h)^4 - 81}{h}$

$= \dfrac{((3 + h)^2 - 9)((3 + h)^2 + 9)}{h}$

$= \dfrac{(9 + 6h + h^2 - 9)(9 + 6h + h^2 + 9)}{h}$

$= (6 + h)(18 + 6h + h^2)$

$= 108 + 54h + 12h^2 + h^3$

c. $\dfrac{\frac{1}{1 + h} - 1}{h} = \dfrac{1 - 1 - h}{h(1 + h)} = -\dfrac{1}{1 + h}$

d. $\dfrac{3(1 + h)^2 - 3}{h} = \dfrac{3((1 + h)^2 - 1)}{h}$

$= \dfrac{3(1 + 2h + h^2 - 1)}{h}$

$$= \frac{3(2h + h^2)}{h}$$
$$= 6 + 3h$$

e. $$\frac{\frac{3}{4+h} - \frac{3}{4}}{h} = \frac{\frac{12 - 12 - 3h}{4(4+h)}}{h}$$
$$= \frac{-3}{4(4+h)}$$

f. $$\frac{\frac{-1}{2+h} + \frac{1}{2}}{h} = \frac{\frac{-2 + 2 + h}{2(2+h)}}{h}$$
$$= \frac{h}{2h(2+h)}$$
$$= \frac{1}{4 + 2h}$$

5. a. $$\frac{\sqrt{16+h} - 4}{h} = \frac{16 + h - 16}{h(\sqrt{16+h} + 4)}$$
$$= \frac{1}{\sqrt{16+h} + 4}$$

b. $$\frac{\sqrt{h^2 + 5h + 4} - 2}{h} = \frac{h^2 + 5h + 4 - 4}{h(\sqrt{h^2 + 5h + 4} + 2)}$$
$$= \frac{h + 5}{\sqrt{h^2 + 5h + 4} + 2}$$

c. $$\frac{\sqrt{5+h} - \sqrt{5}}{h} = \frac{5 + h - 5}{h(\sqrt{5+h} + \sqrt{5})}$$
$$= \frac{1}{\sqrt{5+h} + \sqrt{5}}$$

6. a. $P(1, 3)$, $Q(1 + h, f(1 + h))$, $f(x) = 3x^2$
$$m = \frac{3(1+h)^2 - 3}{h}$$
$$= 6 + 3h$$

b. $P(1, 3)$, $Q(1 + h, (1 + h)^3 + 2)$
$$m = \frac{(1+h)^3 + 2 - 3}{h}$$
$$= \frac{1 + 3h + 3h^2 + h^3 - 1}{h}$$
$$= 3 + 3h + h^2$$

c. $P(9, 3)$, $Q(9 + h, \sqrt{9 + h})$
$$m = \frac{\sqrt{9+h} - 3}{h} \cdot \frac{\sqrt{9+h} + 3}{\sqrt{9+h} + 3}$$
$$= \frac{1}{\sqrt{9+h} + 3}$$

7. a.

P	*Q*	Slope of Line *PQ*
(2, 8)	(3, 27)	19
(2, 8)	(2.5, 15.625)	15.25
(2, 8)	(2.1, 9.261)	12.61
(2, 8)	(2.01, 8.120 601)	12.060 1
(2, 8)	(1, 1)	7
(2, 8)	(1.5, 3.375)	9.25
(2, 8)	(1.9, 6.859)	11.41
(2, 8)	(1.99, 7.880 599)	11.940 1

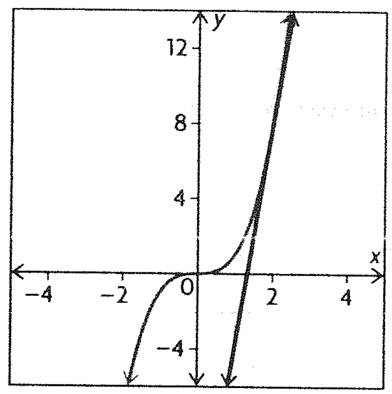
b. 12

c. $(2, 8)$, $((2 + h), (2 + h)^3)$
$$m = \frac{(2+h)^3 - 8}{2 + h - 2}$$
$$= \frac{8 + 12h + 6h^2 + h^3 - 8}{h}$$
$$= 12 + 6h + h^2$$

d. $$m = \lim_{h \to 0}(12 + 6h + h^2)$$
$$= 12$$

e. They are the same.

f.



8. a. $y = 3x^2$, $(-2, 12)$
$$m = \lim_{h \to 0} \frac{3(-2+h)^2 - 12}{h}$$
$$= \lim_{h \to 0} \frac{12 - 12h + 3h^2 - 12}{h}$$
$$= \lim_{h \to 0}(-12 + 3h)$$
$$= -12$$

b. $y = x^2 - x$ at $x = 3$, $y = 6$.
$$m = \lim_{h \to 0} \frac{(3+h)^2 - (3+h) - 6}{h}$$
$$= \lim_{h \to 0} \frac{9 + 6h + h^2 - 3 - h - 6}{h}$$
$$= \lim_{h \to 0}(5 + h)$$
$$= 5$$

c. $y = x^3$ at $x = -2$, $y = -8$.

$$m = \lim_{h\to 0}\frac{(-2+h)^3+8}{h}$$
$$= \lim_{h\to 0}\frac{-8+12h-6h^2+h^3+8}{h}$$
$$= \lim_{h\to 0}(12-6h+h^2)$$
$$= 12$$

9. a. $y = \sqrt{x-2}$; $(3, 1)$

$$m = \lim_{h\to 0}\frac{\sqrt{3+h-2}-1}{h}$$
$$= \lim_{h\to 0}\left[\frac{\sqrt{1+h}-1}{h}\times\frac{\sqrt{1+h}+1}{\sqrt{1+h}+1}\right]$$
$$= \lim_{h\to 0}\frac{1}{\sqrt{1+h}+1}$$
$$= \frac{1}{2}$$

b. $y = \sqrt{x-5}$ at $x = 9$, $y = 2$

$$m = \lim_{h\to 0}\frac{\sqrt{9+h-5}-2}{h}$$
$$= \lim_{h\to 0}\left[\frac{\sqrt{4+h}-2}{h}\times\frac{\sqrt{4+h}+2}{\sqrt{4+h}+2}\right]$$
$$= \lim_{h\to 0}\frac{1}{\sqrt{4+h}+2}$$
$$= \frac{1}{4}$$

c. $y = \sqrt{5x-1}$ at $x = 2$, $y = 3$

$$m = \lim_{h\to 0}\frac{\sqrt{10+5h-1}-3}{h}$$
$$= \lim_{h\to 0}\left[\frac{\sqrt{9+5h}-3}{h}\times\frac{\sqrt{9+5h}+3}{\sqrt{9+5h}+3}\right]$$
$$= \lim_{h\to 0}\frac{5}{\sqrt{9+5h}+3}$$
$$= \frac{5}{6}$$

10. a. $y = \frac{8}{x}$ at $(2, 4)$

$$m = \lim_{h\to 0}\frac{\frac{8}{2+h}-4}{h}$$
$$= \lim_{h\to 0}\frac{-4}{2+h}$$
$$= -2$$

b. $y = \frac{8}{3+x}$ at $x = 1$; $y = 2$

$$m = \lim_{h\to 0}\frac{\frac{8}{4+h}-2}{h}$$
$$= \lim_{h\to 0}\frac{-2}{4+h}$$
$$= -\frac{1}{2}$$

c. $y = \frac{1}{x+2}$ at $x = 3$; $y = \frac{1}{5}$

$$m = \lim_{h\to 0}\frac{\frac{1}{5+h}-\frac{1}{5}}{h}$$
$$= \lim_{h\to 0}\frac{-1}{5(5+h)}$$
$$= -\frac{1}{10}$$

11. a. Let $y = f(x)$.

$f(2) = (2)^2 - 3(2) = 4 - 6 = -2$

$f(2+h) = (2+h)^2 - 3(2+h)$

Using the limit of the difference quotient, the slope of the tangent at $x = 2$ is

$$m = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$
$$= \lim_{h\to 0}\frac{(2+h)^2-3(2+h)-(-2)}{h}$$
$$= \lim_{h\to 0}\frac{4+4h+h^2-6-3h+2}{h}$$
$$= \lim_{h\to 0}\frac{h^2+h}{h}$$
$$= \lim_{h\to 0}(h+1)$$
$$= 0 + 1$$
$$= 1$$

Therefore, the slope of the tangent to $y = f(x) = x^2 - 3x$ at $x = 2$ is 1.

b. $f(-2) = \frac{4}{-2} = -2$

$f(-2+h) = \frac{4}{-2+h}$

Using the limit of the difference quotient, the slope of the tangent at $x = -2$ is

$$m = \lim_{h\to 0}\frac{f(-2+h)-f(-2)}{h}$$
$$= \lim_{h\to 0}\frac{\frac{4}{-2+h}-(-2)}{h}$$
$$= \lim_{h\to 0}\frac{\frac{4}{-2+h}+2}{h}$$
$$= \lim_{h\to 0}\left[\frac{4-4+2h}{-2+h}\cdot\frac{1}{h}\right]$$
$$= \lim_{h\to 0}\left[\frac{2h}{-2+h}\cdot\frac{1}{h}\right]$$

$$= \lim_{h \to 0} \frac{2}{-2 + h}$$
$$= \frac{2}{-2 + 0}$$
$$= -1$$

Therefore, the slope of the tangent to $f(x) = \frac{4}{x}$ at $x = -2$ is -1.

c. Let $y = f(x)$.

$$f(1) = 3(1)^3 = 3$$
$$f(1 + h) = 3(1 + h)^3$$

Using the limit of the difference quotient, the slope of the tangent at $x = 1$ is

$$m = \lim_{h \to 0} \frac{f(1 + h) - f(1)}{h}$$
$$= \lim_{h \to 0} \frac{3(1 + h)^3 - 3}{h}$$

Using the binomial formula to expand $(1 + h)^3$ (or one could simply expand using algebra), the slope m is

$$= \lim_{h \to 0} \frac{3(h^3 + 3h^2 + 3h + 1) - (3)}{h}$$
$$= \lim_{h \to 0} \frac{3h^3 + 9h^2 + 9h + 3 - 3}{h}$$
$$= \lim_{h \to 0} \frac{3h^3 + 9h^2 + 9h}{h}$$
$$= \lim_{h \to 0} (3h^2 + 9h + 9)$$
$$= 3(0) + 9(0) + 9$$
$$= 9$$

Therefore, the slope of the tangent to $y = f(x) = 3x^3$ at $x = 1$ is 9.

d. Let $y = f(x)$.

$$f(16) = \sqrt{16 - 7} = \sqrt{9} = 3$$
$$f(16 + h) = \sqrt{16 + h - 7} = \sqrt{h + 9}$$

Using the limit of the difference quotient, the slope of the tangent at $x = 16$ is

$$m = \lim_{h \to 0} \frac{f(16 + h) - f(16)}{h}$$
$$= \lim_{h \to 0} \frac{\sqrt{h + 9} - 3}{h}$$
$$= \lim_{h \to 0} \frac{\sqrt{h + 9} - 3}{h} \cdot \frac{\sqrt{h + 9} + 3}{\sqrt{h + 9} + 3}$$
$$= \lim_{h \to 0} \frac{(h + 9) - 9}{h(\sqrt{h + 9} + 3)}$$
$$= \lim_{h \to 0} \frac{h}{h(\sqrt{h + 9} + 3)}$$
$$= \lim_{h \to 0} \frac{1}{\sqrt{h + 9} + 3}$$
$$= \frac{1}{\sqrt{0 + 9} + 3}$$
$$= \frac{1}{3 + 3}$$
$$= \frac{1}{6}$$

Therefore, the slope of the tangent to $y = f(x) = \sqrt{x - 7}$ at $x = 16$ is $\frac{1}{6}$.

e. Let $y = f(x)$.

$$f(3) = \sqrt{25 - (3)^2} = \sqrt{25 - 9} = 4$$
$$f(3 + h) = \sqrt{25 - (3 + h)^2}$$
$$= \sqrt{25 - (9 + 6h + h^2)}$$
$$= \sqrt{25 - 9 - 6h - h^2}$$
$$= \sqrt{16 - 6h - h^2}$$

Using the limit of the difference quotient, the slope of the tangent at $x = 3$ is

$$m = \lim_{h \to 0} \frac{f(3 + h) - f(3)}{h}$$
$$= \lim_{h \to 0} \frac{\sqrt{16 - 6h - h^2} - 4}{h}$$
$$= \lim_{h \to 0} \left[\frac{\sqrt{16 - 6h - h^2} - 4}{h} \times \frac{\sqrt{16 - 6h - h^2} + 4}{\sqrt{16 - 6h - h^2} + 4}\right]$$
$$= \lim_{h \to 0} \frac{16 - 6h - h^2 - 16}{h(\sqrt{16 - 6h - h^2} + 4)}$$
$$= \lim_{h \to 0} \frac{h(-6 - h)}{h(\sqrt{16 - 6h - h^2} + 4)}$$
$$= \lim_{h \to 0} \frac{-6 - h}{\sqrt{16 - 6h - h^2} + 4}$$
$$= \frac{-6 - 0}{\sqrt{16 - 6(0) - (0)^2} + 4}$$
$$= \frac{-6}{\sqrt{16} + 4}$$
$$= \frac{-6}{8}$$
$$= -\frac{3}{4}$$

Therefore, the slope of the tangent to $y = f(x) = \sqrt{25 - x^2}$ at $x = 3$ is $-\frac{3}{4}$.

f. Let $y = f(x)$.

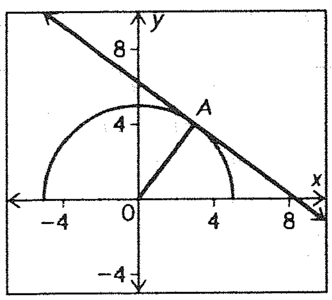
$$f(8) = \frac{4 + 8}{8 - 2} = \frac{12}{6} = 2$$
$$f(8 + h) = \frac{4 + (8 + h)}{(8 + h) - 2} = \frac{12 + h}{6 + h}$$

Using the limit of the difference quotient, the slope of the tangent at $x = 8$ is

$$m = \lim_{h\to 0} \frac{f(8+h) - f(8)}{h}$$
$$= \lim_{h\to 0} \frac{\frac{12+h}{6+h} - 2}{h}$$
$$= \lim_{h\to 0} \frac{12 + h - 12 - 2h}{6+h} \cdot \frac{1}{h}$$
$$= \lim_{h\to 0} \frac{-h}{6+h} \cdot \frac{1}{h}$$
$$= \lim_{h\to 0} \frac{-1}{6+h}$$
$$= \frac{-1}{6+0}$$
$$= -\frac{1}{6}$$

Therefore, the slope of the tangent to $y = f(x) = \frac{4+x}{x-2}$ at $x = 8$ is $-\frac{1}{6}$.

12.



$y = \sqrt{25 - x^2} \rightarrow$ Semi-circle centre $(0, 0)$ rad 5, $y \geq 0$

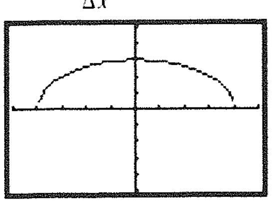
OA is a radius.

The slope of OA is $\frac{4}{3}$.

The slope of tangent is $-\frac{3}{4}$.

13. Take values of x close to the point, then determine $\frac{\Delta y}{\Delta x}$.

14.



Since the tangent is horizontal, the slope is 0.

15. $m = \lim_{h\to 0} \frac{(3+h)^2 - 3(3+h) + 1 - 1}{h}$

$$= \lim_{h\to 0} \frac{9 + 6h + h^2 - 9 - 3h}{h}$$
$$= \lim_{h\to 0} \frac{3h + h^2}{h}$$
$$= \lim_{h\to 0} (3 + h)$$
$$= 3$$

The slope of the tangent is 3.

$$y - 1 = 3(x - 3)$$
$$3x - y - 8 = 0$$

16. $m = \lim_{h\to 0} \frac{(2+h)^2 - 7(2+h) + 12 - 2}{h}$

$$= \lim_{h\to 0} \frac{4 + 4h + h^2 - 14 - 7h + 10}{h}$$
$$= \lim_{h\to 0} \frac{-3h + h^2}{h}$$
$$= \lim_{h\to 0} (-3 + h)$$
$$= -3$$

The slope of the tangent is -3.

When $x = 2$, $y = 2$.

$$y - 2 = -3(x - 2)$$
$$3x + y - 8 = 0$$

17. a. $f(3) = 9 - 12 + 1 = -2$; $(3, -2)$

b. $f(5) = 25 - 20 + 1 = 6$; $(5, 6)$

c. The slope of secant AB is

$$m_{AB} = \frac{6 - (-2)}{5 - 3}$$
$$= \frac{8}{2}$$
$$= 4$$

The equation of the secant is

$$y - y_1 = m_{AB}(x - x_1)$$
$$y + 2 = 4(x - 3)$$
$$y = 4x - 14$$

d. Calculate the slope of the tangent.

$$m = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h\to 0} \frac{(x+h)^2 - 4(x+h) + 1 - (x^2 - 4x + 1)}{h}$$
$$= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - 4x - 4h + 1 - x^2 + 4x - 1}{h}$$
$$= \lim_{h\to 0} \frac{2xh + h^2 - 4h}{h}$$
$$= \lim_{h\to 0} (2x + h - 4)$$
$$= 2x + 0 - 4$$
$$= 2x - 4$$

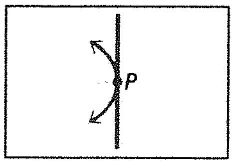
When $x = 3$, the slope is $2(3) - 4 = 2$. So the equation of the tangent at $A(3, -2)$ is

$$y - y_1 = m(x - x_1)$$
$$y + 2 = 2(x - 3)$$
$$y = 2x - 8$$

e. When $x = 5$, the slope of the tangent is $2(5) - 4 = 6$.
So the equation of the tangent at $B(5, 6)$ is

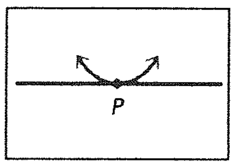
$$y - y_1 = m(x - x_1)$$
$$y - 6 = 6(x - 5)$$
$$y = 6x - 24$$

18. a.



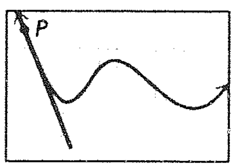
The slope is undefined.

b.



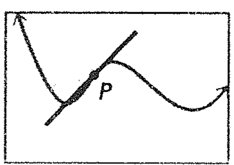
The slope is 0.

c.



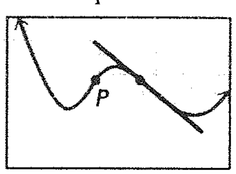
The slope is about −2.5.

d.



The slope is about 1.

e.



The slope is about $-\frac{7}{8}$.

f. There is no tangent at this point.

19. $D(p) = \dfrac{20}{\sqrt{p - 1}}, p > 1$ at $(5, 10)$

$$m = \lim_{h \to 0} \frac{\frac{20}{\sqrt{4 + h}} - 10}{h}$$
$$= 10 \lim_{h \to 0} \frac{2 - \sqrt{4 + h}}{h\sqrt{4 + h}} \times \frac{2 + \sqrt{4 + h}}{2 + \sqrt{4 + h}}$$
$$= 10 \lim_{h \to 0} \frac{4 - 4 - h}{h\sqrt{4 + h}\left(2 + \sqrt{4 + h}\right)}$$
$$= -\frac{10}{8}$$
$$= -\frac{5}{4}$$

20. $C(t) = 100t^2 + 400t + 5000$
Slope at $t = 6$

$$C'(t) = 200t + 400$$
$$C'(6) = 1200 + 400 = 1600$$

Increasing at a rate of 1600 papers per month.

21. Point on $f(x) = 3x^2 - 4x$ tangent parallel to $y = 8x$. Therefore, tangent line has slope 8.

$$m = \lim_{h \to 0} \frac{3(h + a)^2 - 4(h + a) - 3(a^2 + 4a)}{h} = 8$$
$$\lim_{h \to 0} \frac{3h^2 + 6ah - 4h}{h} = 8$$
$$6a - 4 = 8$$
$$a = 2$$

The point has coordinates $(2, 4)$.

22. $y = \frac{1}{3}x^3 - 5x - \frac{4}{x}$

$$\frac{1}{3}(a + h)^3 - \frac{1}{3}a^3 = a^2h + ah^2 + \frac{1}{3}h^3$$
$$\lim_{h \to 0}\left(a^2 + ah + \frac{1}{3}h^3\right) = a^2$$
$$5 \lim_{h \to 0} - \frac{(a + h) - (-a)}{h} = -5$$
$$-\frac{4}{a + h} + \frac{4}{a} = -\frac{4a + 4a + 4h}{a(a + h)}$$
$$\lim_{h \to 0} \frac{4}{a(a + h)} = \frac{4}{a^2}$$
$$m = a^2 - 5 + \frac{4}{a^2} = 0$$
$$a^4 - 5a^2 + 4 = 0$$
$$(a^2 - 4)(a^2 - 1) = 0$$
$$a = \pm 2, a = \pm 1$$

Points on the graph for horizontal tangents are:
$\left(-2, \frac{28}{3}\right), \left(-1, \frac{26}{3}\right), \left(1, -\frac{26}{3}\right), \left(2, -\frac{28}{3}\right)$.

23. $y = x^2$ and $y = \frac{1}{2} - x^2$

$$x^2 = \frac{1}{2} - x^2$$
$$x^2 = \frac{1}{4}$$
$$x = \frac{1}{2} \text{ or } x = -\frac{1}{2}$$

The points of intersection are
$P\left(\frac{1}{2}, \frac{1}{4}\right), Q\left(-\frac{1}{2}, \frac{1}{4}\right)$.
Tangent to $y = x^2$:

$$m = \lim_{h \to 0} \frac{(a + h)^2 - a^2}{h}$$
$$= \lim_{h \to 0} \frac{2ah + h^2}{h}$$
$$= 2a.$$

The slope of the tangent at $a = \frac{1}{2}$ is $1 = m_p$,
at $a = -\frac{1}{2}$ is $-1 = m_q$.
Tangents to $y = \frac{1}{2} - x^2$:

$$m = \lim_{h \to 0} \frac{\left[\frac{1}{2} - (a + h)^2\right] - \left[\frac{1}{2} - a^2\right]}{h}$$
$$= \lim_{h \to 0} \frac{-2ah - h^2}{h}$$
$$= -2a.$$

The slope of the tangents at $a = \frac{1}{2}$ is $-1 = M_p$;
at $a = -\frac{1}{2}$ is $1 = M_q$
$m_pM_p = -1$ and $m_qM_q = -1$
Therefore, the tangents are perpendicular at the points of intersection.

24. $y = -3x^3 - 2x$, $(-1, 5)$

$$m = \lim_{h \to 0} \frac{-3(-1 + h)^3 - 2(-1 + h) - 5}{h}$$
$$= \lim_{h \to 0} \frac{-3(-1 + 3h - 3h^2 + h^3) + 2 - 2h - 5}{h}$$
$$= \lim_{h \to 0} \frac{-3(-1 + 3h - 3h^2 + h^3) + 2 - 2h - 5}{h}$$
$$= \lim_{h \to 0} \frac{3 - 9h + 9h^2 - 3h^3 + 2 - 2h - 5}{h}$$
$$= \lim_{h \to 0} \frac{-11h + 9h^2 - 3h^3}{h}$$
$$= \lim_{h \to 0} (-11 + 9h - 3h^2)$$
$$= -11$$

The slope of the tangent is -11.
We want the line that is parallel to the tangent (i.e. has slope -11) and passes through $(2, 2)$. Then,

$$y - 2 = -11(x - 2)$$
$$y = -11x + 24$$

25. a. Let $y = f(x)$.

$$f(a) = 4a^2 + 5a - 2$$
$$f(a + h) = 4(a + h)^2 + 5(a + h) - 2$$
$$= 4(a^2 + 2ah + h^2) + 5a + 5h - 2$$
$$= 4a^2 + 8ah + 4h^2 + 5a + 5h - 2$$

Using the limit of the difference quotient, the slope of the tangent at $x = a$ is

$$m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$
$$= \lim_{h \to 0} \left[\frac{4a^2 + 8ah + 4h^2 + 5a + 5h - 2}{h} - \frac{(4a^2 + 5a - 2)}{h}\right]$$
$$= \lim_{h \to 0} \left[\frac{4a^2 + 8ah + 4h^2 + 5a + 5h - 2}{h} + \frac{-4a^2 - 5a + 2}{h}\right]$$
$$= \lim_{h \to 0} \frac{8ah + 4h^2 + 5h}{h}$$
$$= \lim_{h \to 0} (8a + 4h + 5)$$
$$= 8a + 4(0) + 5$$
$$= 8a + 5$$

b. To be parallel, the point on the parabola and the line must have the same slope. So, first find the slope of the line. The line $10x - 2y - 18 = 0$ can be rewritten as

$$-2y = 18 - 10x$$
$$y = \frac{18 - 10x}{-2}$$
$$y = -9 + 5x$$
$$y = 5x - 9$$

So, the slope, m, of the line $10x - 2y - 18 = 0$ is 5.
To be parallel, the slope at a must equal 5. From part a., the slope of the tangent to the parabola at $x = a$ is $8a + 5$.

$$8a + 5 = 5$$
$$8a = 0$$
$$a = 0$$

Therefore, at the point $(0, -2)$ the tangent line is parallel to the line $10x - 2y - 18 = 0$.

c. To be perpendicular, the point on the parabola and the line must have slopes that are negative reciprocals of each other. That is, their product must equal -1. So, first find the slope of the line. The line $x - 35y + 7 = 0$ can be rewritten as

$$-35y = -x - 7$$
$$y = \frac{-x - 7}{-35}$$
$$y = \frac{1}{35}x + \frac{7}{35}$$

So, the slope, m, of the line $x - 35y + 7 = 0$ is $\frac{1}{35}$.
To be perpendicular, the slope at a must equal the negative reciprocal of the slope of the line $x - 35y + 7 = 0$. That is, the slope of a must equal -35. From part a., the slope of the tangent to the parabola at $x = a$ is $8a + 5$.

$$8a + 5 = -35$$
$$8a = -40$$
$$a = -5$$

Therefore, at the point $(-5, 73)$ the tangent line is perpendicular to the line $x - 35y + 7 = 0$.

1.3 Rates of Change, pp. 29–31

1. $v(t) = 0$ when $t = 0$ or $t = 4$.

2. a. $\frac{s(9) - s(2)}{7}$. Slope of the secant between the points $(2, s(2))$ and $(9, s(9))$.

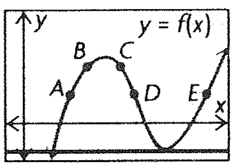
b. $\lim_{h\to 0}\frac{s(6 + h) - s(6)}{h}$. Slope of the tangent at the point $(6, s(6))$.

3. $\lim_{h\to 0}\frac{\sqrt{4 + h} - 2}{h}$. Slope of the tangent to the function with equation $y = \sqrt{x}$ at the point $(4, 2)$.

4. a. A and B

b. greater; the secant line through these two points is steeper than the tangent line at B.

c.



5. Speed is represented only by a number, not a direction.

6. Yes, velocity needs to be described by a number and a direction. Only the speed of the school bus was given, not the direction, so it is not correct to use the word "velocity."

7. $s(t) = 320 - 5t^2, 0 \le t \le 8$

a. Average velocity during the first second:

$\frac{s(1) - s(0)}{1} = 5$ m/s;

third second:

$\frac{s(3) - s(2)}{1} = \frac{45 - 20}{1} = 25$ m/s;

eighth second:

$\frac{s(8) - s(7)}{1} = \frac{320 - 245}{1} = 75$ m/s.

b. Average velocity $3 \le t \le 8$

$\frac{s(8) - s(3)}{8 - 3} = \frac{320 - 45}{5} = \frac{275}{5} = 55$ m/s

c. $s(t) = 320 - 5t^2$

$$v(t) = \lim_{h\to 0}\frac{320 - 5(2 + h)^2 - (320 - 5(2)^2)}{h}$$
$$= 5\lim_{h\to 0}\frac{-4h + h^2}{h}$$
$$= -20$$

Velocity at $t = 2$ is 20 m/s downward.

8. $s(t) = 8t(t + 2), 0 \le t \le 5$

a. i. from $t = 3$ to $t = 4$

Average velocity $\frac{s(4) - s(3)}{1}$

$= 32(6) - 24(5)$
$= 24(8 - 5)$
$= 72$ km/h

ii. from $t = 3$ to $t = 3.1$

$$\frac{s(3.1) - s(3)}{0.1}$$
$$= \frac{126.48 - 120}{0.1}$$
$= 64.8$ km/h

iii. $3 \le t \le 3.01$

$$\frac{s(3.01) - s(3)}{0.01}$$
$= 64.08$ km/h

b. Instantaneous velocity is approximately 64 km/h.

c. At $t = 3$

$s(t) = 8t^2 + 16t$
$v(t) = 16t + 16$
$v(3) = 48 + 16$
$= 64$ km/h

9. a. $N(t) = 20t - t^2$

$$\frac{N(3) - N(2)}{1}$$
$$= \frac{51 - 36}{1}$$
$$= 15$$

15 terms are learned between $t = 2$ and $t = 3$.

b. $\lim_{h\to 0}\frac{20(2 + h) - (2 + h)^2 - 36}{h}$

$$= \lim_{h\to 0}\frac{40 + 20h - 4 - 4h - h^2 - 36}{h}$$
$$= \lim_{h\to 0}\frac{16h - h^2}{h}$$
$$= \lim_{h\to 0}(16 - h)$$
$$= 16$$

At $t = 2$, the student is learning at a rate of 16 terms/h.

10. a. M in mg in 1 mL of blood t hours after the injection.

$M(t) = -\frac{1}{3}t^2 + t; 0 \le t \le 3$

Calculate the instantaneous rate of change when $t = 2$.

$$\lim_{h\to 0}\frac{-\frac{1}{3}(2 + h)^2 + (2 + h) - (-\frac{4}{3} + 2)}{h}$$
$$= \lim_{h\to 0}\frac{-\frac{4}{3} - \frac{4}{3}h - \frac{1}{3}h^2 + 2 + h + \frac{4}{3} - 2}{h}$$
$$= \lim_{h\to 0}\frac{-\frac{1}{3}h - \frac{1}{3}h^2}{h}$$
$$= \lim_{h\to 0}\left(-\frac{1}{3} - \frac{1}{3}h\right)$$
$$= -\frac{1}{3}$$

Rate of change is $-\frac{1}{3}$ mg/h.

b. Amount of medicine in 1 mL of blood is being dissipated throughout the system.

11. $t = \sqrt{\frac{s}{5}}$

Calculate the instantaneous rate of change when $s = 125$.

$$\lim_{h\to 0} \frac{\sqrt{\frac{125 + h}{5}} - \sqrt{\frac{125}{5}}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{\frac{125 + h}{5}} - 5}{h}$$

$$= \lim_{h\to 0} \left[\frac{\sqrt{\frac{125 + h}{5}} - 5}{h} \cdot \frac{\sqrt{\frac{125 + h}{5}} + 5}{\sqrt{\frac{125 + h}{5}} + 5}\right]$$

$$= \lim_{h\to 0} \left[\frac{\frac{125 + h}{5} - 25}{h\left(\sqrt{\frac{125 + h}{5}} + 5\right)}\right]$$

$$= \lim_{h\to 0} \left[\frac{\frac{125 + h - 125}{5}}{h\left(\sqrt{\frac{125 + h}{5}} + 5\right)}\right]$$

$$= \lim_{h\to 0} \frac{1}{5\left(\sqrt{\frac{125 + h}{5}} + 5\right)}$$

$$= \frac{1}{5\left(\sqrt{\frac{125}{5}} + 5\right)}$$

$$= \frac{1}{5(5 + 5)}$$

$$= \frac{1}{50}$$

At $s = 125$, rate of change of time with respect to height is $\frac{1}{50}$ s/m.

12. $T(h) = \frac{60}{h + 2}$

Calculate the instantaneous rate of change when $h = 3$.

$$\lim_{k\to 0} \frac{\frac{60}{(3 + k) + 2} - \frac{60}{(3 + 2)}}{k}$$

$$= \lim_{k\to 0} \frac{\frac{60}{5 + k} - 12}{k}$$

$$= \lim_{k\to 0} \frac{\frac{60}{5 + k} - \frac{60 + 12k}{5 + k}}{k}$$

$$= \lim_{k\to 0} \frac{-12k}{k(5 + k)}$$

$$= \lim_{k\to 0} \frac{-12}{(5 + k)}$$

$$= -\frac{12}{5}$$

Temperature is decreasing at $\frac{12}{5}$ °C/km.

13. $h = 25t^2 - 100t + 100$

When $h = 0$, $25t^2 - 100t + 100 = 0$

$$t^2 - 4t + 4 = 0$$

$$(t - 2)^2 = 0$$

$$t = 2$$

Calculate the instantaneous rate of change when $t = 2$.

$$\lim_{h\to 0} \frac{25(2 + h)^2 - 100(2 + h) + 100 - 0}{h}$$

$$= \lim_{h\to 0} \frac{100 + 100h + 25h^2 - 200 - 100h + 100}{h}$$

$$= \lim_{h\to 0} \frac{25h^2}{h}$$

$$= \lim_{h\to 0} 25h$$

$$= 0$$

It hit the ground in 2 s at a speed of 0 m/s.

14. Sale of x balls per week:

$P(x) = 160x - x^2$ dollars.

a. $P(40) = 160(40) - (40)^2$
$= 4800$

Profit on the sale of 40 balls is $4800.

b. Calculate the instantaneous rate of change when $x = 40$.

$$\lim_{h\to 0} \frac{160(40 + h) - (40 + h)^2 - 4800}{h}$$

$$= \lim_{h\to 0} \frac{6400 + 160h - 1600 - 80h - h^2 - 4800}{h}$$

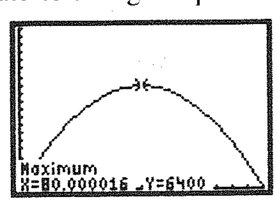
$$= \lim_{h\to 0} \frac{80h - h^2}{h}$$

$$= \lim_{h\to 0} (80 - h)$$

$$= 80$$

Rate of change of profit is $80 per ball.

c.



Rate of change of profit is positive when the sales level is less than 80.

15. a. $f(x) = -x^2 + 2x + 3$; $(-2, -5)$

$$\lim_{x \to -2} \frac{f(x) - f(-2)}{x + 2}$$
$$= \lim_{x \to -2} \frac{-x^2 + 2x + 3 + 5}{x + 2}$$
$$= \lim_{x \to -2} \frac{-(x^2 - 2x - 8)}{x + 2}$$
$$= -\lim_{x \to -2} \frac{(x - 4)(x + 2)}{x + 2}$$
$$= -\lim_{x \to -2} (x - 4)$$
$$= 6$$

b. $f(x) = \frac{x}{x - 1}$, $x = 2$

$$\lim_{x \to 2} \frac{\frac{x}{x - 1} - 2}{x - 2}$$
$$= \lim_{x \to 2} \frac{x - 2x + 2}{(x - 1)(x - 2)}$$
$$= \lim_{x \to 2} \frac{-(x - 2)}{(x - 1)(x - 2)}$$
$$= -1$$

c. $f(x) = \sqrt{x + 1}$, $x = 24$

$$= \lim_{x \to 24} \frac{f(x) - f(24)}{x - 24}$$
$$= \lim_{x \to 24} \frac{\sqrt{x + 1} - 5}{x - 24} \cdot \frac{\sqrt{x + 1} + 5}{\sqrt{x + 1} + 5}$$
$$= \lim_{x \to 24} \frac{x - 24}{(x - 24)(\sqrt{x + 1} + 5)}$$
$$= \frac{1}{10}$$

16. $S(x) = 246 + 64x - 8.9x^2 + 0.95x^3$

Calculate the instantaneous rate of change.

$$= \lim_{h \to 0} \frac{S(x + h) - S(x)}{h}$$
$$= \lim_{h \to 0} \frac{246 + 64(x + h) - 8.9(x + h)^2 + 0.95(x + h)^3 - (246 - 64x - 8.9x^2 + 0.95x^3)}{h}$$
$$= \lim_{h \to 0} \frac{246 - 246 + 64(x + h - x) - 8.9(x^2 + 2xh + h^2 - x^2) + 0.95(x^3 + 3x^2h + 3xh^2 + h^3 - x^3)}{h}$$
$$= \lim_{h \to 0} \frac{64h - 8.9(2xh + h^2) + 0.95(3x^2h + 3xh^2 + h^3)}{h}$$
$$= \lim_{h \to 0} [64 - 8.9(2x + h) + 0.95(3x^2 + 3xh + h^2)]$$
$$= 64 - 8.9(2x + 0) + 0.95[3x^2 + 3x(0) + (0)^2]$$
$$= 64 - 17.8x + 2.85x^2$$

For the year 2005, $x = 2005 - 1982 = 23$. Hence, the rate at which the average annual salary is changing in 2005 is

$P'(23) = 64 - 17.8(23) + 2.85(23)^2 =$

\$1 162 250/years since 1982

17. $s(t) = 3t^2$

a. The distance travelled from 0 s to 5 s is

$s(5) = 3(5)^2 = 75$ m

b. $s(10) = 3(10)^2 = 300$ m

The rate at which the avalanche is moving from 0 s to 10 s is

$$\frac{\Delta s}{\Delta t} = \frac{300 - 0}{10 - 0}$$
$$= 30 \text{ m/s}$$

c. Calculate the instantaneous rate of change when $t = 10$.

$$\lim_{h \to 0} \frac{3(10 + h)^2 - 300}{h}$$
$$= \lim_{h \to 0} \frac{300 + 60h + 3h^2 - 300}{h}$$
$$= \lim_{h \to 0} \frac{60h + 3h^2}{h}$$
$$= \lim_{h \to 0} (60 + 3h)$$
$$= 60$$

At 10 s the avalanche is moving at 60 m/s.

d. Set $s(t) = 600$:

$$3t^2 = 600$$
$$t^2 = 200$$
$$t = \pm 10\sqrt{2}$$

Since $t \geq 0$, $t = 10\sqrt{2} \doteq 14$ s.

18. The coordinates of the point are $\left(a, \frac{1}{a}\right)$. The slope of the tangent is $-\frac{1}{a^2}$. The equation of the tangent is $y - \frac{1}{a} = -\frac{1}{a^2}(x - a)$ or $y = -\frac{1}{a^2}x + \frac{2}{a}$. The intercepts are $\left(0, \frac{2}{a}\right)$ and $(-2a, 0)$. The tangent line and the axes form a right triangle with legs of length $\frac{2}{a}$ and $2a$. The area of the triangle is $\frac{1}{2}\left(\frac{2}{a}\right)(2a) = 2$.

19. $C(x) = F + V(x)$
$C(x + h) = F + V(x + h)$
Rate of change of cost is

$$\lim_{x \to R} \frac{C(x + h) - C(x)}{h}$$
$$= \lim_{x \to h} \frac{V(x + h) - V(x)}{h} h.$$

which is independent of F (fixed costs).

20. $A(r) = \pi r^2$
Rate of change of area is

$$\lim_{h \to 0} \frac{A(r + h) - A(r)}{h}$$
$$= \lim_{h \to 0} \frac{\pi(r + h)^2 - \pi r^2}{h}$$
$$= \pi \lim_{h \to 0} \frac{(r + h - r)(r + h + r)}{h}$$
$$= 2\pi r$$

$r = 100$ m
Rate is 200π m^2/m.

21. Cube of dimensions x by x by x has volume $V = x^3$. Surface area is $6x^2$.
$V'(x) = 3x^2 = \frac{1}{2}$ surface area.

22. a. The surface area of a sphere is given by $A(r) = 4\pi r^2$.
The question asks for the instantaneous rate of change of the surface when $r = 10$. This is

$$\lim_{h \to 0} \frac{A(10 + h) - A(10)}{h}$$
$$= \lim_{h \to 0} \frac{4\pi(10 + h)^2 - 4\pi(10)^2}{h}$$
$$= \lim_{h \to 0} \frac{4\pi(100 + 20h + h^2) - 4\pi(100)}{h}$$
$$= \lim_{h \to 0} \frac{400\pi + 80\pi h + 4\pi h^2 - 400\pi}{h}$$
$$= \lim_{h \to 0} \frac{80\pi h + 4\pi h^2}{h}$$
$$= \lim_{h \to 0} (80\pi + 4\pi h)$$
$$= 80\pi + 4\pi(0)$$
$$= 80\pi$$

Therefore, the instantaneous rate of change of the surface area of a spherical balloon as it is inflated when the radius reaches 10 cm is 80π cm^2/unit of time.

b. The volume of a sphere is given by $V(r) = \frac{4}{3}\pi r^3$.
The question asks for the instantaneous rate of change of the volume when $r = 5$.
Note that the volume is deflating. So, find the rate of the change of the volume when $r = 5$ and then make the answer negative to symbolize a deflating spherical balloon.

$$\lim_{h \to 0} \frac{V(5 + h) - V(5)}{h}$$
$$= \lim_{h \to 0} \frac{\frac{4}{3}\pi(5 + h)^3 - \frac{4}{3}\pi(5)^3}{h}$$

Using the binomial formula to expand $(5 + h)^3$ (or one could simply expand using algebra), the limit is

$$= \lim_{h \to 0} \frac{\frac{4}{3}\pi(h^3 + 15h^2 + 75h + 125) - \frac{4}{3}\pi(5)^3}{h}$$
$$= \lim_{h \to 0} \frac{\frac{4}{3}\pi h^3 + 20\pi h^2 + 100\pi h + \frac{4}{3}\pi(125)}{h}$$
$$\frac{- \frac{4}{3}\pi(125)}{h}$$
$$= \lim_{h \to 0} \frac{\frac{4}{3}\pi h^3 + 20\pi h^2 + 100\pi h}{h}$$
$$= \lim_{h \to 0} \left(\frac{4}{3}\pi h^2 + 20\pi h + 100\pi\right)$$
$$= \frac{4}{3}\pi(0)^2 + 20\pi(0) + 100\pi$$
$$= 100\pi$$

Because the balloon is deflating, the instantaneous rate of change of the volume of the spherical balloon when the radius reaches 5 cm is -100π cm^3/unit of time.

Mid-Chapter Review pp. 32–33

1. a. Corresponding conjugate: $\sqrt{5} + \sqrt{2}$.
$(\sqrt{5} - \sqrt{2})(\sqrt{5} + \sqrt{2})$
$= (\sqrt{25} + \sqrt{10} - \sqrt{10} - \sqrt{4})$
$= 5 - 2$
$= 3$

b. Corresponding conjugate: $3\sqrt{5} - 2\sqrt{2}$.
$(3\sqrt{5} + 2\sqrt{2})(3\sqrt{5} - 2\sqrt{2})$
$= (9\sqrt{25} - 6\sqrt{10} + 6\sqrt{10} - 4\sqrt{4})$
$= 9(5) - 4(2)$
$= 45 - 8$
$= 37$

c. Corresponding conjugate: $9 - 2\sqrt{5}$.

$(9 + 2\sqrt{5})(9 - 2\sqrt{5})$

$= (81 - 18\sqrt{5} + 18\sqrt{5} - 4\sqrt{25})$

$= 81 - 4(5)$

$= 81 - 20$

$= 61$

d. Corresponding conjugate: $3\sqrt{5} + 2\sqrt{10}$.

$(3\sqrt{5} - 2\sqrt{10})(3\sqrt{5} + 2\sqrt{10})$

$= (9\sqrt{25} + 6\sqrt{50} - 6\sqrt{50} - 4\sqrt{100})$

$= 9(5) - 4(10)$

$= 45 - 40$

$= 5$

2. a. $\frac{6 + \sqrt{2}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$

$= \frac{6\sqrt{3} + \sqrt{6}}{\sqrt{9}}$

$= \frac{6\sqrt{3} + \sqrt{6}}{3}$

b. $\frac{2\sqrt{3} + 4}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$

$= \frac{2\sqrt{9} + 4\sqrt{3}}{\sqrt{9}}$

$= \frac{6 + 4\sqrt{3}}{3}$

c. $\frac{5}{\sqrt{7} - 4} \cdot \frac{\sqrt{7} + 4}{\sqrt{7} + 4}$

$= \frac{5(\sqrt{7} + 4)}{\sqrt{49} + 4\sqrt{7} - 4\sqrt{7} - 16}$

$= \frac{5(\sqrt{7} + 4)}{7 - 16}$

$= -\frac{5(\sqrt{7} + 4)}{9}$

d. $\frac{2\sqrt{3}}{\sqrt{3} - 2} \cdot \frac{\sqrt{3} + 2}{\sqrt{3} + 2}$

$= \frac{2\sqrt{9} + 4\sqrt{3}}{\sqrt{9} + 2\sqrt{3} - 2\sqrt{3} - 4}$

$= \frac{6 + 4\sqrt{3}}{3 - 4}$

$= \frac{6 + 4\sqrt{3}}{-1}$

$= -2(3 + 2\sqrt{3})$

e. $\frac{5\sqrt{3}}{2\sqrt{3} + 4} \cdot \frac{2\sqrt{3} - 4}{2\sqrt{3} - 4}$

$= \frac{10\sqrt{9} - 20\sqrt{3}}{4\sqrt{9} - 8\sqrt{3} + 8\sqrt{3} - 16}$

$= \frac{30 - 20\sqrt{3}}{12 - 16}$

$= \frac{30 - 20\sqrt{3}}{-4}$

$= \frac{10\sqrt{3} - 15}{2}$

f. $\frac{3\sqrt{2}}{2\sqrt{3} - 5} \cdot \frac{2\sqrt{3} + 5}{2\sqrt{3} + 5}$

$= \frac{3\sqrt{2}(2\sqrt{3} + 5)}{4\sqrt{9} + 10\sqrt{3} - 10\sqrt{3} - 25}$

$= \frac{3\sqrt{2}(2\sqrt{3} + 5)}{4(3) - 25}$

$= \frac{3\sqrt{2}(2\sqrt{3} + 5)}{12 - 25}$

$= \frac{3\sqrt{2}(2\sqrt{3} + 5)}{-13}$

$= -\frac{3\sqrt{2}(2\sqrt{3} + 5)}{13}$

3. a. $\frac{\sqrt{2}}{5} \cdot \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{\sqrt{4}}{5\sqrt{2}}$

$= \frac{2}{5\sqrt{2}}$

b. $\frac{\sqrt{3}}{6 + \sqrt{2}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$

$= \frac{\sqrt{9}}{\sqrt{3}(6 + \sqrt{2})}$

$= \frac{3}{\sqrt{3}(6 + \sqrt{2})}$

c. $\frac{\sqrt{7} - 4}{5} \cdot \frac{\sqrt{7} + 4}{\sqrt{7} + 4}$

$= \frac{\sqrt{49} + 4\sqrt{7} - 4\sqrt{7} - 16}{5(\sqrt{7} + 4)}$

$= \frac{7 - 16}{5(\sqrt{7} + 4)}$

$= -\frac{9}{5(\sqrt{7} + 4)}$

d. $\frac{2\sqrt{3} - 5}{3\sqrt{2}} \cdot \frac{2\sqrt{3} + 5}{2\sqrt{3} + 5}$

$= \frac{4\sqrt{9} + 10\sqrt{3} - 10\sqrt{3} - 25}{3\sqrt{2}(2\sqrt{3} + 5)}$

$= \frac{4(3) - 25}{3\sqrt{2}(2\sqrt{3} + 5)}$

$= \frac{12 - 25}{3\sqrt{2}(2\sqrt{3} + 5)} = -\frac{13}{3\sqrt{2}(2\sqrt{3} + 5)}$

e. $\dfrac{\sqrt{3}-\sqrt{7}}{4}\cdot\dfrac{\sqrt{3}+\sqrt{7}}{\sqrt{3}+\sqrt{7}}$

$= \dfrac{\sqrt{9}+\sqrt{21}-\sqrt{21}-\sqrt{49}}{4(\sqrt{3}+\sqrt{7})}$

$= \dfrac{3-7}{4(\sqrt{3}+\sqrt{7})}$

$= -\dfrac{4}{4(\sqrt{3}+\sqrt{7})}$

$= -\dfrac{1}{(\sqrt{3}+\sqrt{7})}$

f. $\dfrac{2\sqrt{3}+\sqrt{7}}{5}\cdot\dfrac{2\sqrt{3}-\sqrt{7}}{2\sqrt{3}-\sqrt{7}}$

$= \dfrac{4\sqrt{9}-2\sqrt{21}+2\sqrt{21}-\sqrt{49}}{5(2\sqrt{3}-\sqrt{7})}$

$= \dfrac{4(3)-7}{5(2\sqrt{3}-\sqrt{7})}$

$= \dfrac{12-7}{5(2\sqrt{3}-\sqrt{7})}$

$= \dfrac{1}{(2\sqrt{3}-\sqrt{7})}$

4. a. $m = -\dfrac{2}{3};$

$y - 6 = -\dfrac{2}{3}(x - 0)$

$y - 6 = -\dfrac{2}{3}x$

$\dfrac{2}{3}x + y - 6 = 0$

b. $m = \dfrac{11-7}{6-2} = \dfrac{4}{4} = 1$

$y - 7 = 1(x - 2)$

$y - 7 = x - 2$

$-x + y - 5 = 0$

$x - y + 5 = 0$

c. $m = 4$

$y - 6 = 4(x - 2)$

$y - 6 = 4x - 8$

$-4x + y + 2 = 0$

$4x - y - 2 = 0$

d. $m = \dfrac{1}{5}$

$y - (-2) = \dfrac{1}{5}(x - (-1))$

$y + 2 = \dfrac{1}{5}x + \dfrac{1}{5}$

$-\dfrac{1}{5}x + y + \dfrac{10}{5} - \dfrac{1}{5} = 0$

$-\dfrac{1}{5}x + y + \dfrac{9}{5} = 0$

$\dfrac{1}{5}x - y - \dfrac{9}{5} = 0$

$x - 5y - 9 = 0$

5. The slope of PQ is

$m = \lim_{h\to 0}\dfrac{f(1+h)-(-1)}{(1+h)-1}$

$= \lim_{h\to 0}\dfrac{-(1+h)^2+1}{h}$

$= \lim_{h\to 0}\dfrac{-(1+2h+h^2)+1}{h}$

$= \lim_{h\to 0}\dfrac{-1-2h-h^2+1}{h}$

$= \lim_{h\to 0}\dfrac{-2h-h^2}{h}$

$= \lim_{h\to 0}(-2-h)$

$= -2-(0)$

$= -2$

So, the slope of PQ with $f(x) = -x^2$ is -2.

6. a. Unlisted y-coordinates for Q are found by substituting the x-coordinates into the given function. The slope of the line PQ with the given points is given by the following: Let $P = (x_1, y_1)$ and $Q = (y_1, y_2)$. Then, the slope $= m = \dfrac{y_2 - y_1}{x_2 - x_1}$.

P	Q	Slope of Line PQ
(−1, 1)	(−2, 6)	−5
(−1, 1)	(−1.5, 3.25)	−4.5
(−1, 1)	(−1.1, 1.41)	−4.1
(−1, 1)	(−1.01, 1.040 1)	−4.01
(−1, 1)	(−1.001, 1.004 001)	−4.001

P	Q	Slope of Line PQ
(−1, 1)	(0, −2)	−3
(−1, 1)	(−0.5, −0.75)	−3.5
(−1, 1)	(−0.9, 0.61)	−3.9
(−1, 1)	(−0.99, 0.9601)	−3.99
(−1, 1)	(−0.999, 0.996 001)	−3.999

b. The slope from the right and from the left appear to approach -4. The slope of the tangent to the graph of $f(x)$ at point P is about -4.

c. With the points $P = (-1, 1)$ and $Q = (-1+h, f(-1+h))$, the slope, m, of PQ is the following:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{[(-1+h)^2 - 2(-1+h) - 2] - (1)}{(-1+h) - (-1)}$$
$$= \frac{1 - 2h + h^2 + 2 - 2h - 2 - 1}{-1 + h + 1}$$
$$= \frac{h^2 - 4h}{h}$$
$$= h - 4$$

d. The slope of the tangent is $\lim_{h\to 0} f(x)$.

In this case, as h goes to zero, $h - 4$ goes to $h - 4 = 0 - 4 = -4$. The slope of the tangent to the graph of $f(x)$ at the point P is -4.

e. The answers are equal.

7. a. $m = \lim_{h\to 0} \frac{f(-3+h) - f(-3)}{h}$
$$= \lim_{h\to 0} \frac{[(-3+h)^2 + 3(-3+h) - 5] - [(-3)^2 + 3(-3) - 5]}{h}$$
$$= \lim_{h\to 0} \frac{9 - 6h + h^2 - 9 + 3h - 5 - (9 - 9 - 5)}{h}$$
$$= \lim_{h\to 0} \frac{h^2 - 3h - 5 - (-5)}{h}$$
$$= \lim_{h\to 0} \frac{h^2 - 3h}{h}$$
$$= \lim_{h\to 0} (h - 3)$$
$$= 0 - 3$$
$$= -3$$

b. $y = f(x) = \frac{1}{x}$
$$m = \lim_{h\to 0} \frac{f(\frac{1}{3} + h) - f(\frac{1}{3})}{h}$$
$$= \lim_{h\to 0} \frac{\frac{1}{\frac{1}{3} + h} - \frac{1}{\frac{1}{3}}}{h}$$
$$= \lim_{h\to 0} \frac{\frac{(\frac{1}{3}) - (\frac{1}{3} + h)}{\frac{1}{3}(\frac{1}{3} + h)}}{h}$$
$$= \lim_{h\to 0} \left(\frac{-h}{\frac{1}{9} + \frac{1}{3}h}\right)\frac{1}{h}$$
$$= \lim_{h\to 0} \frac{-1}{\frac{1}{9} + \frac{1}{3}h}$$
$$= \frac{-1}{\frac{1}{9} + \frac{1}{3}(0)}$$
$$= -9$$

c. $y = f(x) = \frac{4}{x - 2}$
$$m = \lim_{h\to 0} \frac{f(6+h) - f(6)}{h}$$
$$= \lim_{h\to 0} \frac{\frac{4}{6+h-2} - \frac{4}{6-2}}{h}$$
$$= \lim_{h\to 0} \frac{\frac{4}{h+4} - \frac{4}{4}}{h}$$
$$= \lim_{h\to 0} \frac{\frac{4}{h+4} - 1}{h}$$
$$= \lim_{h\to 0} \left(\frac{4 - (h+4)}{h+4}\right)\frac{1}{h}$$
$$= \lim_{h\to 0} \left(\frac{-h}{h+4}\right)\frac{1}{h}$$
$$= \lim_{h\to 0} \frac{-1}{h+4}$$
$$= \frac{-1}{0+4}$$
$$= -\frac{1}{4}$$

d. $m = \lim_{h\to 0} \frac{f(5+h) - f(5)}{h}$
$$= \lim_{h\to 0} \frac{\sqrt{5+h+4} - \sqrt{5+4}}{h}$$
$$= \lim_{h\to 0} \frac{\sqrt{9+h} - \sqrt{9}}{h}$$
$$= \lim_{h\to 0} \frac{\sqrt{9+h} - 3}{h}$$
$$= \lim_{h\to 0} \frac{\sqrt{9+h} - 3}{h} \cdot \frac{\sqrt{9+h} + 3}{\sqrt{9+h} + 3}$$
$$= \lim_{h\to 0} \frac{9 + h + 3\sqrt{9+h} - 3\sqrt{9+h} - 9}{h(\sqrt{9+h} + 3)}$$
$$= \lim_{h\to 0} \frac{h}{h(\sqrt{9+h} + 3)}$$
$$= \lim_{h\to 0} \frac{1}{\sqrt{9+h} + 3}$$

$= \frac{1}{\sqrt{9+0}+3}$

$= \frac{1}{6}$

8. $s(t) = 6t(t+1) = 6t^2 + 6t$

a. i. average velocity $= \frac{s(3) - s(2)}{3 - 2}$

$= [6(3)^2 + 6(3)] - [6(2)^2 + 6(2)]$

$= 6(9) + 18 - (24 + 12)$

$= 54 + 18 - 36$

$= 36$ km/h

ii. average velocity $= \frac{s(2.1) - s(2)}{2.1 - 2}$

$= \frac{[6(2.1)^2 + 6(2.1)] - [6(2)^2 + 6(2)]}{0.1}$

$= \frac{[26.46 + 12.6] - [24 + 12]}{0.1}$

$= \frac{39.06 - 36}{0.1}$

$= \frac{3.06}{0.1}$

$= 30.6$ km/h

iii. average velocity $= \frac{s(2.01) - s(2)}{2.01 - 2}$

$= \frac{[6(2.01)^2 + 6(2.01)] - [6(2)^2 + 6(2)]}{0.01}$

$= \frac{[24.2406 + 12.06] - [6(2)^2 + 6(2)]}{0.01}$

$= \frac{36.3006 - [24 + 12]}{0.01}$

$= \frac{36.3006 - 36}{0.01}$

$= \frac{0.3006}{0.01}$

$= 30.06$ km/h

b. At the time $t = 2$, the velocity of the car appears to approach 30 km/h.

c. average velocity $= \frac{f(2+h) - f(2)}{(2+h) - (2)}$

$= \frac{[6(2+h)^2 + 6(2+h)] - [6(2)^2 + 6(2)]}{h}$

$= \frac{[6(4 + 4h + h^2) + 12 + 6h] - [24 + 12]}{h}$

$= \frac{[24 + 24h + 6h^2 + 12 + 6h] - 36}{h}$

$= \frac{6h^2 + 30h + 36 - 36}{h}$

$= \frac{6h^2 + 30h}{h}$

$= (6h + 30)$ km/h

d. When $t = 2$, the velocity is the limit as h approaches 0.

velocity $= \lim_{h \to 0} (6h + 30)$

$= 6(0) + 30$

$= 30$

Therefore, when $t = 2$ the velocity is 30 km/h.

9. a. The instantaneous rate of change of $f(x)$ with respect to x at $x = 2$ is given by

$\lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$

$= \lim_{h \to 0} \frac{[5 - (2+h)^2] - [5 - (2)^2]}{h}$

$= \lim_{h \to 0} \frac{5 - (4 + 4h + h^2) - 1}{h}$

$= \lim_{h \to 0} \frac{5 - 4 - 4h - h^2 - 1}{h}$

$= \lim_{h \to 0} \frac{-h^2 - 4h}{h}$

$= \lim_{h \to 0} (-h - 4)$

$= -(0) - 4$

$= -4$

b. The instantaneous rate of change of $f(x)$ with respect to x at $x = \frac{1}{2}$ is given by

$\lim_{h \to 0} \frac{f(\frac{1}{2} + h) - f(\frac{1}{2})}{h}$

$= \lim_{h \to 0} \frac{\frac{3}{\frac{1}{2} + h} - \frac{3}{\frac{1}{2}}}{h}$

$= \lim_{h \to 0} \frac{\frac{3}{\frac{1}{2} + h} - 6}{h}$

$= \lim_{h \to 0} \frac{3 - 6(\frac{1}{2} + h)}{\frac{1}{2} + h} \cdot \frac{1}{h}$

$= \lim_{h \to 0} \frac{3 - 3 - 6h}{\frac{1}{2} + h} \cdot \frac{1}{h}$

$= \lim_{h \to 0} \frac{-6h}{\frac{1}{2} + h} \cdot \frac{1}{h}$

$= \lim_{h \to 0} \frac{-6}{\frac{1}{2} + h}$

$= \frac{-6}{\frac{1}{2} + 0}$

$= -12$

10. a. The average rate of change of $V(t)$ with respect to t during the first 20 minutes is given by

$$\frac{f(20) - f(0)}{20 - 0}$$
$$= \frac{[50(30 - 20)^2] - [50(30 - 0)^2]}{20}$$
$$= \frac{5000 - 45\,000}{20}$$
$$= -\frac{40\,000}{20}$$
$$= -2000 \text{ L/min}$$

b. The rate of change of $V(t)$ with respect to t at the time $t = 20$ is given by

$$\lim_{h\to 0} \frac{f(20 + h) - f(20)}{h}$$
$$= \lim_{h\to 0} \frac{[50(30 - (20 + h))^2] - [50(30 - 20)^2]}{h}$$
$$= \lim_{h\to 0} \frac{[50(10 - h)^2] - [50(10)^2]}{h}$$
$$= \lim_{h\to 0} \frac{[50(100 - 20h + h^2)] - [50(100)]}{h}$$
$$= \lim_{h\to 0} \frac{5000 - 1000h + 50h^2 - 5000}{h}$$
$$= \lim_{h\to 0} \frac{50h^2 - 1000h}{h}$$
$$= \lim_{h\to 0} 50h - 1000$$
$$= 50(0) - 1000$$
$$= -1000 \text{ L/min}$$

11. a. Let $y = f(x)$.

$$f(4) = (4)^2 + (4) - 3 = 16 + 1 = 17$$
$$f(4 + h) = (4 + h)^2 + (4 + h) - 3$$
$$= 16 + 8h + h^2 + h + 1$$
$$= h^2 + 9h + 17$$

Using the limit of the difference quotient, the slope of the tangent at $x = 4$ is

$$m = \lim_{h\to 0} \frac{f(4 + h) - f(4)}{h}$$
$$= \lim_{h\to 0} \frac{h^2 + 9h + 17 - (17)}{h}$$
$$= \lim_{h\to 0} \frac{h^2 + 9h}{h}$$
$$= \lim_{h\to 0} (h + 9)$$
$$= 0 + 9$$
$$= 9$$

Therefore, the slope of the tangent to $y = f(x) = x^2 + x - 3$ at $x = 4$ is 9.

So an equation of the tangent at $x = 4$ is given by

$$y - 17 = 9(x - 4)$$
$$y - 17 = 9x - 36$$
$$-9x + y - 17 + 36 = 0$$
$$-9x + y + 19 = 0$$

b. Let $y = f(x)$.

$$f(-2) = 2(-2)^2 - 7 = 2(4) - 7 = 1$$
$$f(-2 + h) = 2(-2 + h)^2 - 7$$
$$= 2(4 - 4h + h^2) - 7$$
$$= 8 - 8h + 2h^2 - 7$$
$$= 2h^2 - 8h + 1$$

Using the limit of the difference quotient, the slope of the tangent at $x = 4$ is

$$m = \lim_{h\to 0} \frac{f(-2 + h) - f(-2)}{h}$$
$$= \lim_{h\to 0} \frac{2h^2 - 8h + 1 - (1)}{h}$$
$$= \lim_{h\to 0} \frac{2h^2 - 8h}{h}$$
$$= \lim_{h\to 0} (2h - 8)$$
$$= 2(0) - 8$$
$$= -8$$

Therefore, the slope of the tangent to $y = f(x) = 2x^2 - 7$ at $x = -2$ is -8.
So an equation of the tangent at $x = -2$ is given by

$$y - 1 = -8(x - (-2))$$
$$y - 1 = -8x - 16$$
$$8x + y - 1 + 16 = 0$$
$$8x + y + 15 = 0$$

c. $f(-1) = 3(-1)^2 + 2(-1) - 5 = 3 - 2 - 5$
$= -4$

$$f(-1 + h) = 3(-1 + h)^2 + 2(-1 + h) - 5$$
$$= 3(1 - 2h + h^2) - 2 + 2h - 5$$
$$= 3 - 6h + 3h^2 - 7 + 2h$$
$$= 3h^2 - 4h - 4$$

Using the limit of the difference quotient, the slope of the tangent at $x = 4$ is

$$m = \lim_{h\to 0} \frac{f(-1 + h) - f(-1)}{h}$$
$$= \lim_{h\to 0} \frac{3h^2 - 4h - 4 - (-4)}{h}$$
$$= \lim_{h\to 0} \frac{3h^2 - 4h}{h}$$
$$= \lim_{h\to 0} (3h - 4)$$
$$= 3(0) - 4$$
$$= -4$$

Therefore, the slope of the tangent to $y = f(x) = 3x^2 + 2x - 5$ at $x = -1$ is -4.
So an equation of the tangent at $x = -4$ is given by

$$y - (-4) = -4(x - (-1))$$
$$y + 4 = -4(x + 1)$$
$$y + 4 = -4x - 4$$
$$4x + y + 4 + 4 = 0$$
$$4x + y + 8 = 0$$

d. $f(1) = 5(1)^2 - 8(1) + 3 = 5 - 8 + 3 = 0$

$$f(1 + h) = 5(1 + h)^2 - 8(1 + h) + 3$$
$$= 5(1 + 2h + h^2) - 8 - 8h + 3$$
$$= 5 + 10h + 5h^2 - 5 - 8h$$
$$= 5h^2 + 2h$$

Using the limit of the difference quotient, the slope of the tangent at $x = 1$ is

$$m = \lim_{h\to 0} \frac{f(1 + h) - f(1)}{h}$$
$$= \lim_{h\to 0} \frac{5h^2 + 2h - (0)}{h}$$
$$= \lim_{h\to 0} (5h + 2)$$
$$= 5(0) + 2$$
$$= 2$$

Therefore, the slope of the tangent to $y = f(x) = 5x^2 - 8x + 3$ at $x = 1$ is 2.
So an equation of the tangent at $x = 1$ is given by

$$y - 0 = 2(x - 1)$$
$$y = 2x - 2$$
$$-2x + y + 2 = 0$$

12. a. Using the limit of the difference quotient, the slope of the tangent at $x = -5$ is

$$m = \lim_{h\to 0} \frac{f(-5 + h) - f(-5)}{h}$$
$$= \lim_{h\to 0} \left(\frac{-5 + h}{-5 + h + 3} - \frac{-5}{-5 + 3}\right) \cdot \frac{1}{h}$$
$$= \lim_{h\to 0} \left(\frac{-5 + h}{-2 + h} - \frac{5}{2}\right) \cdot \frac{1}{h}$$
$$= \lim_{h\to 0} \left(\frac{-10 + 2h - (-10 + 5h)}{-4 + 2h}\right) \cdot \frac{1}{h}$$
$$= \lim_{h\to 0} \left(\frac{-10 + 2h + 10 - 5h}{-4 + 2h}\right) \cdot \frac{1}{h}$$
$$= \lim_{h\to 0} \left(\frac{-3h}{-4 + 2h}\right) \cdot \frac{1}{h}$$
$$= \lim_{h\to 0} \left(\frac{-3}{-4 + 2h}\right)$$
$$= \frac{-3}{-4 + 2(0)}$$
$$= \frac{3}{4}$$

Therefore, the slope of the tangent to $f(x) = \frac{x}{x + 3}$ at $x = -5$ is $\frac{3}{4}$.
So an equation of the tangent at $x = \frac{3}{4}$ is given by

$$y - \frac{5}{2} = \frac{3}{4}(x - (-5))$$
$$y - \frac{5}{2} = \frac{3}{4}x + \frac{15}{4}$$
$$-\frac{3}{4}x + y - \frac{10}{4} - \frac{15}{4} = 0$$
$$-\frac{3}{4}x + y - \frac{25}{4} = 0$$
$$-3x + 4y - 25 = 0$$

b. Using the limit of the difference quotient, the slope of the tangent at $x = -1$ is

$$m = \lim_{h\to 0} \frac{f(-1 + h) - f(-1)}{h}$$
$$= \lim_{h\to 0} \left(\frac{2(-1 + h) + 5}{5(-1 + h) - 1} - \frac{2(-1) + 5}{5(-1) - 1}\right) \cdot \frac{1}{h}$$
$$= \lim_{h\to 0} \left(\frac{-2 + 2h + 5}{-5 + 5h - 1} - \frac{-2 + 5}{-5 - 1}\right) \cdot \frac{1}{h}$$
$$= \lim_{h\to 0} \left(\frac{2h + 3}{5h - 6} - \frac{3}{-6}\right) \cdot \frac{1}{h}$$
$$= \lim_{h\to 0} \left(\frac{2h + 3}{5h - 6} + \frac{1}{2}\right) \cdot \frac{1}{h}$$
$$= \lim_{h\to 0} \left(\frac{4h + 6 + 5h - 6}{10h - 12}\right) \cdot \frac{1}{h}$$
$$= \lim_{h\to 0} \left(\frac{9h}{10h - 12}\right) \cdot \frac{1}{h}$$
$$= \lim_{h\to 0} \left(\frac{9}{10h - 12}\right)$$
$$= \frac{9}{10(0) - 12}$$
$$= -\frac{9}{12}$$
$$= -\frac{3}{4}$$

Therefore, the slope of the tangent to $f(x) = \frac{2x + 5}{5x - 1}$ at $x = -1$ is $-\frac{3}{4}$.
So an equation of the tangent at $x = -\frac{3}{4}$ is given by

$$y - \left(-\frac{1}{2}\right) = -\frac{3}{4}(x - (-1))$$
$$y + \frac{1}{2} = -\frac{3}{4}x - \frac{3}{4}$$
$$4y + 2 = -3x - 3$$
$$3x + 4y + 2 + 3 = 0$$
$$3x + 4y + 5 = 0$$

1.4 The Limit of a Function, pp. 37–39

1. a. $\frac{27}{99}$

b. π

2. One way to find a limit is to evaluate the function for values of the independent variable that get progressively closer to the given value of the independent variable.

3. a. A right-sided limit is the value that a function gets close to as the values of the independent variable decrease and get close to a given value.

b. A left-sided limit is the value that a function gets close to as the values of the independent variable increase and get close to a given value.

c. A (two-sided) limit is the value that a function gets close to as the values of the independent variable get close to a given value, regardless of whether the values increase or decrease toward the given value.

4. a. -5

b. $3 + 7 = 10$

c. $10^2 = 100$

d. $4 - 3(-2)^2 = -8$

e. 4

f. $2^3 = 8$

5. Even though $f(4) = -1$, the limit is 1, since that is the value that the function approaches from the left and the right of $x = 4$.

6. a. 0

b. 2

c. -1

d. 2

7. a. 2

b. 1

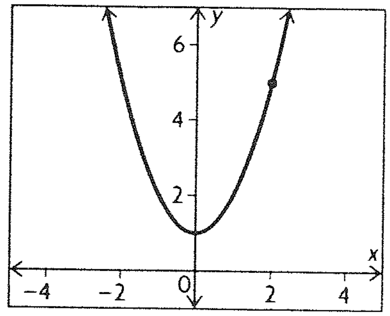
c. does not exist

8. a. $9 - (-1)^2 = 8$

b. $\sqrt{\frac{0 + 20}{0 + 5}} = \sqrt{4}$
$= 2$

c. $\sqrt{5 - 1} = \sqrt{4}$
$= 2$

9. $2^2 + 1 = 5$



10. a. Since 0 is not a value for which the function is undefined, one may substitute 0 in for x to find that

$$\lim_{x \to 0^+} x^4 = \lim_{x \to 0} x^4 = (0)^4 = 0$$

b. Since 2 is not a value for which the function is undefined, one may substitute 2 in for x to find that

$$\lim_{x \to 2^-} (x^2 - 4) = \lim_{x \to 2} (x^2 - 4) = (2)^2 - 4 = 4 - 4 = 0$$

c. Since 3 is not a value for which the function is undefined, one may substitute 3 in for x to find that

$$\lim_{x \to 3^+} (x^2 - 4) = \lim_{x \to 3} (x^2 - 4) = (3)^2 - 4 = 9 - 4 = 5$$

d. Since 1 is not a value for which the function is undefined, one may substitute 1 in for x to find that

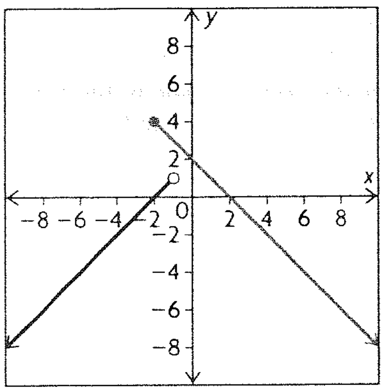
$$\lim_{x \to 1^-} \frac{1}{x - 3} = \lim_{x \to 1} \frac{1}{x - 3} = \frac{1}{1 - 3} = -\frac{1}{2}$$

e. Since 3 is not a value for which the function is undefined, one may substitute 3 in for x to find that

$$\lim_{x \to 3^+} \frac{1}{x + 2} = \lim_{x \to 3} \frac{1}{x + 2} = \frac{1}{3 + 2} = \frac{1}{5}$$

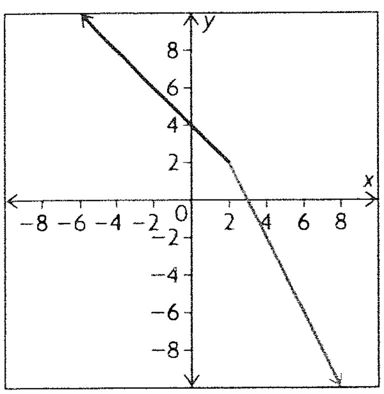
f. If 3 is substituted in the function for x, then the function is undefined because of division by zero. There does not exist a way to divide out the $x - 3$ in the denominator. Also, $\lim_{x\to 3^+} \frac{1}{x-3}$ approaches infinity, while $\lim_{x\to 3^-} \frac{1}{x-3}$ approaches negative infinity. Therefore, since $\lim_{x\to 3^+} \frac{1}{x-3} \neq \lim_{x\to 3^-} \frac{1}{x-3}$, $\lim_{x\to 3} \frac{1}{x-3}$ does not exist.

11. a.



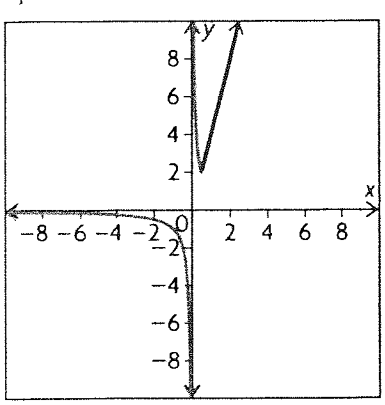
$\lim_{x\to -1^-} f(x) \neq \lim_{x\to -1^+} f(x)$. Therefore, $\lim_{x\to -1} f(x)$ does not exist.

b.



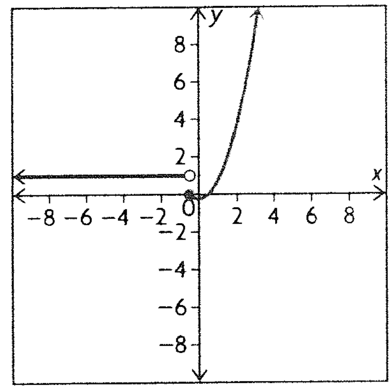
$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$. Therefore, $\lim_{x\to 2} f(x)$ exists and is equal to 2.

c.



$\lim_{x\to \frac{1}{2}^-} f(x) = \lim_{x\to \frac{1}{2}^+} f(x)$. Therefore, $\lim_{x\to \frac{1}{2}} f(x)$ exists and is equal to 2.

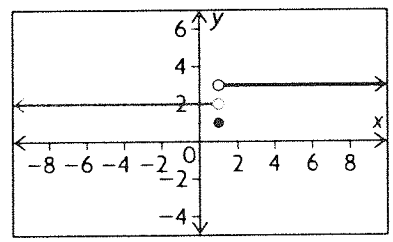
d.



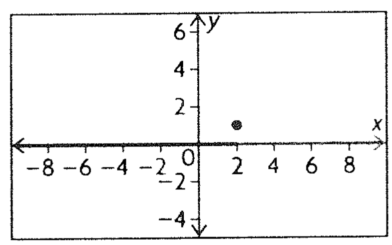
$\lim_{x\to -0.5^-} f(x) \neq \lim_{x\to -0.5^+} f(x)$. Therefore, $\lim_{x\to -0.5} f(x)$ does not exist.

12. Answers may vary. For example:

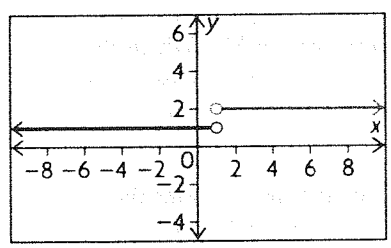
a.



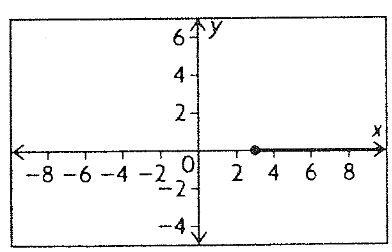
b.



c.



d.



13. $f(x) = mx + b$

$\lim_{x\to 1} f(x) = -2 \quad m + b = -2$

$\lim_{x\to -1} f(x) = 4 \quad -m + b = 4$

$2b = 2$

$b = 1, m = -3$

14. $f(x) = ax^2 + bx + c, a \neq 0$

$f(0) = 0 \qquad c = 0$

$\lim_{x \to 1} f(x) = 5 \qquad a + b = 5$

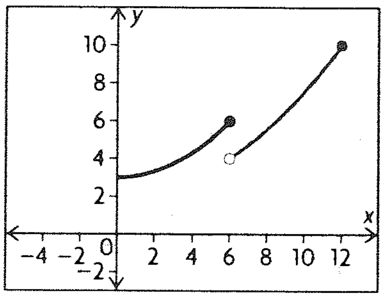
$\lim_{x \to -2} f(x) = 8 \quad 4a - 2b = 8$

$6a = 18$

$a = 3, \qquad b = 2$

Therefore, the values are $a = 3$, $b = 2$, and $c = 0$.

15. a.



b. $\lim_{t \to 6^-} p(t) = 3 + \frac{1}{12}(6)^2$

$= 3 + \frac{36}{12}$

$= 3 + 3$

$= 6$

$\lim_{t \to 6^+} p(t) = 2 + \frac{1}{18}(6)^2$

$= 2 + \frac{36}{18}$

$= 2 + 2$

$= 4$

c. Since $p(t)$ is measured in thousands, right before the chemical spill there were 6000 fish in the lake. Right after the chemical spill there were 4000 fish in the lake. So, $6000 - 4000 = 2000$ fish were killed by the spill.

d. The question asks for the time, t, after the chemical spill when there are once again 6000 fish in the lake. Use the second equation to set up an equation that is modelled by

$6 = 2 + \frac{1}{18}t^2$

$4 = \frac{1}{18}t^2$

$72 = t^2$

$\sqrt{75} = t$

(The question asks for time so the negative answer is disregarded.)

So, at time $t = \sqrt{72} \doteq 8.49$ years the population has recovered to the level before the spill.

1.5 Properties of Limits, pp. 45–47

1. $\lim_{x \to 2}(3 + x)$ and $\lim_{x \to 2}(x + 3)$ have the same value, but $\lim_{x \to 2} 3 + x$ does not. Since there are no brackets around the expression, the limit only applies to 3, and there is no value for the last term, x.

2. Factor the numerator and denominator. Cancel any common factors. Substitute the given value of x.

3. If the two one-sided limits have the same value, then the value of the limit is equal to the value of the one-sided limits. If the one-sided limits do not have the same value, then the limit does not exist.

4. a. $\frac{3(2)}{2^2 + 2} = 1$

b. $(-1)^4 + (-1)^3 + (-1)^2 = 1$

c. $\left[\sqrt{9} + \frac{1}{\sqrt{9}}\right]^2 = \left(3 + \frac{1}{3}\right)^2$

$= \frac{100}{9}$

d. $(2\pi)^3 + \pi^2(2\pi) - 5\pi^3 = 8\pi^3 + 2\pi^3 - 5\pi^3$

$= 5\pi^3$

e. $\sqrt{3 + \sqrt{1 + 0}} = \sqrt{3 + 1}$

$= 2$

f. $\sqrt{\frac{-3 - 3}{2(-3) + 4}} = \sqrt{\frac{-6}{-2}}$

$= \sqrt{3}$

5. a. $\frac{(-2)^3}{-2 - 2} = -2$

b. $\frac{2}{\sqrt{1} + 1} = \frac{2}{\sqrt{2}}$

$= \sqrt{2}$

6. Since substituting $t = 1$ does not make the denominator 0, direct substitution works.

$\frac{1 - 1 - 5}{6 - 1} = \frac{-5}{5}$

$= -1$

7. a. $\lim_{x \to 2} \frac{4 - x^2}{2 - x} = \lim_{x \to 2} \frac{(2 - x)(2 + x)}{(2 - x)}$

$= \lim_{x \to 2}(2 + x)$

$= 4$

b. $\lim_{x \to -1} \frac{2x^2 + 5x + 3}{x + 1} = \lim_{x \to -1} \frac{(x + 1)(2x + 3)}{x + 1}$

$= 5$

c. $\lim_{x \to 3} \frac{x^3 - 27}{x - 3} = \lim_{x \to 3} \frac{(x - 3)(x^2 + 3x + 9)}{x - 3}$

$= 9 + 9 + 9$

$= 27$

d. $\lim\limits_{x\to 0}\left[\frac{2-\sqrt{4+x}}{x}\times\frac{2+\sqrt{4+x}}{2+\sqrt{4+x}}\right]$

$= \lim\limits_{x\to 0}\frac{-1}{2+\sqrt{4+x}}$

$= -\frac{1}{4}$

e. $\lim\limits_{x\to 4}\frac{\sqrt{x}-2}{x-4} = \lim\limits_{x\to 4}\frac{\sqrt{x}-2}{(\sqrt{x}-2)(\sqrt{x}+2)}$

$= \frac{1}{4}$

f. $\lim\limits_{x\to 0}\left[\frac{\sqrt{7-x}-\sqrt{7+x}}{x}\times\frac{\sqrt{7-x}+\sqrt{7+x}}{\sqrt{7-x}+\sqrt{7+x}}\right]$

$= \lim\limits_{x\to 0}\frac{7-x-7-x}{x(\sqrt{7-x}+\sqrt{7+x})}$

$= -\frac{1}{\sqrt{7}}$

8. a. $\lim\limits_{x\to 8}\frac{\sqrt[3]{x}-2}{x-8}$

Let $u=\sqrt[3]{x}$. Therefore, $u^3 = x$ as $x\to 8$, $u\to 2$.

Here, $\lim\limits_{x\to 2}\frac{u-2}{u^3-8} = \lim\limits_{x\to 2}\frac{1}{u^2+2u+4}$

$= \frac{1}{12}$

b. $\lim\limits_{x\to 27}\frac{27-x}{x^{\frac{1}{3}}-3}$

Let $x^{\frac{1}{3}} = u$
$x = u^3$
$x\to 27, u\to 3.$

$= \lim\limits_{x\to 3}\frac{u^3-27}{u-3}$

$= -\lim\limits_{x\to 3}\frac{(u-3)(u^2+3u+9)}{u-3}$

$= -(9+9+9)$

$= -27$

c. $\lim\limits_{x\to 1}\frac{x^{\frac{1}{6}}-1}{x-1}$

$x^{\frac{1}{6}} = u, x = u^6$
$x\to 1, u\to 1$

$= \lim\limits_{x\to 1}\frac{u-1}{u^6-1}$

$= \lim\limits_{x\to 1}\frac{(u-1)}{(u-1)(u^5+u^4+u^3+u^2+u+1)}$

$= \frac{1}{6}$

d. $\lim\limits_{x\to 1}\frac{x^{\frac{1}{6}}-1}{x^{\frac{1}{3}}-1}$

Let $x^{\frac{1}{6}} = u$
$u^6 = x$
$x^{\frac{1}{3}} = u^2$
As $x\to 1, u\to 1$

$= \lim\limits_{x\to 1}\frac{u-1}{u^2-1}$

$= \lim\limits_{x\to 1}\frac{u-1}{(u-1)(u+1)}$

$= \frac{1}{2}$

e. $\lim\limits_{x\to 4}\frac{\sqrt{x}-2}{\sqrt{x^3}-8}$

Let $x^{\frac{1}{2}} = u$
$x^{\frac{3}{2}} = u^3$
$x\to 4, u\to 2$

$= \lim\limits_{x\to 2}\frac{u-2}{u^3-8}$

$= \lim\limits_{x\to 2}\frac{u-2}{(u-2)(u^2+2u+4)}$

$= \frac{1}{12}$

f. $\lim\limits_{x\to 0}\frac{(x+8)^{\frac{1}{3}}-2}{x}$

Let $(x+8)^{\frac{1}{3}} = u$
$x+8 = u^3$
$x = u^3-8$
$x\to 0, u\to 2$

$\lim\limits_{x\to 2}\frac{u-2}{u^3-8}$

$= \frac{1}{12}$

9. a. $\frac{16-16}{64+64} = 0$

b. $\frac{16-16}{16-20+6} = 0$

c. $\lim\limits_{x\to -1}\frac{x^2+x}{x+1} = \lim\limits_{x\to -1}\frac{x(x+1)}{x+1}$

$= -1$

d. $\lim\limits_{x\to 0}\frac{\sqrt{x+1}-1}{x} = \lim\limits_{x\to 0}\frac{\sqrt{x+1}-1}{x+1-1}$

$= \lim\limits_{x\to 0}\frac{\sqrt{x+1}-1}{(\sqrt{x+1}-1)(\sqrt{x+1}+1)}$

$= \frac{1}{2}$

e. $\lim\limits_{h\to 0}\frac{(x+h)^2-x^2}{h} = \lim\limits_{h\to 0}\frac{2xh+h^2}{h}$

$= 2x$

f. $\lim\limits_{x\to 1}\left(\frac{1}{x-1}\right)\left(\frac{1}{x+3}-\frac{2}{3x+5}\right)$

$= \lim\limits_{x\to 1}\left(\frac{1}{x-1}\right)\left(\frac{3x+5-2x-6}{(x+3)(3x+5)}\right)$

$= \lim\limits_{x\to 1}\frac{1}{(x+3)(3x+5)}$

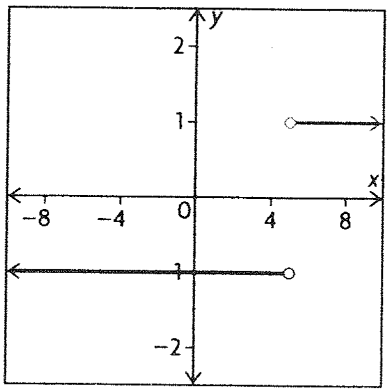
$= \frac{1}{4(8)}$

$= \frac{1}{32}$

10. a. $\lim\limits_{x \to 5} \dfrac{|x-5|}{x-5}$ does not exist.

$$\lim_{x \to 5^+} \frac{|x-5|}{x-5} = \lim_{x \to 5^+} \frac{x-5}{x-5} = 1$$

$$\lim_{x \to 5^-} \frac{|x-5|}{x-5} = \lim_{x \to 5^-} -\left(\frac{x-5}{x-5}\right) = -1$$



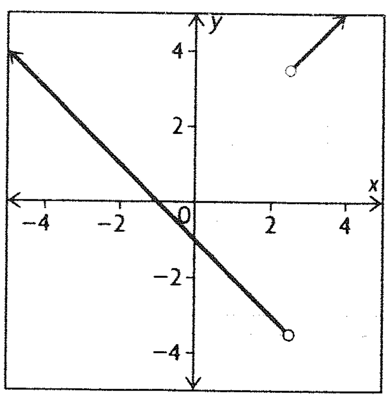
b. $\lim\limits_{x \to \frac{5}{2}} \dfrac{|2x-5|(x+1)}{2x-5}$ does not exist.

$$|2x-5| = 2x-5,\ x \geq \frac{5}{2}$$

$$\lim_{x \to \frac{5}{2}^+} \frac{(2x-5)(x+1)}{2x-5} = x+1$$

$$|2x-5| = -(2x-5),\ x < \frac{5}{2}$$

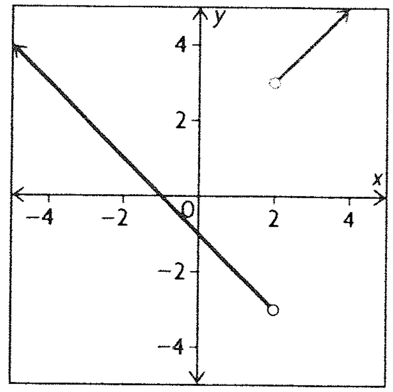
$$\lim_{x \to \frac{5}{2}^-} \frac{-(2x-5)(x+1)}{2x-5} = -(x+1)$$



c. $\lim\limits_{x \to 2} \dfrac{x^2-x-2}{|x-2|} = \lim\limits_{x \to 2} \dfrac{(x-2)(x+1)}{|x-2|}$

$$\lim_{x \to 2^+} \frac{(x-2)(x+1)}{|x-2|} = \lim_{x \to 2^+} \frac{(x-2)(x+1)}{x-2} = \lim_{x \to 2^+} x+1 = 3$$

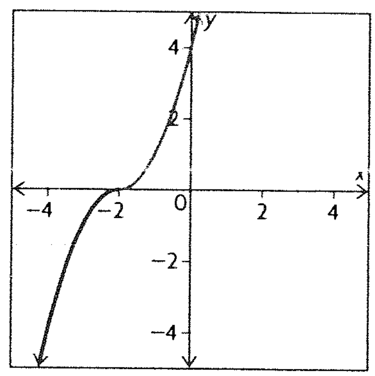
$$\lim_{x \to 2^-} \frac{(x-2)(x+1)}{|x-2|} = \lim_{x \to 2^-} -\frac{(x-2)(x+1)}{(x-2)} = \lim_{x \to 2^-} -(x+1) = -3$$



d. $|x+2| = x+2$ if $x > -2$
$= -(x+2)$ if $x < -2$

$$\lim_{x \to -2^+} \frac{(x+2)(x+2)^2}{x+2} = \lim_{x \to -2^+} (x+2)^2 = 0$$

$$\lim_{x \to -2^-} \frac{(x+2)(x+2)^2}{-(x+2)} = 0$$



11. a.

ΔT	T	V	ΔV
	−40	19.1482	
20			1.6426
	−20	20.7908	
20			1.6426
	0	22.4334	
20			1.6426
	20	24.0760	
20			1.6426
	40	25.7186	
20			1.6426
	60	27.3612	
20			1.6426
	80	29.0038	

ΔV is constant, therefore T and V form a linear relationship.

b. $V = \dfrac{\Delta V}{\Delta T} \cdot T + K$

$$\frac{\Delta V}{\Delta T} = \frac{1.6426}{20} = 0.082\,13$$

$V = 0.082\,13T + K$

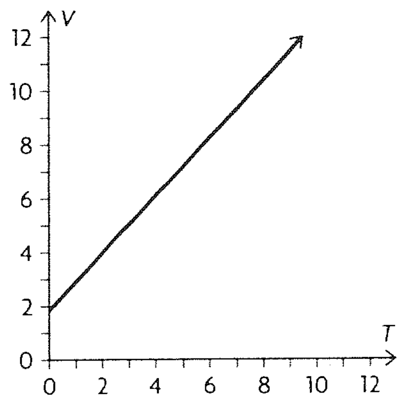
$T = 0 \qquad V = 22.4334$

Therefore, $k = 22.4334$ and $V = 0.082\,13T + 22.4334$.

c. $T = \dfrac{V - 22.4334}{0.082\,13}$

d. $\lim\limits_{v \to 0} T = -273.145$

e.



12. $\lim\limits_{x \to 5} \dfrac{x^2 - 4}{f(x)}$

$= \dfrac{\lim\limits_{x \to 5}(x^2 - 4)}{\lim\limits_{x \to 5} f(x)}$

$= \dfrac{21}{3}$

$= 7$

13. $\lim\limits_{x \to 4} f(x) = 3$

a. $\lim\limits_{x \to 4} [f(x)]^3 = 3^3 = 27$

b.

$\lim\limits_{x \to 4} \dfrac{[f(x)]^2 - x^2}{f(x) + x} = \lim\limits_{x \to 4} \dfrac{(f(x) - x)(f(x) + x)}{f(x) + x}$

$= \lim\limits_{x \to 4} (f(x) - x)$

$= 3 - 4$

$= -1$

c. $\lim\limits_{x \to 4} \sqrt{3f(x) - 2x} = \sqrt{3 \times 3 - 2 \times 4}$

$= 1$

14. $\lim\limits_{x \to 0} \dfrac{f(x)}{x} = 1$

a. $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} \left[\dfrac{f(x)}{x} \times x\right] = 0$

b. $\lim\limits_{x \to 0} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to 0} \left[\dfrac{x}{g(x)} \dfrac{f(x)}{x}\right] = 0$

15. $\lim\limits_{x \to 0} \dfrac{f(x)}{x} = 1$ and $\lim\limits_{x \to 0} \dfrac{g(x)}{x} = 2$

a. $\lim\limits_{x \to 0} g(x) = \lim\limits_{x \to 0} x\left(\dfrac{g(x)}{x}\right) = 0 \times 2$

$= 0$

b. $\lim\limits_{x \to 0} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to 0} \dfrac{\frac{f(x)}{x}}{\frac{g(x)}{x}} = \dfrac{1}{2}$

16. $\lim\limits_{x \to 0} \dfrac{\sqrt{x + 1} - \sqrt{2x + 1}}{\sqrt{3x + 4} - \sqrt{2x + 4}}$

$= \lim\limits_{x \to 0} \left[\dfrac{\sqrt{x + 1} - \sqrt{2x + 1}}{\sqrt{x + 1} + \sqrt{2x + 1}}\right.$

$\times \dfrac{\sqrt{x + 1} + \sqrt{2x + 1}}{\sqrt{3x + 4} - \sqrt{2x + 4}}$

$\left.\times \dfrac{\sqrt{3x + 4} + \sqrt{2x + 4}}{\sqrt{3x + 4} + \sqrt{2x + 4}}\right]$

$= \lim\limits_{x \to 0} \left[\dfrac{(x + 1 - 2x - 1)}{(3x + 4 - 2x - 4)} \times \dfrac{\sqrt{3x + 4} + \sqrt{2x + 4}}{\sqrt{x + 1} + \sqrt{2x + 1}}\right]$

$= -\dfrac{2 + 2}{1 + 1}$

$= -2$

17. $\lim\limits_{x \to 1} \dfrac{x^2 + |x - 1| - 1}{|x - 1|}$

$x \to 1^+ \quad |x - 1| = x - 1$

$\dfrac{x^2 + x - 2}{x - 1} = \dfrac{(x + 2)(x - 1)}{x - 1}$

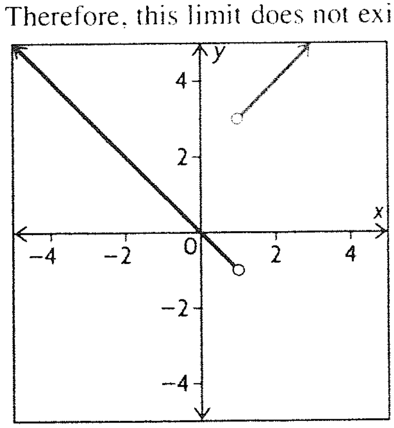
$\lim\limits_{x \to 1^+} \dfrac{x^2 + |x - 1| - 1}{|x - 1|} = 3$

$x \to 1^- \, |x - 1| = -x + 1$

$\lim\limits_{x \to 1} \dfrac{x^2 - x}{-x + 1} = \lim\limits_{x \to 1} \dfrac{x(x - 1)}{-x + 1}$

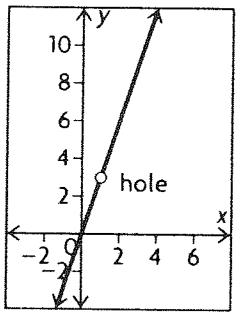
$= -1$

Therefore, this limit does not exist.

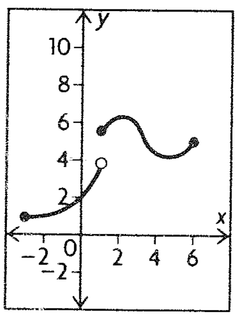


1.6 Continuity, pp. 51–53

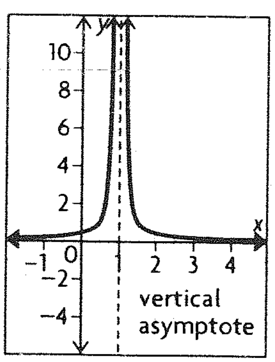
1. Anywhere that you can see breaks or jumps is a place where the function is not continuous.
2. It means that on that domain, you can trace the graph of the function without lifting your pencil.
3. point discontinuity



jump discontinuity

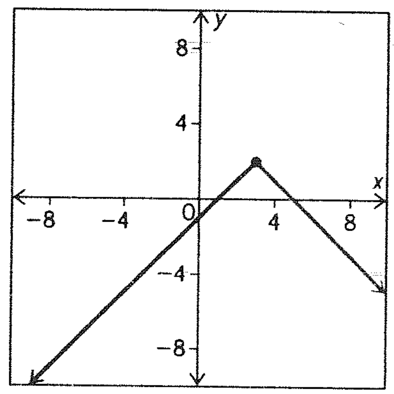


infinite discontinuity

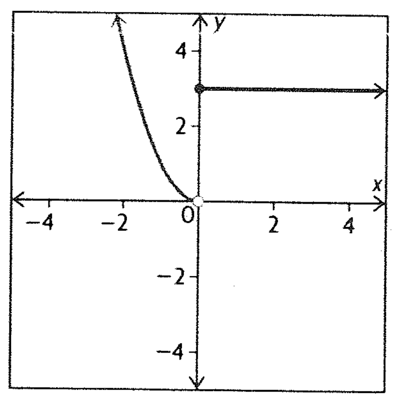


4. a. $x = 3$ makes the denominator 0.
b. $x = 0$ makes the denominator 0.
c. $x = 0$ makes the denominator 0.
d. $x = 3$ and $x = -3$ make the denominator 0.
e. $x^2 + x - 6 = (x + 3)(x - 2)$
$x = -3$ and $x = 2$ make the denominator 0.
f. The function has different one-sided limits at $x = 3$.

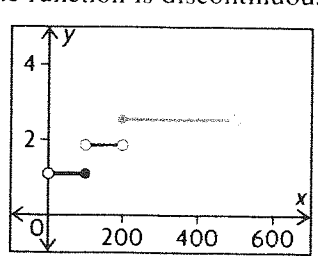
5. a. The function is a polynomial, so the function is continuous for all real numbers.
b. The function is a polynomial, so the function is continuous for all real numbers.
c. $x^2 - 5x = x(x - 5)$
The is continuous for all real numbers except 0 and 5.
d. The is continuous for all real numbers greater than or equal to -2.
e. The is continuous for all real numbers.
f. The is continuous for all real numbers.
6. $g(x)$ is a linear function (a polynomial), and so is continuous everywhere, including $x = 2$.
7.



The function is continuous everywhere.
8.



The function is discontinuous at $x = 0$.
9.



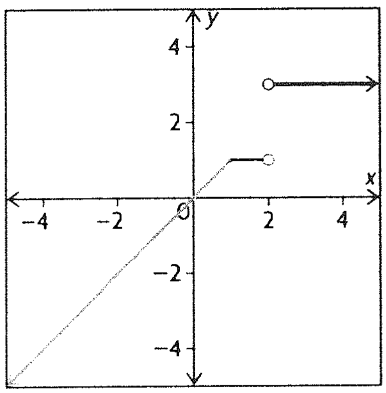
10. $\lim\limits_{x\to3} f(x) = \lim\limits_{x\to3}\dfrac{x^2 - x - 6}{x - 3}$

$= \lim\limits_{x\to3}\dfrac{(x - 3)(x + 2)}{x - 3}$

$= 5$

Function is discontinuous at $x = 3$.

11. Discontinuous at $x = 2$



12. $g(x) = \begin{cases} x + 3, \text{ if } x \neq 3 \\ 2 + \sqrt{k}, \text{ if } x = 3 \end{cases}$

$g(x)$ is continuous.

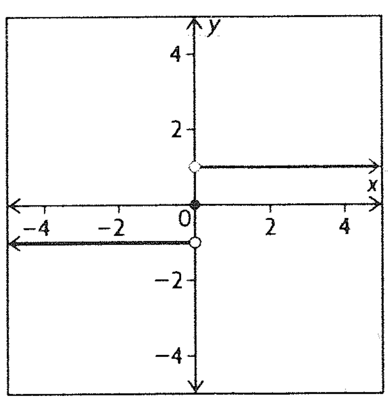
$2 + \sqrt{k} = 6$

$\sqrt{k} = 4, k = 16$

13.

$f(x) = \begin{cases} -1, \text{ if } x < 0 \\ 0, \text{ if } x = 0 \\ 1, \text{ if } x > 0 \end{cases}$

a.



b. i. From the graph, $\lim\limits_{x\to0^-} f(x) = -1$.

ii. From the graph, $\lim\limits_{x\to0^+} f(x) = 1$.

iii. Since the one-sided limits differ, $\lim\limits_{x\to0} f(x)$ does not exist.

c. f is not continuous since $\lim\limits_{x\to0} f(x)$ does not exist.

14. a. From the graph, $f(3) = 2$.

b. From the graph, $\lim\limits_{x\to3^-} f(x) = 4$.

c. $\lim\limits_{x\to3^+} f(x) = 4 = \lim\limits_{x\to3^-} f(x)$

Thus, $\lim\limits_{x\to3} f(x) = 4$. But, $f(3) = 2$. Hence f is not continuous at $x = 2$ (and also not continuous over $-3 < x < 8$).

15. The function is to be continuous at $x = 1$ and discontinuous at $x = 2$.

$f(x) = \begin{cases} \dfrac{Ax - B}{x - 2}, \text{ if } x \leq 1 \\ 3x, \text{ if } 1 < x < 2 \\ Bx^2 - A, \text{ if } x \geq 2 \end{cases}$

For $f(x)$ to be continuous at $x = 1$:

$\dfrac{A(1) - B}{1 - 2} = 3(1)$

$A(1) - B = -3$

$A = B - 3$

For $f(x)$ to be discontinuous at $x = 2$:

$B(2)^2 - A \neq 3(2)$

$4B - A \neq 6$

If $4B - A > 6$, then

$4B - (B - 3) > 6$

$3B + 3 > 6$

$3B > 3$

$B > 1$ and

$A > -2$

if $4B - A < 6$, then

$4B - B + 3 < 6$

$3B + 3 < 6$

$3B < 3$

$B < 1$ and

$A < -2$

This shows that A and B can be any set of real numbers such that

(1) $A = B - 3$

(2) $4B - A \neq 6$ (if $B > 1$, then $A > -2$ if $B < 1$, then $A < -2$)

$A = 1$ and $B = -2$ is not a solution because then the graph would be continuous at $x = 2$.

16. $f(x) = \begin{cases} -x, \text{ if } -3 \leq x \leq -2 \\ ax^2 + b, \text{ if } -2 < x < 0 \\ 6, \text{ if } x = 0 \end{cases}$

at $x = -2$, $4a + b = 2$

at $x = 0$, $b = 6$.

$a = -1$

$f(x) = \begin{cases} -x, \text{ if } -3 \leq x \leq -2 \\ -x^2 + b, \text{ if } -2 < x < 0 \\ 6, \text{ if } x = 0 \end{cases}$

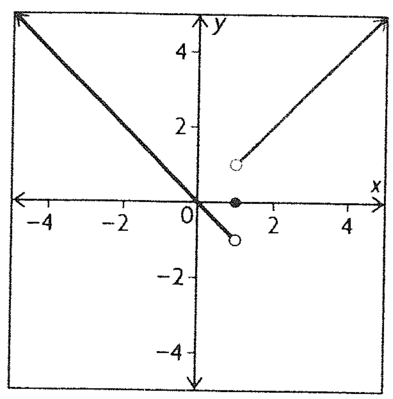
if $a = -1$, $b = 6$, $f(x)$ is continuous.

17. $g(x) = \begin{cases} \dfrac{x|x - 1|}{x - 1}, \text{ if } x \neq 1 \\ 0, \text{ if } x = 1 \end{cases}$

a. $\left.\begin{array}{l} \lim\limits_{x\to1^-} g(x) = -1 \\ \lim\limits_{x\to1^+} g(x) = 1 \end{array}\right\} \lim\limits_{x\to1} g(x)$

$\lim\limits_{x\to1} g(x)$ does not exist.

b.



$g(x)$ is discontinuous at $x = 1$.

Review Exercise, pp. 56–59

1. a. $f(-2) = 36, f(3) = 21$

$$m = \frac{21 - 36}{3 - (-2)}$$
$$= -3$$

b. $f(-1) = 13, f(4) = 48$

$$m = \frac{48 - 13}{4 - (-1)}$$
$$= 7$$

c. $f(1) = -3$

$$m = \lim_{h \to 0} \frac{5(1 + 2h + h^2) - (-3)}{h}$$
$$= \lim_{h \to 0} \frac{2h + h^2}{h}$$
$$= \lim_{h \to 0} 2 + h$$
$$= 2$$
$$y - (-3) = 2(x - 1)$$
$$2x - y - 5 = 0$$

2. a. $f(x) = \dfrac{3}{x + 1}, P(2, 1)$

$$m = \frac{\frac{3}{3 + h} - 1}{h}$$
$$= \lim_{h \to 0} -\frac{1}{3 + h}$$
$$= -\frac{1}{3}$$

b. $g(x) = \sqrt{x + 2}, P(-1, 1)$

$$m = \lim_{h \to 0} \frac{\sqrt{-1 + h + 2} - 1}{h}$$
$$= \lim_{h \to 0} \left[\frac{\sqrt{h + 1} - 1}{x} \times \frac{\sqrt{h + 1} + 1}{\sqrt{h + 1} + 1}\right]$$
$$= \lim_{h \to 0} \frac{1}{\sqrt{h + 1} + 1}$$
$$= \frac{1}{2}$$

c. $h(x) = \dfrac{2}{\sqrt{x + 5}}, P\left(4, \dfrac{2}{3}\right)$

$$m = \lim_{h \to 0} \frac{\frac{2}{\sqrt{4 + h + 5}} - \frac{2}{3}}{h}$$
$$= 2 \lim_{h \to 0} \left[\frac{3 - \sqrt{9 + h}}{3h\sqrt{9 + h}} \times \frac{3 + \sqrt{9 + h}}{3 + \sqrt{9 + h}}\right]$$
$$= 2 \lim_{h \to 0} \left[-\frac{1}{3\sqrt{9 + h}(3 + \sqrt{9 + h})}\right]$$
$$= -\frac{2}{9(6)}$$
$$= -\frac{1}{27}$$

d. $f(x) = \dfrac{5}{x - 2}, P\left(4, \dfrac{5}{2}\right)$

$$m = \lim_{h \to 0} \frac{\frac{5}{4 + h - 2} - \frac{5}{2}}{h}$$
$$= \lim_{h \to 0} \frac{10 - 5(2 + h)}{h(2 + h)(2)}$$
$$= \lim_{h \to 0} - \frac{-5h}{h(2 + h)(2)}$$
$$= -\frac{5}{4}$$

3. $f(x) = \begin{cases} 4 - x^2, \text{ if } x \le 1 \\ 2x + 1, \text{ if } x > 1 \end{cases}$

a. Slope at $P(-1, 3)$ $f(x) = 4 - x^2$

$$m = \lim_{h \to 0} \frac{4 - (-1 + h)^2 - 3}{h}$$
$$= \lim_{h \to 0} \frac{4 - 1 + 2h - h^2 - 3}{h}$$
$$= \lim_{h \to 0} (2 - h)$$
$$= 2$$

Slope of the graph at $P(-1, 3)$ is 2.

b. Slope at $P(2, 0.5)$

$$f(x) = 2x + 1$$
$$f(2 + h) - f(2) = 2(2 + h) + 1 - 5$$
$$= 2h$$
$$m = \lim_{h \to 0} \frac{2h}{h} = 2$$

Slope of the graph at $P(2, 0.5)$ is 2.

4. $s(t) = -5t^2 + 180$

a. $s(0) = 180, s(1) = 175, s(2) = 160$

Average velocity during the first second is

$$\frac{s(1) - s(0)}{1} = -5 \text{ m/s}.$$

Average velocity during the second second is

$\frac{s(2) - s(1)}{1} = -15$ m/s.

b. At $t = 4$:

$s(4 + h) - s(4)$

$= -5(4 + h)^2 + 180 - (-5(16) + 180)$

$= -80 - 40h - 5h^2 + 180 + 80 - 180$

$\frac{s(4 + h) - s(4)}{h} = \frac{-40h - 5h^2}{h}$

$v(4) = \lim_{h \to 0}(-40 - 5h) = -40$

Velocity is -40 m/s.

c. Time to reach ground is when $s(t) = 0$.

Therefore, $-5t^2 + 180 = 0$

$t^2 = 36$

$t = 6, t > 0.$

Velocity at $t = 6$:

$s(6 + h) = -5(36 + 12h + h^2) + 180$

$= -60h - 5h^2$

$s(6) = 0$

Therefore, $v(6) = \lim_{h \to 0}(-60 - 5h) = -60.$

5. $M(t) = t^2$ mass in grams

a. Growth during $3 \le t \le 3.01$

$M(3.01) = (3.01)^2 = 9.0601$

$M(3) = 3^2$

$= 9$

Grew 0.0601 g during this time interval.

b. Average rate of growth is

$\frac{0.0601}{0.01} = 6.01$ g/min.

c. $s(3 + h) = 9 + 6h + h^2$

$s(3) = 9$

$\frac{s(3 + h) - s(3)}{h} = \frac{6h + h^2}{h}$

Rate of growth is $\lim_{h \to 0}(6 + h) = 6$ g/min.

6. $Q(t) = 10^4(t^2 + 15t + 70)$ tonnes of waste,

$0 \le t \le 10$

a. At $t = 0$,

$Q(t) = 70 \times 10^4$

$= 700\,000.$

700 000 t have accumulated up to now.

b. Over the next three years, the average rate of change:

$Q(3) = 10^4(9 + 45 + 70)$

$= 124 \times 10^4$

$Q(0) = 70 \times 10^4$

$\frac{Q(3) - Q(0)}{3} = \frac{54 \times 10^4}{3}$

$= 18 \times 10^4$ t per year.

c. Present rate of change:

$Q(h) = 10^4(h^2 + 15h + 70)$

$Q(0) = 10^4 + 70$

$\lim_{h \to 0}\frac{Q(h) - Q(0)}{h} = \lim_{h \to 0} 10^4(h + 15)$

$= 15 \times 10^4$ t per year.

d. $Q(a + h)$

$= 10^4[a^2 + 2ah + h^2 + 15a + 15h + 70]$

$Q(a) = 10^4[a^2 + 15a + 70]$

$\frac{Q(a + h) - Q(a)}{h} = \frac{10^4[2ah + h^2 + 15h]}{h}$

$\lim_{h \to 0}\frac{Q(a + h) - Q(a)}{h} = \lim_{h \to 0} 10^4(2a + h + 15)$

$= (2a + 15)10^4$

Now,

$(2a + 15)10^4 = 3 \times 10^5$

$2a + 15 = 30$

$a = 7.5$

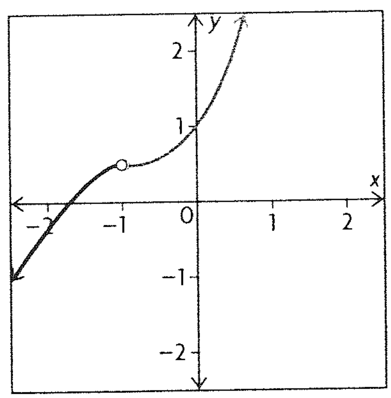
It will take 7.5 years to reach a rate of 3.0×10^5 t per year.

7. a. From the graph, the limit is 10.

b. 7; 0

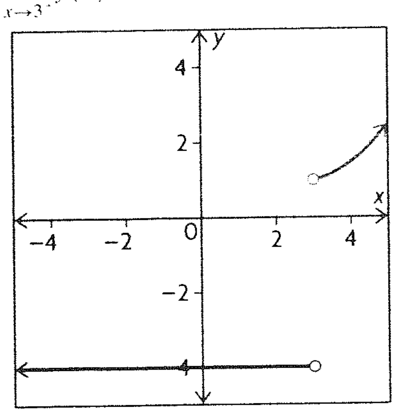
c. $p(t)$ is discontinuous for $t = 3$ and $t = 4$.

8. a. Answers will vary. $\lim_{x \to -1} f(x) = 0.5$. f is discontinuous at $x = -1$

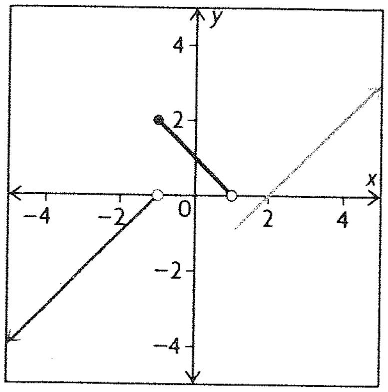


b. $f(x) = -4$ if $x < 3$; f is increasing for $x > 3$

$\lim_{x \to 3^+} f(x) = 1$



9. a.



b. $f(x) = \begin{cases} x + 1, \text{ if } x < -1 \\ -x + 1, \text{ if } -1 \le x < 1 \\ x - 2, \text{ if } x > 1 \end{cases}$

Discontinuous at $x = -1$ and $x = 1$.

c. They do not exist.

10. The function is not continuous at $x = -4$ because the function is not defined at $x = -4$. ($x = -4$ makes the denominator 0.)

11. $f(x) = \dfrac{2x - 2}{x^2 + x - 2}$

$= \dfrac{2(x - 1)}{(x - 1)(x + 2)}$

a. f is discontinuous at $x = 1$ and $x = -2$.

b. $\lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1} \dfrac{2}{x + 2}$

$= \dfrac{2}{3}$

$\lim\limits_{x\to -2} f(x)$: $= \lim\limits_{x\to -2^+} \dfrac{2}{x + 2} = +\infty$

$\lim\limits_{x\to -2^-} \dfrac{2}{x + 2} = -\infty$

$\lim\limits_{x\to -2} f(x)$ does not exist.

12. a. $f(x) = \dfrac{1}{x^2}$, $\lim\limits_{x\to 0} f(x)$ does not exist.

b. $g(x) = x(x - 5)$, $\lim\limits_{x\to 0} g(x) = 0$

c. $h(x) = \dfrac{x^3 - 27}{x^2 - 9}$,

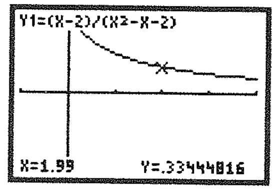
$\lim\limits_{x\to 4} h(x) = \dfrac{37}{7} = 5.2857$

$\lim\limits_{x\to -3} h(x)$ does not exist.

13. a.

x	1.9	1.99	1.999	2.001	2.01	2.1
$\dfrac{x-2}{x^2-x-2}$	0.344 83	0.334 45	0.333 44	0.333 22	0.332 23	0.322 58

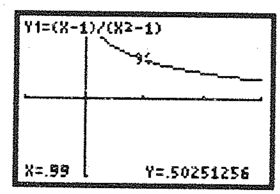
$\dfrac{1}{3}$



b.

x	0.9	0.99	0.999	1.001	1.01	1.1
$\dfrac{x-1}{x^2-1}$	0.526 32	0.502 51	0.500 25	0.499 75	0.497 51	0.476 19

$\dfrac{1}{2}$



14.

x	−0.1	−0.01	−0.001	0.001	0.01	0.1
$\dfrac{\sqrt{x+3}-\sqrt{3}}{x}$	0.291 12	0.288 92	0.2887	0.288 65	0.288 43	0.286 31

$\lim\limits_{x\to 0}\left[\dfrac{\sqrt{x+3}-\sqrt{3}}{x}\cdot\dfrac{\sqrt{x+3}+\sqrt{3}}{\sqrt{x+3}+\sqrt{3}}\right]$

$= \lim\limits_{x\to 0} \dfrac{x + 3 - 3}{x(\sqrt{x+3}+\sqrt{3})}$

$= \lim\limits_{x\to 0} \dfrac{x}{x(\sqrt{x+3}+\sqrt{3})}$

$= \lim\limits_{x\to 0} \dfrac{1}{\sqrt{x+3}+\sqrt{3}}$

$= \dfrac{1}{2\sqrt{3}}$

This agrees well with the values in the table.

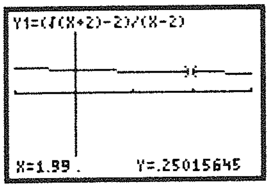
15. a. $f(x) = \dfrac{\sqrt{x+2} - 2}{x - 2}$

x	2.1	2.01	2.001	2.0001
$f(x)$	0.248 46	0.249 84	0.249 98	0.25

$x = 2.0001$

$f(x) \doteq 0.25$

b.



$\lim_{x\to 2} f(x) = 0.25$

c. $\lim_{x\to 2}\left[\dfrac{\sqrt{x+2}-2}{x-2} \times \dfrac{\sqrt{x+2}+2}{\sqrt{x+2}+2}\right]$

$= \lim_{x\to 2}\dfrac{1}{\sqrt{x+2}+2}$

$= \dfrac{1}{4} = 0.25$

16. a. $\lim_{h\to 0}\dfrac{(5+h)^2-25}{h}$

$= \lim_{h\to 0}(10+h)$

$= 10$

Slope of the tangent to $y = x^2$ at $x = 5$ is 10.

b. $\lim_{h\to 0}\dfrac{\sqrt{4+h}-2}{h} = \lim_{h\to 0}\dfrac{\sqrt{4+h}-2}{4+h-4}$

$= \lim_{h\to 0}\dfrac{1}{\sqrt{4+h}+2}$

$= \dfrac{1}{4}$

Slope of the tangent to $y = \sqrt{x}$ at $x = 4$ is $\frac{1}{4}$.

c. $\lim_{h\to 0}\dfrac{\frac{1}{4+h}-\frac{1}{4}}{h} = \lim_{h\to 0}\dfrac{4-4-h}{4(4+h)(h)}$

$= \lim_{h\to 0} -\dfrac{1}{4(4+h)}$

$= -\dfrac{1}{16}$

Slope of the tangent to $y = \dfrac{1}{x}$ at $(x = 4)$ is $-\frac{1}{16}$.

17. a. $\lim_{x\to -4}\dfrac{(x+4)(x+8)}{x+4} = \lim_{x\to -4}(x+8)$

$= (-4) + 8$

$= 4$

b. $\lim_{x\to a}\dfrac{(x+4a)^2-25a^2}{x-a} = \lim_{x\to a}\dfrac{(x-a)(x+9a)}{x-a}$

$= 10a$

c. $\lim_{x\to 0}\left[\dfrac{\sqrt{x+5}-\sqrt{5-x}}{x} \times \dfrac{\sqrt{x+5}+\sqrt{5-x}}{\sqrt{x+5}+\sqrt{5-x}}\right]$

$= \lim_{x\to 0}\dfrac{x+5-5+x}{x(\sqrt{x+5}+\sqrt{5-x})}$

$= \lim_{x\to 0}\dfrac{1}{(\sqrt{x+5}+\sqrt{5-x})}$

$= \dfrac{1}{\sqrt{5}}$

d. $\lim_{x\to 2}\dfrac{(x-2)(x+2)}{(x-2)(x^2+2x+4)}$

$= \lim_{x\to 2}\dfrac{x+2}{x^2+2x+4}$

$= \dfrac{(2)+2}{(2)^2+2(2)+4}$

$= \dfrac{4}{12}$

$= \dfrac{1}{3}$

e. $\lim_{x\to 4}\left[\dfrac{4-\sqrt{12+x}}{x-4} \cdot \dfrac{4+\sqrt{12+x}}{4+\sqrt{12+x}}\right]$

$= \lim_{x\to 4}\dfrac{16-(12+x)}{(x-4)(4+\sqrt{12+x})}$

$= \lim_{x\to 4}\dfrac{4-x}{(x-4)(4+\sqrt{12+x})}$

$= \lim_{x\to 4}\dfrac{-(x-4)}{(x-4)(4+\sqrt{12+x})}$

$= \lim_{x\to 4}\dfrac{-1}{4+\sqrt{12+x}}$

$= \dfrac{-1}{4+\sqrt{12+(4)}}$

$= \dfrac{-1}{4+4}$

$= -\dfrac{1}{8}$

f. $\lim_{x\to 0}\dfrac{1}{x}\left(\dfrac{1}{2+x}-\dfrac{1}{2}\right)$

$= \lim_{x\to 0}\left[\dfrac{1}{x} \times -\dfrac{x}{2(2+x)}\right]$

$= \lim_{x\to 0}\left[-\dfrac{1}{2(2+x)}\right]$

$= -\dfrac{1}{4}$

18. a. The function is not defined for $x < 3$, so there is no left-side limit.

b. Even after dividing out common factors from numerator and denominator, there is a factor of $x - 2$ in the denominator; the graph has a vertical asymptote at $x = 2$.

c. $f(x) = \begin{cases} -5, \text{ if } x < 1 \\ 2, \text{ if } x \geq 1 \end{cases}$

$\lim_{x\to 1^-} f(x) = -5 \neq \lim_{x\to 1^+} f(x) = 2$

d. The function has a vertical asymptote at $x = 2$.

e. $\lim_{x\to 0} \frac{|x|}{x}$

$x \to 0^- \; |x| = -x$

$\lim_{x\to 0^-} \frac{|x|}{x} = -1$

$\lim_{x\to 0^+} \frac{|x|}{x} = 1$

$\lim_{x\to 0^+} \frac{|x|}{x} \neq \lim_{x\to 0^-} \frac{|x|}{x}$

f. $f(x) = \begin{cases} 5x^2, \text{ if } x < -1 \\ 2x + 1, \text{ if } x \geq -1 \end{cases}$

$\lim_{x\to -1^+} f(x) = -1$

$\lim_{x\to -1^-} f(x) = 5$

$\lim_{x\to -1^+} f(x) \neq \lim_{x\to -1^-} f(x)$

Therefore, $\lim_{x\to -1} f(x)$ does not exist.

19. a.

$$m = \lim_{h\to 0} \frac{-3(1+h)^2 + 6(1+h) + 4 - (-3+6+4)}{h}$$
$$= \lim_{h\to 0} \frac{-3 - 6h - h^2 + 6 + 6h + 4 - 7}{h}$$
$$= \lim_{h\to 0} \frac{-h^2}{h}$$
$$= \lim_{h\to 0} -h$$
$$= 0$$

When $x = 1$, $y = 7$.

The equation of the tangent is $y - 7 = 0(x - 1)$

$y = 7$

b.

$$m = \lim_{h\to 0} \frac{(-2+h)^2 - (-2+h) - 1 - (4+2-1)}{h}$$
$$= \lim_{h\to 0} \frac{4 - 4h + h^2 + 2 - h - 1 - 5}{h}$$
$$= \lim_{h\to 0} \frac{-5h + h^2}{h}$$
$$= \lim_{h\to 0} (-5 + h)$$
$$= -5$$

When $x = -2$, $y = 5$.

The equation of the tangent is $y - 5 = -5(x + 2)$

$y = -5x - 5$

c. $$m = \lim_{h\to 0} \frac{6(-1+h)^3 - 3 - (-6 - 3)}{h}$$
$$= \lim_{h\to 0} \frac{6(-1 + 3h - 3h^2 + h^3) - 3 + 9}{h}$$
$$= \lim_{h\to 0} \frac{18h - 18h^2 + 6h^3}{h}$$
$$= \lim_{h\to 0} (18 - 18h + 6h^2)$$
$$= 18$$

When $x = -1$, $y = -9$.

The equation of the tangent is

$y - (-9) = 18(x - (-1))$

$y = 18x + 9$

d. $$m = \lim_{h\to 0} \frac{-2(3+h)^4 - (-162)}{h}$$
$$= \lim_{h\to 0} \frac{-2(81 + 108h + 54h^2 + 12h^3 + h^4) + 162}{h}$$
$$= \lim_{h\to 0} \frac{-216h - 108h^2 - 24h^3 - 2h^4}{h}$$
$$= \lim_{h\to 0} (-216 - 108h - 24h^2 - 2h^3)$$
$$= -216$$

When $x = 3$, $y = -162$.

The equation of the tangent is

$y - (-162) = -216(x - 3)$

$y = -216x + 486$

20. $P(t) = 20 + 61t + 3t^2$

a. $P(8) = 20 + 61(8) + 3(8)^2$

$= 700000$

b.

$$\lim_{h\to 0} \frac{20 + 61(8+h) + 3(8+h)^2 - (20 + 488 + 192)}{h}$$
$$= \lim_{h\to 0} \frac{20 + 488 + 61h + 3(64 + 16h + h^2) - 700}{h}$$
$$= \lim_{h\to 0} \frac{20 + 488 + 61h + 192 + 48h + 3h^2 - 700}{h}$$
$$= \lim_{h\to 0} \frac{109h + 3h^2}{h}$$
$$= \lim_{h\to 0} (109 + 3h)$$
$$= 109$$

The population is changing at the rate of 109000/h.

Chapter 1 Test, p. 60

1. $\lim_{x\to 1} \frac{1}{x - 1}$ does not exist since

$\lim_{x\to 1^+} \frac{1}{x-1} = +\infty \neq \lim_{x\to 1^-} \frac{1}{x-1} = -\infty.$

2. $f(x) = 5x^2 - 8x$

$f(-2) = 5(4) - 8(-2) = 20 + 16 = 36$

$f(1) = 5 - 8 = -3$

Slope of secant is $\frac{36 + 3}{-2 - 1} = -\frac{39}{3}$

$= -13$

3. a. $\lim\limits_{x\to1} f(x)$ does not exist.

b. $\lim\limits_{x\to2} f(x) = 1$

c. $\lim\limits_{x\to4} f(x) = 1$

d. f is discontinuous at $x = 1$ and $x = 2$.

4. a. Average velocity from $t = 2$ to $t = 5$:

$$\frac{s(5) - s(2)}{3} = \frac{(40 - 25) - (16 - 4)}{3} = \frac{15 - 12}{3} = 1$$

Average velocity from $t = 2$ to $t = 5$ is 1 km/h.

b. $s(3 + h) - s(3)$

$= 8(3 + h) - (3 + h)^2 - (24 - 9)$

$= 24 + 8h - 9 - 6h - h^2 - 15$

$= 2h - h^2$

$$v(3) = \lim_{h\to0}\frac{2h - h^2}{h} = 2$$

Velocity at $t = 3$ is 2 km/h.

5. $f(x) = \sqrt{x + 11}$

Average rate of change from $x = 5$ to $x = 5 + h$:

$$\frac{f(5 + h) - f(5)}{h} = \frac{\sqrt{16 + h} - \sqrt{16}}{h}$$

6. $f(x) = \dfrac{x}{x^2 - 15}$

Slope of the tangent at $x = 4$:

$$f(4 + h) = \frac{4 + h}{(4 + h)^2 - 15} = \frac{4 + h}{1 + 8h + h^2}$$

$$f(4) = \frac{4}{1}$$

$$f(4 + h) - f(4) = \frac{4 + h}{1 + 8h + h^2} - 4 = \frac{4 + h - 4 - 32h - 4h^2}{1 + 2h + h^2} = -\frac{31h - 4h^2}{(1 + 2h + h^2)}$$

$$\lim_{h\to0}\frac{f(4 + h) - f(4)}{h} = \lim_{h\to0}\frac{(-31 - 4h)}{1 + 2h + h^2} = -31$$

Slope of the tangent at $x = 4$ is -31.

7. a. $$\lim_{x\to3}\frac{4x^2 - 36}{2x - 6} = \lim_{x\to3}\frac{2(x - 3)(x + 3)}{(x - 3)} = 12$$

b. $$\lim_{x\to2}\frac{2x^2 - x - 6}{3x^2 - 7x + 2} = \lim_{x\to2}\frac{(2x + 3)(x - 2)}{(x - 2)(3x - 1)} = \frac{7}{5}$$

c. $$\lim_{x\to5}\frac{x - 5}{\sqrt{x - 1} - 2} = \lim_{x\to5}\frac{(x - 1) - 4}{\sqrt{x - 1} - 2}$$

$$= \lim_{x\to5}\frac{(\sqrt{x - 1} - 2)(\sqrt{x - 1} + 2)}{\sqrt{x - 1} - 2}$$

$$= 4$$

d. $$\lim_{x\to-1}\frac{x^3 + 1}{x^4 - 1} = \lim_{x\to-1}\frac{(x + 1)(x^2 - x + 1)}{(x - 1)(x + 1)(x^2 + 1)} = \frac{3}{-2(2)} = -\frac{3}{4}$$

e. $$\lim_{x\to3}\left(\frac{1}{x - 3} - \frac{6}{x^2 - 9}\right) = \lim_{x\to3}\frac{(x + 3) - 6}{(x - 3)(x + 3)} = \lim_{x\to3}\frac{1}{x + 3} = \frac{1}{6}$$

f. $$\lim_{x\to0}\frac{(x + 8)^{\frac{1}{3}} - 2}{x} = \lim_{x\to0}\frac{(x + 8)^{\frac{1}{3}} - 2}{(x + 8) - 8}$$

$$= \lim_{x\to0}\frac{(x + 8)^{\frac{1}{3}} - 2}{((x + 8)^{\frac{1}{3}} - 2)((x + 8)^{\frac{2}{3}} + 2(x + 8)^{\frac{1}{3}} + 4)}$$

$$= \frac{1}{4 + 4 + 4} = \frac{1}{12}$$

8. $$f(x) = \begin{cases} ax + 3, \text{ if } x > 5 \\ 8, \text{ if } x = 5 \\ x^2 + bx + a, \text{ if } x < 5 \end{cases}$$

$f(x)$ is continuous.

Therefore, $5a + 3 = 8$ $a = 1$

$25 + 5b + a = 8$ $5b = -18$

$b = -\dfrac{18}{5}$

Chapter 2
Derivatives

Review of Prerequisite Skills, pp. 62–63

1. a. $a^5 \times a^3 = a^{5+3}$
$= a^8$

b. $(-2a^2)^3 = (-2)^3(a^2)^3$
$= -8(a^{2\times 3})$
$= -8a^6$

c. $\dfrac{4p^7 \times 6p^9}{12p^{15}} = \dfrac{24p^{7+9}}{12p^{15}}$
$= 2p^{16-15}$
$= 2p$

d. $(a^4b^{-5})(a^{-6}b^{-2}) = (a^{4-6})(b^{-5-2})$
$= a^{-2}b^{-7}$
$= \dfrac{1}{a^2b^7}$

e. $(3e^6)(2e^3)^4 = (3)(e^6)(2^4)(e^3)^4$
$= (3)(2^4)(e^6)(e^{3\times 4})$
$= (3)(16)(e^{6+12})$
$= 48e^{18}$

f. $\dfrac{(3a^{-4})[2a^3(-b)^3]}{12a^5b^2} = \dfrac{(3)(2)(-1)^3(a^{-4+3})(b^3)}{12a^5b^2}$
$= \dfrac{-6(a^{-1-5})(b^{3-2})}{12}$
$= \dfrac{-1(a^{-6})(b)}{2}$
$= -\dfrac{b}{2a^6}$

2. a. $(x^{\frac{1}{2}})(x^{\frac{2}{3}}) = x^{\frac{1}{2}+\frac{2}{3}}$
$= x^{\frac{7}{6}}$

b. $(8x^6)^{\frac{2}{3}} = 8^{\frac{2}{3}}x^{6\times\frac{2}{3}}$
$= 4x^4$

c. $\dfrac{\sqrt{a}\sqrt[3]{a}}{\sqrt{a}} = \dfrac{(a^{\frac{1}{2}})(a^{\frac{1}{3}})}{a^{\frac{1}{2}}}$
$= a^{\frac{1}{3}}$

3. A perpendicular line will have a slope that is the negative reciprocal of the slope of the given line:

a. slope $= \dfrac{-1}{\frac{2}{3}}$
$= -\dfrac{3}{2}$

b. slope $= \dfrac{-1}{-\frac{1}{2}}$
$= 2$

c. slope $= \dfrac{-1}{\frac{5}{3}}$
$= -\dfrac{3}{5}$

d. slope $= \dfrac{-1}{-1}$
$= 1$

4. a. This line has slope $m = \dfrac{-4-(-2)}{-3-9}$
$= \dfrac{-2}{-12}$
$= \dfrac{1}{6}$

The equation of the desired line is therefore $y + 4 = \frac{1}{6}(x+3)$ or $x - 6y - 21 = 0$.

b. The equation $3x - 2y = 5$ can be rewritten as $2y = 3x - 5$ or $y = \frac{3}{2}x - \frac{5}{2}$, which has slope $\frac{3}{2}$. The equation of the desired line is therefore $y + 5 = \frac{3}{2}(x+2)$ or $3x - 2y - 4 = 0$.

c. The line perpendicular to $y = \frac{3}{4}x - 6$ will have slope $m = \dfrac{-1}{\frac{3}{4}} = -\frac{4}{3}$. The equation of the desired line is therefore $y + 3 = -\frac{4}{3}(x-4)$ or $4x + 3y - 7 = 0$.

5. a. $(x-3y)(2x+y) = 2x^2 + xy - 6xy - 3y^2$
$= 2x^2 - 5xy - 3y^2$

b. $(x-2)(x^2-3x+4)$
$= x^3 - 3x^2 + 4x - 2x^2 + 6x - 8$
$= x^3 - 5x^2 + 10x - 8$

c. $(6x-3)(2x+7) = 12x^2 + 42x - 6x - 21$
$= 12x^2 + 36x - 21$

d. $2(x+y) - 5(3x-8y) = 2x + 2y - 15x + 40y$
$= -13x + 42y$

e. $(2x-3y)^2 + (5x+y)^2$
$= 4x^2 - 12xy + 9y^2 + 25x^2 + 10xy + y^2$
$= 29x^2 - 2xy + 10y^2$

f. $3x(2x-y)^2 - x(5x-y)(5x+y)$
$= 3x(4x^2 - 4xy + y^2) - x(25x^2 - y^2)$
$= 12x^3 - 12x^2y + 3xy^2 - 25x^3 + xy^2$
$= -13x^3 - 12x^2y + 4xy^2$

6. a. $\frac{3x(x+2)}{x^2} \times \frac{5x^3}{2x(x+2)} = \frac{15x^4(x+2)}{2x^3(x+2)}$
$= \frac{15}{2}x^{4-3}$
$= \frac{15}{2}x$

$x \neq 0, -2$

b. $\frac{y}{(y+2)(y-5)} \times \frac{(y-5)^2}{4y^3}$
$= \frac{y(y-5)(y-5)}{4y^3(y+2)(y-5)}$
$= \frac{y-5}{4y^2(y+2)}$
$y \neq -2, 0, 5$

c. $\frac{4}{h+k} \div \frac{9}{2(h+k)} = \frac{4}{h+k} \times \frac{2(h+k)}{9}$
$= \frac{8(h+k)}{9(h+k)}$
$= \frac{8}{9}$

$h \neq -k$

d. $\frac{(x+y)(x-y)}{5(x-y)} \div \frac{(x+y)^3}{10}$
$= \frac{(x+y)(x-y)}{5(x-y)} \times \frac{10}{(x+y)^3}$
$= \frac{10(x+y)(x-y)}{5(x-y)(x+y)^3}$
$= \frac{2}{(x+y)^2}$
$x \neq -y, +y$

e. $\frac{x-7}{2x} + \frac{5x}{x-1} = \frac{(x-7)(x-1)}{2x(x-1)} + \frac{(5x)(2x)}{2x(x-1)}$
$= \frac{x^2 - 7x - x + 7 + 10x^2}{2x(x-1)}$
$= \frac{11x^2 - 8x + 7}{2x(x-1)}$

$x \neq 0, 1$

f. $\frac{x+1}{x-2} - \frac{x+2}{x+3}$
$= \frac{(x+1)(x+3)}{(x-2)(x+3)} - \frac{(x+2)(x-2)}{(x+3)(x-2)}$
$= \frac{x^2 + x + 3x + 3 - x^2 + 4}{(x+3)(x-2)}$
$= \frac{4x+7}{(x+3)(x-2)}$
$x \neq -3, 2$

7. a. $4k^2 - 9 = (2k+3)(2k-3)$

b. $x^2 + 4x - 32 = x^2 + 8x - 4x - 32$
$= x(x+8) - 4(x+8)$
$= (x-4)(x+8)$

c. $3a^2 - 4a - 7 = 3a^2 - 7a + 3a - 7$
$= a(3a-7) + 1(3a-7)$
$= (a+1)(3a-7)$

d. $x^4 - 1 = (x^2+1)(x^2-1)$
$= (x^2+1)(x+1)(x-1)$

e. $x^3 - y^3 = (x-y)(x^2 + xy + y^2)$

f. $r^4 - 5r^2 + 4 = r^4 - 4r^2 - r^2 + 4$
$= r^2(r^2-4) - 1(r^2-4)$
$= (r^2-1)(r^2-4)$
$= (r+1)(r-1)(r+2)(r-2)$

8. a. Letting $f(a) = a^3 - b^3$, $f(b) = b^3 - b^3$
$= 0$

So b is a root of $f(a)$, and so by the factor theorem, $a - b$ is a factor of $a^3 - b^3$. Polynomial long division provides the other factor:

$$\begin{array}{r} a^2 + ab + b^2 \\ a - b\overline{)a^3 + 0a^2 + 0a - b^3} \\ \underline{a^3 - a^2b} \\ a^2b + 0a - b^3 \\ \underline{a^2b - ab^2} \\ ab^2 - b^3 \\ \underline{ab^2 - b^3} \\ 0 \end{array}$$

So $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$.

b. Using long division or recognizing a pattern from the work in part a.:
$a^5 - b^5 = (a-b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4)$.

c. Using long division or recognizing a pattern from the work in part a.: $a^7 - b^7$
$= (a-b)(a^6 + a^5b + a^4b^2 + a^3b^3$
$+ a^2b^4 + ab^5 + b^6)$.

d. Using the pattern from the previous parts:
$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots$
$+ a^2b^{n-3} + ab^{n-2} + b^{n-1})$.

9. a. $f(2) = -2(2^4) + 3(2^2) + 7 - 2(2)$
$= -32 + 12 + 7 - 4$
$= -17$

b. $f(-1) = -2(-1)^4 + 3(-1)^2 + 7 - 2(-1)$
$= -2 + 3 + 7 + 2$
$= 10$

c. $f\left(\frac{1}{2}\right) = -2\left(\frac{1}{2}\right)^4 + 3\left(\frac{1}{2}\right)^2 + 7 - 2\left(\frac{1}{2}\right)$
$= -\frac{1}{8} + \frac{3}{4} + 7 - 1$
$= \frac{53}{8}$

d. $f(-0.25) = f\left(-\frac{1}{4}\right)$

$$= 2\left(-\frac{1}{4}\right)^4 + 3\left(-\frac{1}{4}\right)^2 + 7 - 2\left(-\frac{1}{4}\right)$$
$$= -\frac{1}{128} + \frac{3}{16} + 7 + \frac{1}{2}$$
$$= \frac{983}{128}$$
$$\doteq 7.68$$

10. a. $$\frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{(\sqrt{2})(\sqrt{2})}$$
$$= \frac{3\sqrt{2}}{2}$$

b. $$\frac{4 - \sqrt{2}}{\sqrt{3}} = \frac{(4 - \sqrt{2})(\sqrt{3})}{(\sqrt{3})(\sqrt{3})}$$
$$= \frac{4\sqrt{3} - \sqrt{6}}{3}$$

c. $$\frac{2 + 3\sqrt{2}}{3 - 4\sqrt{2}} = \frac{(2 + 3\sqrt{2})(3 + 4\sqrt{2})}{(3 - 4\sqrt{2})(3 + 4\sqrt{2})}$$
$$= \frac{6 + 9\sqrt{2} + 8\sqrt{2} + 12(2)}{3^2 - (4\sqrt{2})^2}$$
$$= \frac{30 + 17\sqrt{2}}{9 - 16(2)}$$
$$= -\frac{30 + 17\sqrt{2}}{23}$$

d. $$\frac{3\sqrt{2} - 4\sqrt{3}}{3\sqrt{2} + 4\sqrt{3}} = \frac{(3\sqrt{2} - 4\sqrt{3})(3\sqrt{2} - 4\sqrt{3})}{(3\sqrt{2} + 4\sqrt{3})(3\sqrt{2} - 4\sqrt{3})}$$
$$= \frac{(3\sqrt{2})^2 - 24\sqrt{6} + (4\sqrt{3})^2}{(3\sqrt{2})^2 - (4\sqrt{3})^2}$$
$$= \frac{9(2) - 24\sqrt{6} + 16(3)}{9(2) - 16(3)}$$
$$= -\frac{66 - 24\sqrt{6}}{30}$$
$$= -\frac{11 - 4\sqrt{6}}{5}$$

11. a. $f(x) = 3x^2 - 2x$

When $a = 2$,

$$\frac{f(a + h) - f(a)}{h} = \frac{f(2 + h) - f(2)}{h}$$
$$= \frac{3(2 + h)^2 - 2(2 + h) - [3(2)^2 - 2(2)]}{h}$$
$$= \frac{3(4 + 4h + h^2) - 4 - 2h - 8}{h}$$
$$= \frac{12 + 12h + 3h^2 - 2h - 12}{h}$$
$$= \frac{3h^2 + 10h}{h}$$
$$= 3h + 10$$

This expression can be used to determine the slope of the secant line between $(2, 8)$ and $(2 + h, f(2 + h))$.

b. For $h = 0.01$: $3(0.01) + 10 = 10.03$

c. The value in part b. represents the slope of the secant line through $(2, 8)$ and $(2.01, 8.1003)$.

2.1 The Derivative Function, pp. 73–75

1. A function is not differentiable at a point where its graph has a cusp, a discontinuity, or a vertical tangent:

a. The graph has a cusp at $x = -2$, so f is differentiable on $\{x \in \mathbf{R} \mid x \neq -2\}$.

b. The graph is discontinuous at $x = 2$, so f is differentiable on $\{x \in \mathbf{R} \mid x \neq 2\}$.

c. The graph has no cusps, discontinuities, or vertical tangents, so f is differentiable on $\{x \in \mathbf{R}\}$.

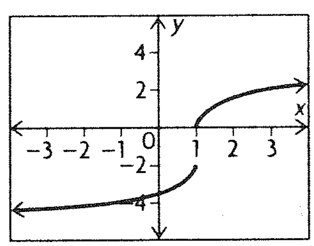
d. The graph has a cusp at $x = 1$, so f is differentiable on $\{x \in \mathbf{R} \mid x \neq 1\}$.

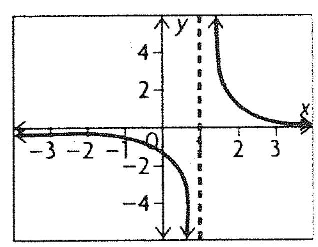
e. The graph has no cusps, discontinuities, or vertical tangents, so f is differentiable on $\{x \in \mathbf{R}\}$.

f. The function does not exist for $x < 2$, but has no cusps, discontinuities, or vertical tangents elsewhere, so f is differentiable on $\{x \in \mathbf{R} \mid x > 2\}$.

2. The derivative of a function represents the slope of the tangent line at a given value of the independent variable or the instantaneous rate of change of the function at a given value of the independent variable.

3.





4. a.
$$f(x) = 5x - 2$$
$$f(a + h) = 5(a + h) - 2$$
$$= 5a + 5h - 2$$
$$f(a + h) - f(a) = 5a + 5h - 2 - (5a - 2)$$
$$= 5h$$

b.
$$f(x) = x^2 + 3x - 1$$
$$f(a + h) = (a + h)^2 + 3(a + h) - 1$$
$$= a^2 + 2ah + h^2 + 3a + 3h - 1$$
$$f(a + h) - f(a) = a^2 + 2ah + h^2 + 3a + 3h - 1 - (a^2 + 3a - 1)$$
$$= 2ah + h^2 + 3h$$

c.
$$\begin{aligned} f(x) &= x^3 - 4x + 1 \\ f(a+h) &= (a+h)^3 - 4(a+h) + 1 \\ &= a^3 + 3a^2h + 3ah^2 + h^3 \\ &\quad - 4a - 4h + 1 \\ f(a+h) - f(a) &= a^3 + 3a^2h + 3ah^2 + h^3 - 4a \\ &\quad - 4h + 1 - (a^3 - 4a + 1) \\ &= 3a^2h + 3ah^2 + h^3 - 4h \end{aligned}$$

d.
$$\begin{aligned} f(x) &= x^2 + x - 6 \\ f(a+h) &= (a+h)^2 + (a+h) - 6 \\ &= a^2 + 2ah + h^2 + a + h - 6 \\ f(a+h) - f(a) &= a^2 + 2ah + h^2 + a + h - 6 \\ &\quad - (a^2 + a - 6) \\ &= 2ah + h^2 + h \end{aligned}$$

e.
$$\begin{aligned} f(x) &= -7x + 4 \\ f(a+h) &= -7(a+h) + 4 \\ &= -7a - 7h + 4 \\ f(a+h) - f(a) &= -7a - 7h + 4 - (-7a + 4) \\ &= -7h \end{aligned}$$

f.
$$\begin{aligned} f(x) &= 4 - 2x - x^2 \\ f(a+h) &= 4 - 2(a+h) - (a+h)^2 \\ &= 4 - 2a - 2h - a^2 - 2ah - h^2 \\ f(a+h) - f(a) &= 4 - 2a - 2h - a^2 - 2ah \\ &\quad - h^2 - 4 + 2a + a^2 \\ &= -2h - h^2 - 2ah \end{aligned}$$

5. a.
$$\begin{aligned} f'(1) &= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h \to 0} \frac{(1+h)^2 - 1^2}{h} \\ &= \lim_{h \to 0} \frac{1 + 2h + h^2 - 1}{h} \\ &= \lim_{h \to 0} \frac{2h + h^2}{h} \\ &= \lim_{h \to 0} (2 + h) \\ &= 2 \end{aligned}$$

b.
$$\begin{aligned} f'(3) &= \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} \\ &= \lim_{h \to 0} \left[\frac{(3+h)^2 + 3(3+h) + 1}{h} \right. \\ &\qquad \left. - \frac{(3^2 + 3(3) + 1)}{h} \right] \\ &= \lim_{h \to 0} \frac{9 + 6h + h^2 + 9 + 3h + 1 - 19}{h} \\ &= \lim_{h \to 0} \frac{9h + h^2}{h} \\ &= \lim_{h \to 0} (9 + h) \\ &= 9 \end{aligned}$$

c.
$$\begin{aligned} f'(0) &= \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h \to 0} \frac{\sqrt{h+1} - \sqrt{0+1}}{h} \\ &= \lim_{h \to 0} \frac{\sqrt{h+1} - 1}{h} \\ &= \lim_{h \to 0} \frac{(\sqrt{h+1} - 1)(\sqrt{h+1} + 1)}{h(\sqrt{h+1} + 1)} \\ &= \lim_{h \to 0} \frac{(\sqrt{h+1})^2 - 1}{h(\sqrt{h+1} + 1)} \\ &= \lim_{h \to 0} \frac{h + 1 - 1}{h(\sqrt{h+1} + 1)} \\ &= \lim_{h \to 0} \frac{1}{(\sqrt{h+1} + 1)} \\ &= \lim_{h \to 0} \frac{1}{(\sqrt{1} + 1)} \\ &= \frac{1}{2} \end{aligned}$$

d.
$$\begin{aligned} f'(-1) &= \lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h} \\ &= \lim_{h \to 0} \frac{\frac{5}{-1+h} - \frac{5}{-1}}{h} \\ &= \lim_{h \to 0} \frac{\frac{5}{-1+h} + 5}{h} \\ &= \lim_{h \to 0} \frac{\frac{5}{-1+h} + \frac{5(-1+h)}{-1+h}}{h} \\ &= \lim_{h \to 0} \frac{5 - 5 + 5h}{h(-1+h)} \\ &= \lim_{h \to 0} \frac{5h}{h(-1+h)} \\ &= \lim_{h \to 0} \frac{5}{(-1+h)} \\ &= \frac{5}{-1} \\ &= -5 \end{aligned}$$

6. a.
$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{-5(x+h) - 8 - (-5x - 8)}{h} \\ &= \lim_{h \to 0} \frac{-5x - 5h - 8 + 5x + 8}{h} \end{aligned}$$

$$= \lim_{h\to 0} \frac{-5h}{h}$$
$$= \lim_{h\to 0} -5$$
$$= -5$$

b. $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h\to 0} \left[\frac{2(x+h)^2 + 4(x+h)}{h} - \frac{(2x^2 + 4x)}{h} \right]$$
$$= \lim_{h\to 0} \left[\frac{2x^2 + 4xh + 2h^2 + 4x}{h} + \frac{4h - 2x^2 - 4x}{h} \right]$$
$$= \lim_{h\to 0} \frac{4xh + 2h^2 + 4h}{h}$$
$$= \lim_{h\to 0} (4x + 2h + 4)$$
$$= 4x + 4$$

c. $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h\to 0} \left[\frac{6(x+h)^3 - 7(x+h)}{h} - \frac{(6x^3 - 7x)}{h} \right]$$
$$= \lim_{h\to 0} \left[\frac{6x^3 + 18x^2h + 18xh^2 + 6h^3}{h} + \frac{-7x - 7h - 6x^3 + 7x}{h} \right]$$
$$= \lim_{h\to 0} \frac{18x^2h + 18xh^2 + 6h^3 - 7h}{h}$$
$$= \lim_{h\to 0} (18x^2 + 18xh + 6h^2 - 7)$$
$$= 18x^2 - 7$$

d. $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h\to 0} \frac{\sqrt{3(x+h)+2} - \sqrt{3x+2}}{h}$$
$$= \lim_{h\to 0} \frac{\sqrt{3x+3h+2} - \sqrt{3x+2}}{h}$$
$$= \lim_{h\to 0} \left[\frac{(\sqrt{3x+3h+2} - \sqrt{3x+2})}{h} \times \frac{(\sqrt{3x+3h+2} + \sqrt{3x+2})}{(\sqrt{3x+3h+2} + \sqrt{3x+2})} \right]$$
$$= \lim_{h\to 0} \frac{(\sqrt{3x+3h+2})^2 - (\sqrt{3x+2})^2}{h(\sqrt{3x+3h+2} + \sqrt{3x+2})}$$
$$= \lim_{h\to 0} \frac{3x + 3h + 2 - 3x - 2}{h(\sqrt{3x+3h+2} + \sqrt{3x+2})}$$
$$= \lim_{h\to 0} \frac{3}{\sqrt{3x+3h+2} + \sqrt{3x+2}}$$
$$= \frac{3}{2\sqrt{3x+2}}$$

7. a. Let $y = f(x)$, then
$$\frac{dy}{dx} = f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h\to 0} \frac{6 - 7(x+h) - (6 - 7x)}{h}$$
$$= \lim_{h\to 0} \frac{6 - 7x - 7h - 6 + 7x}{h}$$
$$= \lim_{h\to 0} \frac{-7h}{h}$$
$$= \lim_{h\to 0} -7$$
$$= -7$$

b. Let $y = f(x)$, then
$$\frac{dy}{dx} = f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h\to 0} \frac{\frac{x+h+1}{x+h-1} - \frac{x+1}{x-1}}{h}$$
$$= \lim_{h\to 0} \left[\frac{\frac{(x+h+1)(x-1)}{(x+h-1)(x-1)}}{h} - \frac{\frac{(x+1)(x+h-1)}{(x-1)(x+h-1)}}{h} \right]$$
$$= \lim_{h\to 0} \left[\frac{\frac{x^2 + hx + x - x - h - 1}{(x+h-1)(x-1)}}{h} - \frac{\frac{x^2 + hx - x + x + h - 1}{(x+h-1)(x-1)}}{h} \right]$$
$$= \lim_{h\to 0} \frac{\frac{-2h}{(x+h-1)(x-1)}}{h}$$
$$= \lim_{h\to 0} \frac{-2}{(x+h-1)(x-1)}$$
$$= -\frac{2}{(x-1)^2}$$

c. Let $y = f(x)$, then

$$\frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{3(x+h)^2 - 3x^2}{h}$$
$$= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 - 3x^2}{h}$$
$$= \lim_{h \to 0} \frac{6xh + 3h^2}{h}$$
$$= \lim_{h \to 0} 6x + 3h$$
$$= 6x$$

8. Let $y = f(x)$, then the slope of the tangent at each point x can be found by calculating $f'(x)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{2(x+h)^2 - 4(x+h) - 2x^2 + 4x}{h}$$
$$= \lim_{h \to 0} \left[\frac{2x^2 + 4xh + 2h^2 - 4x - 4h}{h} + \frac{-2x^2 + 4x}{h}\right]$$
$$= \lim_{h \to 0} \frac{4xh + h^2 - 4h}{h}$$
$$= \lim_{h \to 0} 4x + h - 4$$
$$= 4x - 4$$

So the slope of the tangent at $x = 0$ is

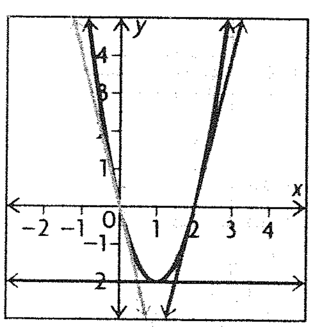
$$f'(0) = 4(0) - 4 = -4$$

At $x = 1$, the slope of the tangent is

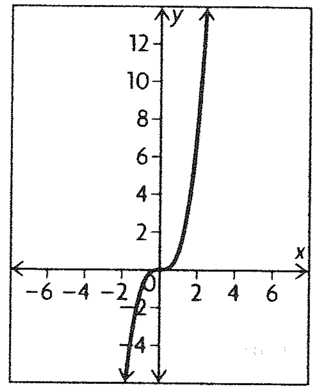
$$f'(1) = 4(1) - 4 = 0$$

At $x = 2$, the slope of the tangent is

$$f'(2) = 4(2) - 4 = 4$$



9. a.



b. Let $y = f(x)$, then the slope of the tangent at each point x can be found by calculating $f(x)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$
$$= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$
$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$$
$$= \lim_{h \to 0} 3x^2 + 3xh + h^2$$
$$= 3x^2$$

So the slope of the tangent at $x = -2$ is

$$f'(-2) = 3(-2)^2 = 12$$

At $x = -1$, the slope of the tangent is

$$f'(-1) = 3(-1)^2 = 3$$

At $x = 0$, the slope of the tangent is

$$f'(0) = 3(0)^2 = 0$$

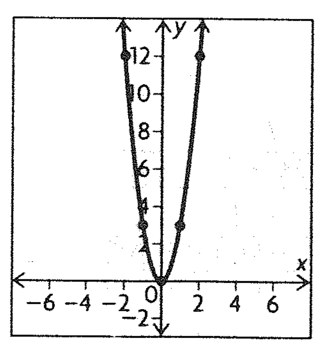
At $x = 1$, the slope of the tangent is

$$f'(1) = 3(1)^2 = 3$$

At $x = 2$, the slope of the tangent is

$$f'(2) = 3(2)^2 = 12$$

c.



d. The graph of $f(x)$ is a cubic. The graph of $f'(x)$ seems to be a parabola.

10. The velocity the particle at time t is given by $s'(t)$

$$s'(t) = \lim_{h\to 0}\frac{s(t+h)-s(t)}{h}$$
$$= \lim_{h\to 0}\frac{-(t+h)^2+8(t+h)-(-t^2+8t)}{h}$$
$$= \lim_{h\to 0}\frac{-t^2-2th-h^2+8t+8h+t^2-8t}{h}$$
$$= \lim_{h\to 0}\frac{-2th-h^2+8h}{h}$$
$$= \lim_{h\to 0} -2t-h+8$$
$$= -2t+8$$

So the velocity at $t=0$ is

$$s'(0) = -2(0)+8 = 8\text{ m/s}$$

At $t=4$, the velocity is

$$s'(4) = -2(4)+8 = 0\text{ m/s}$$

At $t=6$, the velocity is

$$s'(6) = -2(6)+8 = -4\text{ m/s}$$

11. $$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0}\frac{\sqrt{x+h+1}-\sqrt{x+1}}{h}$$
$$= \lim_{h\to 0}\left[\frac{(\sqrt{x+h+1}-\sqrt{x+1})}{h}\times\frac{(\sqrt{x+h+1}+\sqrt{x+1})}{(\sqrt{x+h+1}+\sqrt{x+1})}\right]$$
$$= \lim_{h\to 0}\frac{(\sqrt{x+h+1})^2-(\sqrt{x+1})^2}{h(\sqrt{x+h+1}+\sqrt{x+1})}$$
$$= \lim_{h\to 0}\frac{x+h+1-x-1}{h(\sqrt{x+h+1}+\sqrt{x+1})}$$
$$= \lim_{h\to 0}\frac{h}{h(\sqrt{x+h+1}+\sqrt{x+1})}$$
$$= \lim_{h\to 0}\frac{1}{(\sqrt{x+h+1}+\sqrt{x+1})}$$
$$= \frac{1}{2\sqrt{x+1}}$$

The equation $x-6y+4=0$ can be rewritten as $y=\frac{1}{6}x+\frac{2}{3}$, so this line has slope $\frac{1}{6}$. The value of x where the tangent to $f(x)$ has slope $\frac{1}{6}$ will satisfy $f'(x)=\frac{1}{6}$.

$$\frac{1}{2\sqrt{x+1}} = \frac{1}{6}$$
$$6 = 2\sqrt{x+1}$$
$$3^2 = (\sqrt{x+1})^2$$
$$9 = x+1$$
$$8 = x$$
$$f(8) = \sqrt{8+1} = \sqrt{9} = 3$$

So the tangent passes through the point (8, 3), and its equation is $y-3=\frac{1}{6}(x-8)$ or $x-6y+10=0$.

12. a. Let $y=f(x)$, then

$$\frac{dy}{dx} = f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0}\frac{c-c}{h}$$
$$= \lim_{h\to 0}\frac{0}{h}$$
$$= 0$$

b. Let $y=f(x)$, then

$$\frac{dy}{dx} = f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0}\frac{(x+h)-x}{h}$$
$$= \lim_{h\to 0}\frac{h}{h}$$
$$= \lim_{h\to 0} 1$$
$$= 1$$

c. Let $y=f(x)$, then

$$\frac{dy}{dx} = f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0}\frac{m(x+h)+b-mx-b}{h}$$
$$= \lim_{h\to 0}\frac{mx+mh+b-mx-b}{h}$$
$$= \lim_{h\to 0}\frac{mh}{h}$$
$$= \lim_{h\to 0} m$$
$$= m$$

d. Let $y=f(x)$, then

$$\frac{dy}{dx} = f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0}\left[\frac{a(x+h)^2+b(x+h)+c}{h} - \frac{(ax^2+bx+c)}{h}\right]$$

$$= \lim_{h \to 0} \left[\frac{ax^2 + 2axh + ah^2 + bx + bh}{h} + \frac{-ax^2 - bx - c}{h} \right]$$

$$= \lim_{h \to 0} \frac{2axh + ah^2 + bh}{h}$$

$$= \lim_{h \to 0} (2ax + ah + b)$$

$$= 2ax + b$$

13. The slope of the function at a point x is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$$

$$= \lim_{h \to 0} 3x^2 + 3xh + h^2$$

$$= 3x^2$$

Since $3x^2$ is nonnegative for all x, the original function never has a negative slope.

14. $h(t) = 18t - 4.9t^2$

a. $$h'(t) = \lim_{k \to 0} \frac{h(t+k) - h(t)}{k}$$

$$= \lim_{k \to 0} \frac{18(t+k) - 4.9(t+k)^2}{k} - \frac{(18t - 4.9t^2)}{k}$$

$$= \lim_{k \to 0} \frac{18t + 18k - 4.9t^2 - 9.8tk - 4.9k^2}{k} - \frac{18t + 4.9t^2}{k}$$

$$= \lim_{k \to 0} \frac{18k - 9.8tk - 4.9k^2}{k}$$

$$= \lim_{k \to 0} (18 - 9.8t - 4.9k)$$

$$= 18 - 9.8t - 4.9(0)$$

$$= 18 - 9.8t$$

Then $h'(2) = 18 - 9.8(2) = -1.6$ m/s.

b. $h'(2)$ measures the rate of change in the height of the ball with respect to time when $t = 2$.

15. a. This graph has positive slope for $x < 0$, zero slope at $x = 0$, and negative slope for $x > 0$, which corresponds to graph e.

b. This graph has positive slope for $x < 0$, zero slope at $x = 0$, and positive slope for $x > 0$, which corresponds to graph f.

c. This graph has negative slope for $x < -2$, positive slope for $-2 < x < 0$, negative slope for $0 < x < 2$, positive slope for $x > 2$, and zero slope at $x = -2$, $x = 0$, and $x = 2$, which corresponds to graph d.

16. This function is defined piecewise as $f(x) = -x^2$ for $x < 0$, and $f(x) = x^2$ for $x \geq 0$. The derivative will exist if the left-side and right-side derivatives are the same at $x = 0$:

$$\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{-(0+h)^2 - (-0^2)}{h}$$

$$= \lim_{h \to 0^-} \frac{-h^2}{h}$$

$$= \lim_{h \to 0} (-h)$$

$$= 0$$

$$\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{(0+h)^2 - (0^2)}{h}$$

$$= \lim_{h \to 0^+} \frac{h^2}{h}$$

$$= \lim_{h \to 0^+} (h)$$

$$= 0$$

Since the limits are equal for both sides, the derivative exists and $f'(0) = 0$.

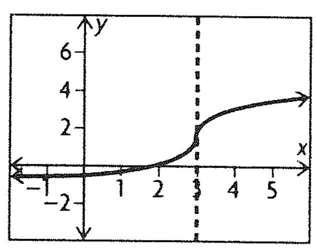
17. Since $f'(a) = 6$ and $f(a) = 0$,

$$6 = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$6 = \lim_{h \to 0} \frac{f(a+h) - 0}{h}$$

$$3 = \lim_{h \to 0} \frac{f(a+h)}{2h}$$

18.



$f(x)$ is continuous.

$f(3) = 2$

But $f'(3) = \infty$.

(Vertical tangent)

19. $y = x^2 - 4x - 5$ has a tangent parallel to $2x - y = 1$.

Let $f(x) = x^2 - 4x - 5$. First, calculate

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0}\left[\frac{(x+h)^2 - 4(x+h) - 5}{h} - \frac{(x^2 - 4x - 5)}{h}\right]$$
$$= \lim_{h\to 0}\left[\frac{x^2 + 2xh + h^2 - 4x - 4h - 5}{h} + \frac{-x^2 + 4x + 5}{h}\right]$$
$$= \lim_{h\to 0}\frac{2xh + h^2 - 4h}{h}$$
$$= \lim_{h\to 0}(2x + h - 4)$$
$$= 2x + 0 - 4$$
$$= 2x - 4$$

Thus, $2x - 4$ is the slope of the tangent to the curve at x. We want the tangent parallel to $2x - y = 1$.
Rearranging, $y = 2x - 1$.
If the tangent is parallel to this line,
$2x - 4 = 2$
$x = 3$
When $x = 3$, $y = (3)^2 - 4(3) - 5 = -8$.
The point is $(3, -8)$.

20. $f(x) = x^2$
The slope of the tangent at any point (x, x^2) is

$$f' = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h\to 0}\frac{(x+h)^2 - x^2}{h}$$
$$= \lim_{h\to 0}\frac{(x+h+x)(x+h-x)}{h}$$
$$= \lim_{h\to 0}\frac{h(2x+h)}{h}$$
$$= \lim_{h\to 0}(2x+h)$$
$$= 2x + 0$$
$$= 2x$$

Let (a, a^2) be a point of tangency. The equation of the tangent is
$y - a^2 = (2a)(x - a)$
$y = (2a)x - a^2$
Suppose the tangent passes through $(1, -3)$. Substitute $x = 1$ and $y = -3$ into the equation of the tangent:
$-3 = (2a)(1) - a^2$
$a^2 - 2a - 3 = 0$
$(a - 3)(a + 1) = 0$
$a = -1, 3$
So the two tangents are $y = -2x - 1$ or $2x + y + 1 = 0$ and $y = 6x - 9$ or $6x - y - 9 = 0$.

2.2 The Derivatives of Polynomial Functions, pp. 82–84

1. Answers may vary. For example:
constant function rule: $\frac{d}{dx}(5) = 0$

power rule: $\frac{d}{dx}(x^3) = 3x^2$

constant multiple rule: $\frac{d}{dx}(4x^3) = 12x^2$

sum rule: $\frac{d}{dx}(x^2 + x) = 2x + 1$

difference rule: $\frac{d}{dx}(x^3 - x^2 + 3x) = 3x^2 - 2x + 3$

2. a. $f'(x) = \frac{d}{dx}(4x) - \frac{d}{dx}(7)$
$$= 4\frac{d}{dx}(x) - \frac{d}{dx}(7)$$
$$= 4(x^0) - 0$$
$$= 4$$

b. $f'(x) = \frac{d}{dx}(x^3) - \frac{d}{dx}(x^2)$
$$= 3x^2 - 2x$$

c. $f'(x) = \frac{d}{dx}(-x^2) + \frac{d}{dx}(5x) + \frac{d}{dx}(8)$
$$= -\frac{d}{dx}(x^2) + 5\frac{d}{dx}(x) + \frac{d}{dx}(8)$$
$$= -(2x) + 5 + 0$$
$$= -2x + 5$$

d. $f'(x) = \frac{d}{dx}\left(\sqrt[3]{x}\right)$
$$= \frac{d}{dx}\left(x^{\frac{1}{3}}\right)$$
$$= \frac{1}{3}\left(x^{\left(\frac{1}{3}-1\right)}\right)$$
$$= \frac{1}{3}\left(x^{-\frac{2}{3}}\right)$$
$$= \frac{1}{3\sqrt[3]{x^2}}$$

e. $f'(x) = \frac{d}{dx}\left(\left(\frac{x}{2}\right)^4\right)$
$$= \left(\frac{1}{2}\right)^4\frac{d}{dx}(x^4)$$
$$= \frac{1}{16}(4x^3)$$
$$= \frac{x^3}{4}$$

f. $f'(x) = \frac{d}{dx}(x^{-3})$
$= (-3)(x^{-3-1})$
$= -3x^{-4}$

3. a. $h'(x) = \frac{d}{dx}((2x + 3)(x + 4))$
$= \frac{d}{dx}(2x^2 + 8x + 3x + 12)$
$= \frac{d}{dx}(2x^2) + \frac{d}{dx}(11x) + \frac{d}{dx}(12)$
$= 2\frac{d}{dx}(x^2) + 11\frac{d}{dx}(x) + \frac{d}{dx}(12)$
$= 2(2x) + 11(1) + 0$
$= 4x + 11$

b. $f'(x) = \frac{d}{dx}(2x^3 + 5x^2 - 4x - 3.75)$
$= \frac{d}{dx}(2x^3) + \frac{d}{dx}(5x^2) - \frac{d}{dx}(4x)$
$- \frac{d}{dx}(3.75)$
$= 2\frac{d}{dx}(x^3) + 5\frac{d}{dx}(x^2) - 4\frac{d}{dx}(x)$
$- \frac{d}{dx}(3.75)$
$= 2(3x^2) + 5(2x) - 4(1) - 0$
$= 6x^2 + 10x - 4$

c. $\frac{ds}{dt} = \frac{d}{dt}(t^2(t^2 - 2t))$
$= \frac{d}{dt}(t^4 - 2t^3)$
$= \frac{d}{dt}(t^4) - \frac{d}{dt}(2t^3)$
$= \frac{d}{dt}(t^4) - 2\frac{d}{dt}(t^3)$
$= 4t^3 - 2(3t^2)$
$= 4t^3 - 6t^2$

d. $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{5}x^5 + \frac{1}{3}x^3 - \frac{1}{2}x^2 + 1\right)$
$= \frac{d}{dx}\left(\frac{1}{5}x^5\right) + \frac{d}{dx}\left(\frac{1}{3}x^3\right) - \frac{d}{dx}\left(\frac{1}{2}x^2\right) + \frac{d}{dx}(1)$
$= \left(\frac{1}{5}\right)\frac{d}{dx}(x^5) + \left(\frac{1}{3}\right)\frac{d}{dx}(x^3) - \left(\frac{1}{2}\right)\frac{d}{dx}(x^2)$
$+ \frac{d}{dx}(1)$
$= \frac{1}{5}(5x^4) + \frac{1}{3}(3x^2) - \frac{1}{2}(2x) + 0$
$= x^4 + x^2 - x$

e. $g'(x) = \frac{d}{dx}(5(x^2)^4)$
$= 5\frac{d}{dx}(x^{2\times 4})$
$= 5\frac{d}{dx}(x^8)$
$= 5(8x^7)$
$= 40x^7$

f. $s'(t) = \frac{d}{dt}\left(\frac{t^5 - 3t^2}{2t}\right)$
$= \left(\frac{1}{2}\right)\frac{d}{dt}(t^4 - 3t)$
$= \left(\frac{1}{2}\right)\left(\frac{d}{dt}(t^4) - \frac{d}{dt}(3t)\right)$
$= \left(\frac{1}{2}\right)\left(\frac{d}{dt}(t^4) - 3\frac{d}{dt}(t)\right)$
$= \left(\frac{1}{2}\right)(4t^3 - 3(1))$
$= 2t^3 - \frac{3}{2}$

4. a. $\frac{dy}{dx} = \frac{d}{dx}(3x^{\frac{5}{3}})$
$= 3\frac{d}{dx}(x^{\frac{5}{3}})$
$= \left(\frac{5}{3}\right)3(x^{(\frac{5}{3}-1)})$
$= 5x^{\frac{2}{3}}$

b. $\frac{dy}{dx} = \frac{d}{dx}\left(4x^{-\frac{1}{2}} - \frac{6}{x}\right)$
$= 4\frac{d}{dx}(x^{-\frac{1}{2}}) - 6\frac{d}{dx}(x^{-1})$
$= 4\left(\frac{-1}{2}\right)(x^{-\frac{1}{2}-1}) - 6(-1)(x^{-1-1})$
$= -2x^{-\frac{3}{2}} + 6x^{-2}$

c. $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{6}{x^3} + \frac{2}{x^2} - 3\right)$
$= 6\frac{d}{dx}(x^{-3}) + 2\frac{d}{dx}(x^{-2}) - \frac{d}{dx}(3)$
$= 6(-3)(x^{-3-1}) + 2(-2)(x^{-2-1}) - 0$
$= -18x^{-4} - 4x^{-3}$
$= \frac{-18}{x^4} - \frac{4}{x^3}$

d. $\frac{dy}{dx} = \frac{d}{dx}(9x^{-2} + 3\sqrt{x})$

$= 9\frac{d}{dx}(x^{-2}) + 3\frac{d}{dx}(x^{\frac{1}{2}})$

$= 9(-2)(x^{-2-1}) + 3\left(\frac{1}{2}\right)(x^{\frac{1}{2}-1})$

$= -18x^{-3} + \frac{3}{2}x^{-\frac{1}{2}}$

e. $\frac{dy}{dx} = \frac{d}{dx}(\sqrt{x} + 6\sqrt{x^3} + \sqrt{2})$

$= \frac{d}{dx}(x^{\frac{1}{2}}) + 6\frac{d}{dx}(x^{\frac{3}{2}}) + \frac{d}{dx}(\sqrt{2})$

$= \frac{1}{2}(x^{\frac{1}{2}-1}) + 6\left(\frac{3}{2}\right)(x^{\frac{3}{2}-1}) + 0$

$= \frac{1}{2}(x^{-\frac{1}{2}}) + 9x^{\frac{1}{2}}$

f. $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1 + \sqrt{x}}{x}\right)$

$= \frac{d}{dx}\left(\frac{1}{x}\right) + \frac{d}{dx}\left(\frac{x^{\frac{1}{2}}}{x}\right)$

$= \frac{d}{dx}(x^{-1}) + \frac{d}{dx}(x^{-\frac{1}{2}})$

$= (-1)x^{-1-1} + \frac{-1}{2}(x^{-\frac{1}{2}-1})$

$= -x^{-2} - \frac{1}{2}x^{-\frac{3}{2}}$

5. a. $\frac{ds}{dt} = \frac{d}{dt}(-2t^2 + 7t)$

$= (-2)\left(\frac{d}{dt}(t^2)\right) + 7\left(\frac{d}{dt}(t)\right)$

$= (-2)(2t) + 7(1)$

$= -4t + 7$

b. $\frac{ds}{dt} = \frac{d}{dt}\left(18 + 5t - \frac{1}{3}t^3\right)$

$= \frac{d}{dt}(18) + 5\frac{d}{dt}(t) - \left(\frac{1}{3}\right)\frac{d}{dt}(t^3)$

$= 0 + 5(1) - \left(\frac{1}{3}\right)(3t^2)$

$= 5 - t^2$

c. $\frac{ds}{dt} = \frac{d}{dt}((t - 3)^2)$

$= \frac{d}{dt}(t^2 - 6t + 9)$

$= \frac{d}{dt}(t^2) - (6)\frac{d}{dt}(t) + \frac{d}{dt}(9)$

$= 2t - 6(1) + 0$

$= 2t - 6$

6. a. $f'(x) = \frac{d}{dx}(x^3 - \sqrt{x})$

$= \frac{d}{dx}(x^3) - \frac{d}{dx}(x^{\frac{1}{2}})$

$= 3x^2 - \frac{1}{2}(x^{\frac{1}{2}-1})$

$= 3x^2 - \frac{1}{2}x^{-\frac{1}{2}}$

so $f'(a) = f'(4) = 3(4)^2 - \frac{1}{2}(4)^{-\frac{1}{2}}$

$= 3(16) - \frac{1}{2}\frac{1}{\sqrt{4}}$

$= 48 - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$

$= 47.75$

b. $f'(x) = \frac{d}{dx}(7 - 6\sqrt{x} + 5x^{\frac{2}{3}})$

$= \frac{d}{dx}(7) - 6\frac{d}{dx}(x^{\frac{1}{2}}) + 5\frac{d}{dx}(x^{\frac{2}{3}})$

$= 0 - 6\left(\frac{1}{2}\right)(x^{\frac{1}{2}-1}) + 5\left(\frac{2}{3}\right)(x^{\frac{2}{3}-1})$

$= -3x^{-\frac{1}{2}} + \left(\frac{10}{3}\right)(x^{-\frac{1}{3}})$

so $f'(a) = f'(64) = -3(64^{-\frac{1}{2}}) + \left(\frac{10}{3}\right)(64^{-\frac{1}{3}})$

$= -3\left(\frac{1}{8}\right) + \frac{10}{3}\left(\frac{1}{4}\right)$

$= \frac{11}{24}$

7. a. $\frac{dy}{dx} = \frac{d}{dx}(3x^4)$

$= 3\frac{d}{dx}(x^4)$

$= 3(4x^3)$

$= 12x^3$

The slope at (1, 3) is found by substituting $x = 1$ into the equation for $\frac{dy}{dx}$. So the slope $= 12(1)^3$

$= 12$

b. $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{x^{-5}}\right)$

$= \frac{d}{dx}(x^5)$

$= 5x^4$

The slope at $(-1, -1)$ is found by substituting $x = -1$ into the equation for $\frac{dy}{dx}$. So the

$$\begin{aligned}\text{slope} &= 5(-1)^4 \\ &= 5\end{aligned}$$

c. $$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{2}{x}\right) \\ &= 2\frac{d}{dx}(x^{-1}) \\ &= 2(-1)x^{-1-1} \\ &= -2x^{-2}\end{aligned}$$

The slope at $(-2, -1)$ is found by substituting $x = -2$ into the equation for $\frac{dy}{dx}$. So the

$$\begin{aligned}\text{slope} &= -2(-2)^{-2} \\ &= -\frac{1}{2}\end{aligned}$$

d. $$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\sqrt{16x^3}\right) \\ &= \sqrt{16}\frac{d}{dx}(x^{\frac{3}{2}}) \\ &= 4\left(\frac{3}{2}\right)x^{\frac{3}{2}-1} \\ &= 6x^{\frac{1}{2}}\end{aligned}$$

The slope at (4, 32) is found by substituting $x = 4$ into the equation for $\frac{dy}{dx}$. So the

$$\begin{aligned}\text{slope} &= 6(4)^{\frac{1}{2}} \\ &= 12\end{aligned}$$

8. a. $y = 2x^3 + 3x$

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(2x^3 + 3x) \\ &= 2\frac{d}{dx}(x^3) + 3\frac{d}{dx}(x) \\ &= 2(3x^2) + 3(1) \\ &= 6x^2 + 3\end{aligned}$$

The slope at $x = 1$ is found by substituting $x = 1$ into the equation for $\frac{dy}{dx}$. So the slope is $6(1)^2 + 3 = 9$.

b. $y = 2\sqrt{x} + 5$

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(2\sqrt{x} + 5) \\ &= 2\frac{d}{dx}(x^{\frac{1}{2}}) + \frac{d}{dx}(5) \\ &= 2\left(\frac{1}{2}\right)(x^{\frac{1}{2}-1}) + 0 \\ &= x^{\frac{-1}{2}}\end{aligned}$$

The slope at $x = 4$ is found by substituting $x = 4$ into the equation for $\frac{dy}{dx}$. So the slope is $(4)^{\frac{-1}{2}} = \frac{1}{2}$.

c. $y = \frac{16}{x^2}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{16}{x^2}\right) \\ &= 16\frac{d}{dx}(x^{-2}) \\ &= 16(-2)x^{-2-1} \\ &= -32x^{-3}\end{aligned}$$

The slope at $x = -2$ is found by substituting $x = -2$ into the equation for $\frac{dy}{dx}$. So the slope is $-32(-2)^{-3} = \frac{(-32)}{(-2)^3} = 4$.

d. $$\begin{aligned}y &= x^{-3}(x^{-1} + 1) \\ &= x^{-4} + x^{-3}\end{aligned}$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(x^{-4} + x^{-3}) \\ &= -4x^{-5} - 3x^{-4} \\ &= -\frac{4}{x^5} - \frac{3}{x^4}\end{aligned}$$

The slope at $x = 1$ is found by substituting $x = 1$ into the equation for $\frac{dy}{dx}$. So the slope is $-\frac{4}{1^5} - \frac{3}{1^4} = -7$.

9. a. $$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(2x - \frac{1}{x}\right) \\ &= 2\frac{d}{dx}(x) - \frac{d}{dx}(x^{-1}) \\ &= 2(1) - (-1)x^{-1-1} \\ &= 2 + x^{-2}\end{aligned}$$

The slope at $x = 0.5$ is found by substituting $x = 0.5$ into the equation for $\frac{dy}{dx}$.
So the slope is $2 + (0.5)^{-2} = 6$.
The equation of the tangent line is therefore $y + 1 = 6(x - 0.5)$ or $6x - y - 4 = 0$.

b. $$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{3}{x^2} - \frac{4}{x^3}\right) \\ &= 3\frac{d}{dx}(x^{-2}) - 4\frac{d}{dx}(x^{-3}) \\ &= 3(-2)x^{-2-1} - 4(-3)x^{-3-1} \\ &= 12x^{-4} - 6x^{-3}\end{aligned}$$

The slope at $x = -1$ is found by substituting $x = -1$ into the equation for $\frac{dy}{dx}$. So the slope is $12(-1)^{-4} - 6(-1)^{-3} = 18$.

The equation of the tangent line is therefore
$y - 7 = 18(x + 1)$ or $18x - y + 25 = 0$.

c. $\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{3x^3}\right)$

$= \sqrt{3}\frac{d}{dx}\left(x^{\frac{3}{2}}\right)$

$= \sqrt{3}\left(\frac{3}{2}\right)x^{\frac{3}{2}-1}$

$= \frac{3\sqrt{3}x^{\frac{1}{2}}}{2}$

The slope at $x = 3$ is found by substituting $x = 3$ into the equation for $\frac{dy}{dx}$.

So the slope is $\frac{3\sqrt{3}(3)^{\frac{1}{2}}}{2} = \frac{9}{2}$.

The equation of the tangent line is therefore
$y - 9 = \frac{9}{2}(x - 3)$ or $9x - 2y - 9 = 0$.

d. $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{x}\left(x^2 + \frac{1}{x}\right)\right)$

$= \frac{d}{dx}\left(x + \frac{1}{x^2}\right)$

$= \frac{d}{dx}(x) + \frac{d}{dx}(x^{-2})$

$= 1 + (-2)x^{-2-1}$

$= 1 - 2x^{-3}$

The slope at $x = 1$ is found by substituting into the equation for $\frac{dy}{dx}$.

So the slope is $1 - 2(1)^{-3} = -1$.

The equation of the tangent line is therefore
$y - 2 = -(x - 1)$ or $x + y - 3 = 0$.

e. $\frac{dy}{dx} = \frac{d}{dx}\left(\left(\sqrt{x} - 2\right)\left(3\sqrt{x} + 8\right)\right)$

$= \frac{d}{dx}\left(3\left(\sqrt{x}\right)^2 + 8\sqrt{x} - 6\sqrt{x} - 16\right)$

$= \frac{d}{dx}\left(3x + 2\sqrt{x} - 16\right)$

$= \frac{d}{dx}(3x) + 2\frac{d}{dx}\left(x^{\frac{1}{2}}\right) - \frac{d}{dx}(16)$

$= 3(1) + 2\left(\frac{1}{2}\right)x^{\frac{1}{2}-1} - 0$

$= 3 + x^{-\frac{1}{2}}$

The slope at $x = 4$ is found by substituting $x = 4$ into the equation for $\frac{dy}{dx}$.

So the slope is $3 + (4)^{-\frac{1}{2}} = 3.5$.

The equation of the tangent line is therefore
$y = 3.5(x - 4)$ or $7x - 2y - 28 = 0$.

f. $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\sqrt{x} - 2}{\sqrt[3]{x}}\right)$

$= \frac{d}{dx}\left(\frac{x^{\frac{1}{2}} - 2}{x^{\frac{1}{3}}}\right)$

$= \frac{d}{dx}\left(x^{\frac{1}{2}-\frac{1}{3}} - 2x^{-\frac{1}{3}}\right)$

$= \frac{d}{dx}\left(x^{\frac{1}{6}}\right) - 2\frac{d}{dx}\left(x^{-\frac{1}{3}}\right)$

$= \frac{1}{6}\left(x^{\frac{1}{6}-1}\right) - 2\left(-\frac{1}{3}\right)x^{-\frac{1}{3}-1} - 0$

$= \frac{1}{6}\left(x^{-\frac{5}{6}}\right) + \frac{2}{3}x^{-\frac{4}{3}}$

The slope at $x = 1$ is found by substituting $x = 1$ into the equation for $\frac{dy}{dx}$.

So the slope is $\frac{1}{6}(1)^{-\frac{5}{6}} + \frac{2}{3}(1)^{-\frac{4}{3}} = \frac{5}{6}$.

The equation of the tangent line is therefore
$y + 1 = \frac{5}{6}(x - 1)$ or $5x - 6y - 11 = 0$.

10. A normal to the graph of a function at a point is a line that is perpendicular to the tangent at the given point.

$y = \frac{3}{x^2} - \frac{4}{x^3}$ at $P(-1, 7)$

Slope of the tangent is 18, therefore, the slope of the normal is $-\frac{1}{18}$.

Equation is $y - 7 = -\frac{1}{18}(x + 1)$.

$x + 18y - 125 = 0$

11. $y = \frac{3}{\sqrt[3]{x}} = 3x^{-\frac{1}{3}}$

Parallel to $x + 16y + 3 - 0$

Slope of the line is $-\frac{1}{16}$.

$\frac{dy}{dx} = -x^{-\frac{4}{3}}$

$x^{-\frac{4}{3}} = \frac{1}{16}$

$\frac{1}{x^{\frac{4}{3}}} = \frac{1}{16}$

$x^{\frac{4}{3}} = 16$

$x = (16)^{\frac{3}{4}} = 8$

12. $y = \frac{1}{x} = x^{-1}$; $y = x^3$

$\frac{dy}{dx} = -\frac{1}{x^2}$; $\frac{dy}{dx} = 3x^2$

Now, $-\frac{1}{x^2} = 3x^2$

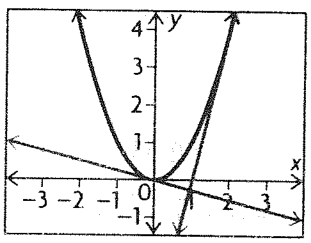
$x^4 = -\frac{1}{3}$.

No real solution. They never have the same slope.

13. $y = x^2$, $\frac{dy}{dx} = 2x$

The slope of the tangent at $A(2, 4)$ is 4 and at $B\left(-\frac{1}{8}, \frac{1}{64}\right)$ is $-\frac{1}{4}$.

Since the product of the slopes is -1, the tangents at $A(2, 4)$ and $B\left(-\frac{1}{8}, \frac{1}{64}\right)$ will be perpendicular.



14. $y = -x^2 + 3x + 4$

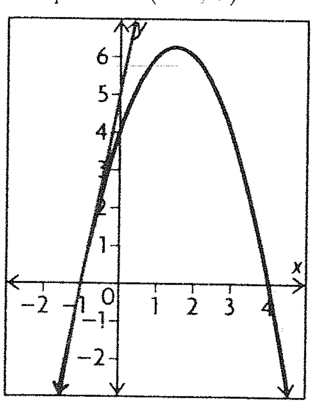
$\frac{dy}{dx} = -2x + 3$

For $\frac{dy}{dx} = 5$

$5 = -2x + 3$

$x = -1$.

The point is $(-1, 0)$.



15. $y = x^3 + 2$

$\frac{dy}{dx} = 3x^2$, slope is 12

$x^2 = 4$

$x = 2$ or $x = -2$

Points are $(2, 10)$ and $(-2, -6)$.

16. $y = \frac{1}{5}x^5 - 10x$, slope is 6

$\frac{dy}{dx} = x^4 - 10 = 6$

$x^4 = 16$

$x^2 = 4$ or $x^2 = -4$

$x = \pm 2$ non-real

Tangents with slope 6 are at the points $\left(2, -\frac{68}{5}\right)$ and $\left(-2, \frac{68}{5}\right)$.

17. $y = 2x^2 + 3$

a. Equation of tangent from $A(2, 3)$:

If $x = a$, $y = 2x^2 + 3$.

Let the point of tangency be $P(a, 2a^2 + 3)$.

Now, $\frac{dy}{dx} = 4x$ and when $x = a$, $\frac{dy}{dx} = 4a$.

The slope of the tangent is the slope of AP.

$\frac{2a^2}{a - 2} = 4a$.

$2a^2 = 4a^2 - 8a$

$2a^2 - 8a = 0$

$2a(a - 4) = 0$

$a = 0$ or $a = 4$.

Point $(2, 3)$:

Slope is 0.	Slope is 16.
Equation of tangent is	Equation of tangent is
$y - 3 = 0$.	$y - 3 = 16(x - 2)$ or
	$16x - y - 29 = 0$.

b. From the point $B(2, -7)$:

Slope of BP: $\frac{2a^2 + 10}{a - 2} = 4a$

$2a^2 + 10 = 4a^2 - 8a$

$2a^2 - 8a - 10 = 0$

$a^2 - 4a - 5 = 0$

$(a - 5)(a + 1) = 0$

$a = 5$	$a = -1$
Slope is $4a = 20$.	Slope is $4a = -4$.
Equation is	Equation is
$y + 7 = 20(x - 2)$	$y + 7 = -4(x - 2)$
or $20x - y - 47 = 0$.	or $4x + y - 1 = 0$.

18. $ax - 4y + 21 = 0$ is tangent to $y = \frac{a}{x^2}$ at $x = -2$.

Therefore, the point of tangency is $\left(-2, \frac{a}{4}\right)$.

This point lies on the line, therefore,

$a(-2) - 4\left(\frac{a}{4}\right) + 21 = 0$

$-3a + 21 = 0$

$a = 7$.

19. a. When $h = 200$,

$d = 3.53\sqrt{200}$

$\doteq 49.9$

Passengers can see about 49.9 km.

b. $d = 3.53\sqrt{h} = 3.53h^{\frac{1}{2}}$

$$d' = 3.53\left(\frac{1}{2}h^{-\frac{1}{2}}\right)$$

$$= \frac{3.53}{2\sqrt{h}}$$

When $h = 200$,

$$d' = \frac{3.53}{2\sqrt{200}}$$

$\doteq 0.12$

The rate of change is about 0.12 km/m.

20. $d(t) = 4.9t^2$

a. $d(2) = 4.9(2)^2 = 19.6$ m

$d(5) = 4.9(5)^2 = 122.5$ m

The average rate of change of distance with respect to time from 2 s to 5 s is

$$\frac{\Delta d}{\Delta t} = \frac{122.5 - 19.6}{5 - 2}$$

$= 34.3$ m/s

b. $d'(t) = 9.8t$

Thus, $d'(4) = 9.8(4) = 39.2$ m/s.

c. When the object hits the ground, $d = 150$.

Set $d(t) = 150$:

$4.9t^2 = 150$

$$t^2 = \frac{1500}{49}$$

$$t = \pm\frac{10}{7}\sqrt{15}$$

Since $t \geq 0$, $t = \frac{10}{7}\sqrt{15}$

Then,

$$d'\left(\frac{10}{7}\sqrt{15}\right) = 9.8\left(\frac{10}{7}\sqrt{15}\right)$$

$\doteq 54.2$ m/s

21. $v(t) = s'(t) = 2t - t^2$

$0.5 = 2t - t^2$

$t^2 - 2t + 0.5 = 0$

$2t^2 - 4t + 1 = 0$

$$t = \frac{4 \pm \sqrt{8}}{4}$$

$t \doteq 1.71, 0.29$

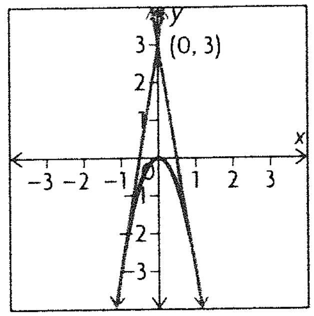
The train has a velocity of 0.5 km/min at about 0.29 min and 1.71 min.

22. $v(t) = R'(t) = -10t$

$v(2) = -20$

The velocity of the bolt at $t = 2$ is -20 m/s.

23.



Let the coordinates of the points of tangency be $A(a, -3a^2)$.

$\frac{dy}{dx} = -6x$, slope of the tangent at A is $-6a$

Slope of PA: $\frac{-3a^2 - 3}{a} = -6a$

$-3a^2 - 3 = -6a^2$

$3a^2 = 3$

$a = 1$ or $a = -1$

Coordinates of the points at which the tangents touch the curve are $(1, -3)$ and $(-1, -3)$.

24. $y = x^3 - 6x^2 + 8x$, tangent at $A(3, -3)$

$$\frac{dy}{dx} = 3x^2 - 12x + 8$$

When $x = 3$,

$$\frac{dy}{dx} = 27 - 36 + 8 = -1$$

The slope of the tangent at $A(3, -3)$ is -1.

Equation will be

$y + 3 = -1(x - 3)$

$y = -x$.

$-x = x^3 - 6x^2 + 8x$

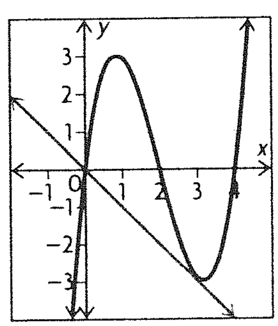
$x^3 - 6x^2 + 9x = 0$

$x(x^2 - 6x + 9) = 0$

$x(x - 3)^2 = 0$

$x = 0$ or $x = 3$

Coordinates are $B(0, 0)$.



25. a. i. $f(x) = 2x - 5x^2$

$f'(x) = 2 - 10x$

Set $f'(x) = 0$:

$2 - 10x = 0$

$10x = 2$

$x = \frac{1}{5}$

Then,

$$f\left(\frac{1}{5}\right) = 2\left(\frac{1}{5}\right) - 5\left(\frac{1}{5}\right)^2$$

$$= \frac{2}{5} - \frac{1}{5}$$

$$= \frac{1}{5}$$

Thus the point is $\left(\frac{1}{5}, \frac{1}{5}\right)$.

ii. $f(x) = 4x^2 + 2x - 3$

$f'(x) = 8x + 2$

Set $f'(x) = 0$:

$8x + 2 = 0$

$8x = -2$

$x = -\frac{1}{4}$

Then,

$$f\left(-\frac{1}{4}\right) = 4\left(-\frac{1}{4}\right)^2 + 2\left(-\frac{1}{4}\right) - 3$$

$$= \frac{1}{4} - \frac{2}{4} - \frac{12}{4}$$

$$= -\frac{13}{4}$$

Thus the point is $\left(-\frac{1}{4}, -\frac{13}{4}\right)$.

iii. $f(x) = x^3 - 8x^2 + 5x + 3$

$f'(x) = 3x^2 - 16x + 5$

Set $f'(x) = 0$:

$3x^2 - 16x + 5 = 0$

$3x^2 - 15x - x + 5 = 0$

$3x(x - 5) - (x - 5) = 0$

$(3x - 1)(x - 5) = 0$

$x = \frac{1}{3}, 5$

$$f\left(\frac{1}{3}\right) = \left(\frac{1}{3}\right)^3 - 8\left(\frac{1}{3}\right)^2 + 5\left(\frac{1}{3}\right) + 3$$

$$= \frac{1}{27} - \frac{24}{27} + \frac{45}{27} + \frac{81}{27}$$

$$= \frac{103}{27}$$

$$f(5) = (5)^3 - 8(5)^2 + 5(5) + 3$$

$$= 25 - 200 + 25 + 3$$

$$= -47$$

Thus the two points are $\left(\frac{1}{3}, \frac{103}{27}\right)$ and $(5, -47)$.

b. At these points, the slopes of the tangents are zero, meaning that the rate of change of the value of the function with respect to the domain is zero. These points are also local maximum or minimum points.

26. $\sqrt{x} + \sqrt{y} = 1$

$P(a, b)$ is on the curve, therefore $a \geq 0, b \geq 0$.

$\sqrt{y} = 1 - \sqrt{x}$

$y = 1 - 2\sqrt{x} + x$

$\frac{dy}{dx} = -\frac{1}{2} \cdot 2x^{-\frac{1}{2}} + 1$

At $x = a$, slope is $-\frac{1}{\sqrt{a}} + 1 = \frac{-1 + \sqrt{a}}{\sqrt{a}}$.

But $\sqrt{a} + \sqrt{b} = 1$

$-\sqrt{b} = \sqrt{a} - 1$.

Therefore, slope is $-\frac{\sqrt{b}}{\sqrt{a}} = -\sqrt{\frac{b}{a}}$.

27. $f(x) = x^n, f'(x) = nx^{n-1}$

Slope of the tangent at $x = 1$ is $f'(1) = n$,

The equation of the tangent at $(1, 1)$ is:

$y - 1 = n(x - 1)$

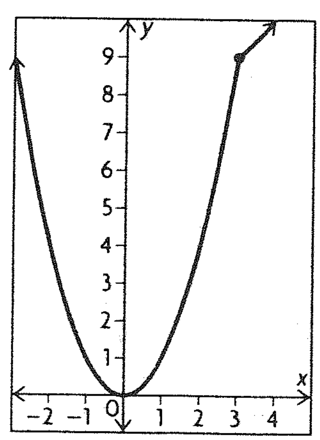
$nx - y - n + 1 = 0$

Let $y = 0, nx = n - 1$

$x = \frac{n-1}{n} = 1 - \frac{1}{n}$.

The x-intercept is $1 - \frac{1}{n}$; as $n \to \infty, \frac{1}{n} \to 0$, and the x-intercept approaches 1. As $n \to \infty$, the slope of the tangent at $(1, 1)$ increases without bound, and the tangent approaches a vertical line having equation $x - 1 = 0$.

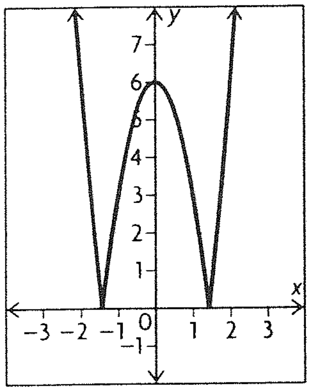
28. a.



$$f(x) = \begin{cases} x^2, \text{ if } x < 3 \\ x + 6, \text{ if } x \geq 3 \end{cases} \qquad f'(x) = \begin{cases} 2x, \text{ if } x < 3 \\ 1, \text{ if } x \geq 3 \end{cases}$$

$f'(3)$ does not exist.

b.

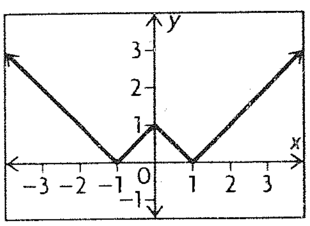


$$f(x) = \begin{cases} 3x^2 - 6, \text{ if } x < -\sqrt{2} \text{ or } x > \sqrt{2} \\ 6 - 3x^2, \text{ if } -\sqrt{2} < x < \sqrt{2} \end{cases}$$

$$f'(x) = \begin{cases} 6x, \text{ if } x < -\sqrt{2} \text{ or } x > \sqrt{2} \\ -6x, \text{ if } -\sqrt{2} \leq x \leq \sqrt{2} \end{cases}$$

$f'(\sqrt{2})$ and $f'(-\sqrt{2})$ do not exist.

c.



$$f(x) = \begin{cases} x - 1, \text{ if } x \geq 1 & \text{since } |x - 1| = x - 1 \\ 1 - x, \text{ if } 0 \leq x < 1 & \text{since } |x - 1| = 1 - x \\ x + 1, \text{ if } -1 < x < 0 & \text{since } |-x - 1| = x + 1 \\ -x - 1, \text{ if } x \leq -1 & \text{since } |-x-1| = -x - 1 \end{cases}$$

$$f'(x) = \begin{cases} 1, \text{ if } x > 1 \\ -1, \text{ if } 0 < x < 1 \\ 1, \text{ if } -1 < x < 0 \\ -1, \text{ if } x < -1 \end{cases}$$

$f'(0)$, $f'(-1)$, and $f'(1)$ do not exist.

2.3 The Product Rule, pp. 90–91

1. a. $h(x) = x(x - 4)$

$h'(x) = x(1) + (1)(x - 4)$

$= 2x - 4$

b. $h(x) = x^2(2x - 1)$

$h'(x) = x^2(2) + (2x)(2x - 1)$

$= 6x^2 - 2x$

c. $h(x) = (3x + 2)(2x - 7)$

$h'(x) = (3x + 2)(2) + (3)(2x - 7)$

$= 12x - 17$

d. $h(x) = (5x^7 + 1)(x^2 - 2x)$

$h'(x) = (5x^7 + 1)(2x - 2) + (35x^6)(x^2 - 2x)$

$= 45x^8 - 80x^7 + 2x - 2$

e. $s(t) = (t^2 + 1)(3 - 2t^2)$

$s'(t) = (t^2 + 1)(-4t) + (2t)(3 - 2t^2)$

$= -8t^3 + 2t$

f. $f(x) = \dfrac{x - 3}{x + 3}$

$f(x) = (x - 3)(x + 3)^{-1}$

$f'(x) = (x - 3)(-1)(x + 3)^{-2} + (1)(x + 3)^{-1}$

$= (x + 3)^{-2}(-x + 3 + x + 3)$

$= \dfrac{6}{(x + 3)^2}$

2. a. $y = (5x + 1)^3(x - 4)$

$\dfrac{dy}{dx} = (5x + 1)^3(1) + 3(5x + 1)^2(5)(x - 4)$

$= (5x + 1)^3 + 15(5x + 1)^2(x - 4)$

b. $y = (3x^2 + 4)(3 + x^3)^5$

$\dfrac{dy}{dx} = (3x^2 + 4)(5)(3 + x^3)^4(3x^2)$

$+ (6x)(3 + x^3)^5$

$= 15x^2(3x^2 + 4)(3 + x^3)^4 + 6x(3 + x^3)^5$

c. $y = (1 - x^2)^4(2x + 6)^3$

$\dfrac{dy}{dx} = 4(1 - x^2)^3(-2x)(2x + 6)^3$

$+ (1 - x^2)^4 3(2x + 6)^2(2)$

$= -8x(1 - x^2)^3(2x + 6)^3$

$+ 6(1 - x^2)^4(2x + 6)^2$

d. $y = (x^2 - 9)^4(2x - 1)^3$

$\dfrac{dy}{dx} = (x^2 - 9)^4(3)(2x - 1)^2(2)$

$+ 4(x^2 - 9)^3(2x)(2x - 1)^3$

$= 6(x^2 - 9)^4(2x - 1)^2$

$+ 8x(x^2 - 9)^3(2x - 1)^3$

3. It is not appropriate or necessary to use the product rule when one of the factors is a constant or when it would be easier to first determine the product of the factors and then use other rules to determine the derivative. For example, it would not be best to use the product rule for $f(x) = 3(x^2 + 1)$ or $g(x) = (x + 1)(x - 1)$.

4. $F(x) = [b(x)][c(x)]$

$F'(x) = [b(x)][c'(x)] + [b'(x)][c(x)]$

5. a. $y = (2 + 7x)(x - 3)$

$\frac{dy}{dx} = (2 + 7x)(1) + 7(x - 3)$

At $x = 2$,

$\frac{dy}{dx} = (2 + 14) + 7(-1)$

$= 16 - 7$

$= 9$

b. $y = (1 - 2x)(1 + 2x)$

$\frac{dy}{dx} = (1 - 2x)(2) + (-2)(1 + 2x)$

At $x = \frac{1}{2}$,

$\frac{dy}{dx} = (0)(2) - 2(2)$

$= -4$

c. $y = (3 - 2x - x^2)(x^2 + x - 2)$

$\frac{dy}{dx} = (3 - 2x - x^2)(2x + 1)$

$+ (-2 - 2x)(x^2 + x - 2)$

At $x = -2$,

$\frac{dy}{dx} = (3 + 4 - 4)(-4 + 1)$

$+ (-2 + 4)(4 - 2 - 2)$

$= (3)(-3) + (2)(0)$

$= -9$

d. $y = x^3(3x + 7)^2$

$\frac{dy}{dx} = 3x^2(3x + 7)^2 + x^3 6(3x + 7)$

At $x = -2$,

$\frac{dy}{dx} = 12(1)^2 + (-8)(6)(1)$

$= 12 - 48$

$= -36$

e. $y = (2x + 1)^5(3x + 2)^4, x = -1$

$\frac{dy}{dx} = 5(2x + 1)^4(2)(3x + 2)^4$

$+ (2x + 1)^5 4(3x + 2)^3(3)$

At $x = -1$,

$\frac{dy}{dx} = 5(-1)^4(2)(-1)^4$

$+ (-1)^5(4)(-1)^3(3)$

$= 10 + 12$

$= 22$

f. $y = x(5x - 2)(5x + 2)$

$= x(25x^2 - 4)$

$\frac{dy}{dx} = x(50x) + (25x^2 - 4)(1)$

At $x = 3$,

$\frac{dy}{dx} = 3(150) + (25 \cdot 9 - 4)$

$= 450 + 221$

$= 671$

6. Tangent to $y = (x^3 - 5x + 2)(3x^2 - 2x)$ at $(1, -2)$

$\frac{dy}{dx} = (3x^2 - 5)(3x^2 - 2x)$

$+ (x^3 - 5x + 2)(6x - 2)$

when $x = 1$,

$\frac{dy}{dx} = (-2)(1) + (-2)(4)$

$= -2 + -8$

$= -10$

Slope of the tangent at $(1, -2)$ is -10.

The equation is $y + 2 = -10(x - 1)$;

$10x + y - 8 = 0$.

7. a. $y = 2(x - 29)(x + 1)$

$\frac{dy}{dx} = 2(x - 29)(1) + 2(1)(x + 1)$

$2x - 58 + 2x + 2 = 0$

$4x - 56 = 0$

$4x = 56$

$x = 14$

Point of horizontal tangency is $(14, -450)$.

b. $y = (x^2 + 2x + 1)(x^2 + 2x + 1)$

$= (x^2 + 2x + 1)^2$

$\frac{dy}{dx} = 2(x^2 + 2x + 1)(2x + 2)$

$(x^2 + 2x + 1)(2x + 2) = 0$

$2(x + 1)(x + 1)(x + 1) = 0$

$x = -1$

Point of horizontal tangency is $(-1, 0)$.

8. a. $y = (x + 1)^3(x + 4)(x - 3)^2$

$\frac{dy}{dx} = 3(x + 1)^2(x + 4)(x - 3)^2$

$+ (x + 1)^3(1)(x - 3)^2$

$+ (x + 1)^3(x + 4)[2(x - 3)]$

b. $y = x^2(3x^2+4)^2(3-x^3)^4$

$$\frac{dy}{dx} = 2x(3x^2+4)^2(3-x^3)^4$$
$$+ x^2[2(3x^2+4)(6x)](3-x^3)^4$$
$$+ x^2(3x^2+4)^2[4(3-x^3)^3(-3x^2)]$$

9. $V(t) = 75\left(1 - \frac{t}{24}\right)^2,\ 0 \le t \le 24$

$75\text{ L} \times 60\% = 45\text{ L}$

$$\text{Set } \frac{45}{75} = \left(1 - \frac{t}{24}\right)^2$$
$$\pm\sqrt{\frac{3}{5}} = 1 - \frac{t}{24}$$
$$t = \left(\pm\sqrt{\frac{3}{5}} - 1\right)(-24)$$

$t \doteq 42.590$ (inadmissable) or $t \doteq 5.4097$

$$V(t) = 75\left(1 - \frac{t}{24}\right)^2$$
$$V(t) = 75\left(1 - \frac{t}{24}\right)\left(1 - \frac{t}{24}\right)$$
$$V'(t) = 75\left[\left(1 - \frac{t}{24}\right)\left(-\frac{1}{24}\right) + \left(-\frac{1}{24}\right)\left(1 - \frac{t}{24}\right)\right]$$
$$= (75)(2)\left(1 - \frac{t}{24}\right)\left(-\frac{1}{24}\right)$$

$V'(5.4097) = -4.84\text{ L/h}$

10. Determine the point of tangency, and then find the negative reciprocal of the slope of the tangent. Use this information to find the equation of the normal.

$$h(x) = 2x(x+1)^3(x^2+2x+1)^2$$
$$h'(x) = 2(x+1)^3(x^2+2x+1)^2$$
$$+ (2x)(3)(x+1)^2(x^2+2x+1)^2$$
$$+ 2x(x+1)^3\,2(x^2+2x+1)(2x+2)$$
$$h'(-2) = 2(-1)^3(1)^2$$
$$+ 2(-2)(3)(-1)^2(1)^2$$
$$+ 2(-2)(-1)^3(2)(1)(-2)$$
$$= -2 - 12 - 16$$
$$= -30$$

11.

a. $f(x) = g_1(x)g_2(x)g_3(x)\ldots\ g_{n-1}(x)g_n(x)$

$$f'(x) = g_1'(x)g_2(x)g_3(x)\ldots\ g_{n-1}(x)g_n(x)$$
$$+ g_1(x)g_2'(x)g_3(x)\ldots\ g_{n-1}(x)g_n(x)$$
$$+ g_1(x)g_2(x)g_3'(x)\ldots\ g_{n-1}(x)g_n(x)$$
$$+ \ldots + g_1(x)g_2(x)g_3(x)\ldots g_{n-1}(x)g_n'(x)$$

b. $f(x) = (1+x)(1+2x)(1+3x)\ldots(1+nx)$

$$f'(x) = 1(1+2x)(1+3x)\ldots(1+nx)$$
$$+ (1+x)(2)(1+3x)\ldots(1+nx)$$
$$+ (1+x)(1+2x)(3)\ldots(1+nx)$$
$$+ \ldots + (1+x)(1+2x)(1+3x)\ldots(n)$$
$$f'(0) = 1(1)(1)(1)\ldots(1)$$
$$+ 1(2)(1)(1)\ldots(1)$$
$$+ 1(1)(3)(1)\ldots(1)$$
$$+ \ldots + (1)(1)(1)\ldots(n)$$
$$= 1 + 2 + 3 + \ldots + n$$
$$f'(0) = \frac{n(n+1)}{2}$$

12. $f(x) = ax^2 + bx + c$

$f'(x) = 2ax + b \qquad (1)$

Horizontal tangent at $(-1, -8)$

$f'(x) = 0$ at $x = -1$

$-2a + b = 0$

Since $(2, 19)$ lies on the curve,

$4a + 2b + c = 19 \qquad (2)$

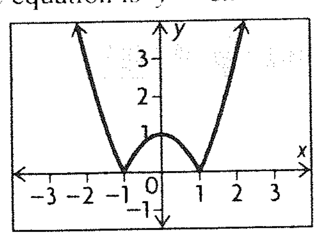
Since $(-1, -8)$ lies on the curve,

$a - b + c = -8 \qquad (3)$

$$4a + 2b + c = 19$$
$$-3a - 3b = -27$$
$$a + b = 9$$
$$-2a + b = 0$$
$$3a = 9$$
$$a = 3, \quad b = 6$$
$$3 - 6 + c = -8$$
$$c = -5$$

The equation is $y = 3x^2 + 6x - 5$.

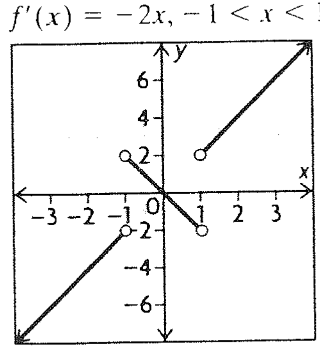
13.



a. $x = 1$ or $x = -1$

b. $f'(x) = 2x,\ x < -1$ or $x > 1$

$f'(x) = -2x,\ -1 < x < 1$



c. $f'(-2) = 2(-2) = -4$

$f'(0) = -2(0) = 0$

$f'(3) = 2(3) = 6$

14. $y = \frac{16}{x^2} - 1$

$\frac{dy}{dx} = -\frac{32}{x^3}$

Slope of the line is 4.

$-\frac{32}{x^3} = 4$

$4x^3 = -32$

$x^3 = -8$

$x = -2$

$y = \frac{16}{4} - 1$

$= 3$

Point is at $(-2, 3)$.

Find intersection of line and curve:

$4x - y + 11 = 0$

$y = 4x + 11$

Substitute,

$4x + 11 = \frac{16}{x^2} - 1$

$4x^3 + 11x^2 = 16 - x^2$ or $4x^3 + 12x^2 - 16 = 0$.

Let $x = -2$

$RS = 4(-2)^3 + 12(-2)^2 - 16$

$= 0$

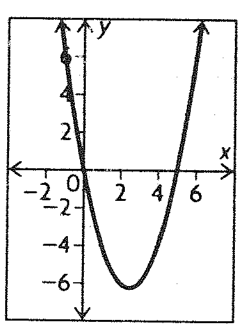
Since $x = -2$ satisfies the equation, therefore it is a solution.

When $x = -2$, $y = 4(-2) + 11 = 3$.

Intersection point is $(-2, 3)$. Therefore, the line is tangent to the curve.

Mid-Chapter Review, pp. 92–93

1. a.



b. $f'(x) = \lim_{h\to0}\frac{((x + h)^2 - 5(x + h)) - (x^2 - 5x)}{h}$

$= \lim_{h\to0}\frac{x^2 + 2hx + h^2 - 5x - 5h - x^2 + 5x}{h}$

$= \lim_{h\to0}\frac{h^2 + 2hx - 5h}{h}$

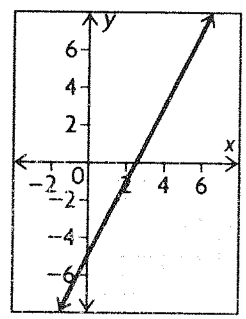
$= \lim_{h\to0}\frac{h(h + 2x - 5)}{h}$

$= 2x - 5$

Use the derivative function to calculate the slopes of the tangents.

x	Slope of Tangent $f'(x)$
0	−5
1	−3
2	−1
3	1
4	3
5	5

c.



d. $f(x)$ is quadratic; $f'(x)$ is linear.

2. a. $f'(x) = \lim_{h\to0}\frac{(6(x + h) + 15) - (6x + 15)}{h}$

$= \lim_{h\to0}\frac{6h}{h}$

$= \lim_{h\to0} 6$

$= 6$

b. $f'(x) = \lim_{h\to0}\frac{(2(x + h)^2 - 4) - (2x^2 - 4)}{h}$

$= \lim_{h\to0} 2\frac{(x + h)^2 - x^2}{h}$

$= \lim_{h\to0} 2\frac{((x + h) - x)((x + h) + x)}{h}$

$= \lim_{h\to0} 2\frac{h(2x + h)}{h}$

$= \lim_{h\to0} 2(2x + h)$

$= 4x$

c. $f'(x) = \lim_{h\to0}\frac{\frac{5}{(x + h) + 5} - \frac{5}{x + 5}}{h}$

$= \lim_{h\to0}\frac{5(x + 5) - 5((x + h) + 5)}{((x + h) + 5)(x + 5)h}$

$= \lim_{h\to0}\frac{-5h}{((x + h) + 5)(x + 5)h}$

$$= \lim_{h\to 0} \frac{-5}{((x+h)+5)(x+5)}$$

$$= \frac{-5}{(x+5)^2}$$

d. $f'(x) = \lim_{h\to 0} \frac{\sqrt{(x+h)-2} - \sqrt{x-2}}{h}$

$$= \lim_{h\to 0}\left[\frac{\sqrt{(x+h)-2} - \sqrt{x-2}}{h} \times \frac{\sqrt{(x+h)-2} + \sqrt{x-2}}{\sqrt{(x+h)-2} + \sqrt{x-2}}\right]$$

$$= \lim_{h\to 0} \frac{((x+h)-2)-(x-2)}{h(\sqrt{(x+h)-2} + \sqrt{x-2})}$$

$$= \lim_{h\to 0} \frac{h}{h(\sqrt{(x+h)-2} + \sqrt{x-2})}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{(x+h)-2} + \sqrt{x-2}}$$

$$= \frac{1}{2\sqrt{x-2}}$$

3. a. $y' = 2x - 4$

When $x = 1$,

$y' = 2(1) - 4$

$= -2.$

When $x = 1$,

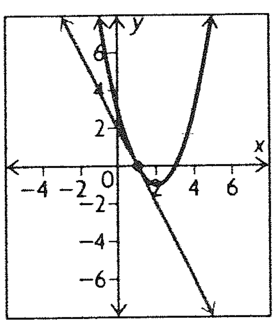
$y = (1)^2 - 4(1) + 3$

$= 0.$

Equation of the tangent line:

$y - 0 = -2(x - 1)$, or $y = -2x + 2$

b.



4. a. $\frac{dy}{dx} = 24x^3$

b. $\frac{dy}{dx} = 5x^{-\frac{1}{2}}$

$$= \frac{5}{\sqrt{x}}$$

c. $g'(x) = -6x^{-4}$

$$= -\frac{6}{x^4}$$

d. $\frac{dy}{dx} = 5 - 6x^{-3}$

$$= 5 - \frac{6}{x^3}$$

e. $\frac{dy}{dt} = 2(11t + 1)(11)$

$= 242t + 22$

f. $y = 1 - \frac{1}{x}$

$= 1 - x^{-1}$

$\frac{dy}{dx} = x^{-2}$

$$= \frac{1}{x^2}$$

5. $f'(x) = 8x^3$

$8x^3 = 1$

$x^3 = \frac{1}{8}$

$x = \frac{1}{2}$

$f\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^4$

$= \frac{1}{8}$

Equation of the tangent line:

$y - \frac{1}{8} = 1\left(x - \frac{1}{2}\right)$, or $y = x - \frac{3}{8}$

6. a. $f'(x) = 8x - 7$

b. $f'(x) = -6x^2 + 8x + 5$

c. $f(x) = 5x^{-2} - 3x^{-3}$

$f'(x) = -10x^{-3} + 9x^{-4}$

$$= -\frac{10}{x^3} + \frac{9}{x^4}$$

d. $f(x) = x^{\frac{1}{2}} + x^{\frac{1}{3}}$

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{3}x^{-\frac{2}{3}}$

$$= \frac{1}{2x^{\frac{1}{2}}} + \frac{1}{3x^{\frac{2}{3}}}$$

e. $f(x) = 7x^{-2} - 3x^{\frac{1}{2}}$

$f'(x) = -14x^{-3} - \frac{3}{2}x^{-\frac{1}{2}}$

$$= -\frac{14}{x^3} - \frac{3}{2x^{\frac{1}{2}}}$$

f. $f'(x) = 4x^{-2} + 5$

$$= \frac{4}{x^2} + 5$$

7. a. $y' = -6x + 6$

When $x = 1$,

$$\begin{aligned} y' &= -6(1) + 6 \\ &= 0. \end{aligned}$$

When $x = 1$,

$$\begin{aligned} y &= -3(1^2) + 6(1) + 4 \\ &= 7. \end{aligned}$$

Equation of the tangent line:

$y - 7 = 0(x - 1)$, or

$y = 7$

b. $y = 3 - 2x^{\frac{1}{2}}$

$$\begin{aligned} y' &= -x^{-\frac{1}{2}} \\ &= \frac{-1}{\sqrt{x}} \end{aligned}$$

When $x = 9$,

$$\begin{aligned} y' &= \frac{-1}{\sqrt{9}} \\ &= -\frac{1}{3}. \end{aligned}$$

When $x = 9$,

$$\begin{aligned} y &= 3 - 2\sqrt{9} \\ &= -3. \end{aligned}$$

Equation of the tangent line:

$y - (-3) = -\frac{1}{3}(x - 9)$, or $y = -\frac{1}{3}x$

c. $f'(x) = -8x^3 + 12x^2 - 4x - 8$

$$\begin{aligned} f'(3) &= -8(3)^3 + 12(3)^2 - 4(3) - 8 \\ &= -216 + 108 - 12 - 8 \\ &= -218 \\ f(3) &= -2(3)^4 + 4(3)^3 - 2(3)^2 - 8(3) + 9 \\ &= -162 + 108 - 18 - 24 + 9 \\ &= -87 \end{aligned}$$

Equation of the tangent line:

$y - (-87) = -128(x - 3)$, or

$y = -128x + 297$

8. a.
$$\begin{aligned} f'(x) &= \frac{d}{dx}(4x^2 - 9x)(3x^2 + 5) \\ &\quad + (4x^2 - 9x)\frac{d}{dx}(3x^2 + 5) \\ &= (8x - 9)(3x^2 + 5) + (4x^2 - 9x)(6x) \\ &= 24x^3 - 27x^2 + 40x - 45 \\ &\quad + 24x^3 - 54x^2 \\ &= 48x^3 - 81x^2 + 40x - 45 \end{aligned}$$

b.
$$\begin{aligned} f'(t) &= \frac{d}{dt}(-3t^2 - 7t + 8)(4t - 1) \\ &\quad + (-3t^2 - 7t + 8)\frac{d}{dt}(4t - 1) \\ &= (-6t - 7)(4t - 1) \\ &\quad + (-3t^2 - 7t + 8)(4) \\ &= -24t^2 - 28t + 6t + 7 - 12t^2 - 28t + 32 \\ &= -36t^2 - 50t + 39 \end{aligned}$$

c.
$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(3x^2 + 4x - 6)(2x^2 - 9) \\ &\quad + (3x^2 + 4x - 6)\frac{d}{dx}(2x^2 - 9) \\ &= (6x + 4)(2x^2 - 9) + (3x^2 + 4x - 6)(4x) \\ &= 12x^3 - 54x + 8x^2 - 36 + 12x^3 \\ &\quad + 16x^2 - 24x \\ &= 24x^3 + 24x^2 - 78x - 36 \end{aligned}$$

d.
$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(3 - 2x^3)^2(3 - 2x^3) \\ &\quad + (3 - 2x^3)^2\frac{d}{dx}(3 - 2x^3) \\ &= \left[\frac{d}{dx}(3 - 2x^3)(3 - 2x^3)\right. \\ &\quad \left. + (3 - 2x^3)\frac{d}{dx}(3 - 2x^3)\right](3 - 2x^3) \\ &\quad + (3 - 2x^3)^2(-6x^2) \\ &= [2(-6x^2)(3 - 2x^3)](3 - 2x^3) \\ &\quad + (3 - 2x^3)^2(-6x^2) \\ &= 3(3 - 2x^3)^2(-6x^2) \\ &= (3 - 2x^3)^2(-18x^2) \\ &= (9 - 12x^3 + 4x^6)(-18x^2) \\ &= -162x^2 + 216x^5 - 72x^8 \end{aligned}$$

9.
$$\begin{aligned} y' &= \frac{d}{dx}(5x^2 + 9x - 2)(-x^2 + 2x + 3) \\ &\quad + (5x^2 + 9x - 2)\frac{d}{dx}(-x^2 + 2x + 3) \\ &= (10x + 9)(-x^2 + 2x + 3) \\ &\quad + (5x^2 + 9x - 2)(2 - 2x) \\ y'(1) &= (10(1) + 9)(-(1)^2 + 2(1) + 3) \\ &\quad + (5(1)^2 + 9(1) - 2)(2 - 2(1)) \\ &= (19)(4) \\ &= 76 \end{aligned}$$

Equation of the tangent line:

$y - 48 = 76(x - 1)$, or $76x - y - 28 = 0$

10.
$$\begin{aligned} \frac{dy}{dx} &= 2\frac{d}{dx}(x - 1)(5 - x) \\ &\quad + 2(x - 1)\frac{d}{dx}(5 - x) \\ &= 2(5 - x) - 2(x - 1) \\ &= 12 - 4x \end{aligned}$$

The tangent line is horizontal when $\frac{dy}{dx} = 0$.

$$\begin{aligned} 12 - 4x &= 0 \\ 12 &= 4x \\ x &= 3 \end{aligned}$$

When $x = 3$,

$y = 2((3) - 1)(5 - (3))$
$= 8$.

Point where tangent line is horizontal: (3, 8)

11. $\frac{dy}{dx} = \lim_{h \to 0}\left[\frac{(5(x+h)^2 - 8(x+h) + 4)}{h} - \frac{(5x^2 - 8x + 4)}{h}\right]$

$= \lim_{h\to 0}\frac{5(x+h)^2 - 5x^2 - 8h}{h}$

$= \lim_{h\to 0}\frac{5((x+h) - x)((x+h)+x) - 8h}{h}$

$= \lim_{h\to 0}\frac{5h(2x+h) - 8h}{h}$

$= \lim_{h\to 0}(5(2x+h) - 8)$

$= 10x - 8$

12. $V(t) = 500\left(1 - \frac{t}{90}\right)^2, 0 \le t \le 90$

a. After 1 h, $t = 60$, and the volume is

$V(60) = 500\left(1 - \frac{60}{90}\right)^2$

$= 500\left(\frac{30}{90}\right)^2$

$= 500\left(\frac{1}{3}\right)^2$

$= \frac{500}{9}$ L

b. $V(0) = 500(1 - 0)^2 = 500$ L

$V(60) = \frac{500}{9}$ L

The average rate of change of volume with respect to time from 0 min to 60 min is

$\frac{\Delta V}{\Delta t} = \frac{\frac{500}{9} - 500}{60 - 0}$

$= \frac{\frac{-8}{9}(500)}{60}$

$= -\frac{200}{27}$ L/min

c. Calculate $V'(t)$:

$V'(t) = \lim_{h\to 0}\frac{V(t+h) - V(t)}{h}$

$= \lim_{h\to 0}\frac{500\left(1 - \frac{t+h}{90}\right)^2 - 500\left(-1 + \frac{t}{90}\right)^2}{h}$

$= \lim_{h\to 0}\frac{500\left(1 - \frac{t+h}{90} - 1 + \frac{t}{90}\right)}{h}$

$\times \frac{\left(1 - \frac{t+h}{90} + 1 - \frac{t}{90}\right)}{h}$

$= \lim_{h\to 0}\frac{500\left(-\frac{h}{90}\right)\left(2 - \frac{2t+h}{90}\right)}{h}$

$= \lim_{h\to 0} -\frac{500}{90}\left(2 - \frac{2t+h}{90}\right)$

$= \frac{-50}{9}\left(2 - \frac{2t}{90}\right)$

$= \frac{-900 + 10t}{81}$

Then,

$V'(30) = \frac{-900 + 10(30)}{81}$

$= -\frac{200}{27}$ L/min

13. $V(r) = \frac{4}{3}\pi r^3$

a. $V(10) = \frac{4}{3}\pi(10)^3$

$= \frac{4}{3}\pi(1000)$

$= \frac{4000}{3}\pi$

$V(15) = \frac{4}{3}\pi(15)^3$

$= \frac{4}{3}\pi(3375)$

$= 4500\pi$

Then, the average rate of change of volume with respect to radius is

$\frac{\Delta V}{\Delta r} = \frac{4500\pi - \frac{4000}{3}\pi}{15 - 10}$

$= \frac{500\pi\left(9 - \frac{8}{3}\right)}{5}$

$= 100\pi\left(\frac{19}{3}\right)$

$= \frac{1900}{3}\pi$ cm³/cm

b. First calculate $V'(r)$:

$V'(r) = \lim_{h\to 0}\frac{V(r+h) - V(r)}{h}$

$= \lim_{h\to 0}\frac{\frac{4}{3}\pi[(r+h)^3 - r^3]}{h}$

$= \lim_{h\to 0}\frac{\frac{4}{3}\pi\left(r^3 + 3r^2h + 3rh^2 + h^3 - r^3\right)}{h}$

$= \lim_{h\to 0}\frac{\frac{4}{3}\pi\left(3r^2h + 3rh^2 + h^3\right)}{h}$

$$= \lim_{h \to 0} \frac{4}{3}\pi(3r^2 + 3rh + h^2)$$
$$= \frac{4}{3}\pi(3r^2 + 3r(0) + (0)^2)$$
$$= 4\pi r^2$$

Then, $V'(8) = 4\pi(8)^2$
$= 4\pi(64)$
$= 256\pi \text{ cm}^3/\text{cm}$

14. This statement is always true. A cubic polynomial function will have the form $f(x) = ax^3 + bx^2 + cx + d$, $a \neq 0$. So the derivative of this cubic is $f'(x) = 3ax^2 + 2bx + c$, and since $3a \neq 0$, this derivative is a quadratic polynomial function. For example, if $f(x) = x^3 + x^2 + 1$,
we get
$f'(x) = 3x^2 + 2x$,
and if
$f(x) = 2x^3 + 3x^2 + 6x + 2$,
we get
$f'(x) = 6x^2 + 6x + 6$

15. $y = \frac{x^{2a+3b}}{x^{a-b}}$, $a, b \in \mathbf{I}$

Simplifying,
$y = x^{2a+3b-(a-b)} = x^{a+4b}$
Then,
$y' = (a + 4b)^{a+4b-1}$

16. a. $f(x) = -6x^3 + 4x - 5x^2 + 10$
$f'(x) = -18x^2 + 4 - 10x$
Then, $f'(x) = -18(3)^2 + 4 - 10(3)$
$= -188$

b. $f'(3)$ is the slope of the tangent line to $f(x)$ at $x = 3$ and the rate of change in the value of $f(x)$ with respect to x at $x = 3$.

17. a. $P(t) = 100 + 120t + 10t^2 + 2t^3$
$P(t) = 100 + 120t + 10t^2 + 2t^3$
$P(0) = 100 + 120(0) + 10(0)^2 + 2(0)^3$
$= 100$ bacteria

b. At 5 h, the population is
$P(5) = 100 + 120(5) + 10(5)^2 + 2(5)^3$
$= 1200$ bacteria

c. $P'(t) = 120 + 20t + 6t^2$
At 5 h, the colony is growing at
$P'(5) = 120 + 20(5) + 6(5)^2$
$= 370$ bacteria/h

18. $C(t) = \frac{100}{t}$, $t > 2$

Simplifying, $C(t) = 100t^{-1}$.

Then, $C'(t) = -100t^{-2} = -\frac{100}{t^2}$.

$C'(5)$	$C'(50)$	$C'(100)$
$= -\frac{100}{(5)^2}$	$= -\frac{100}{(50)^2}$	$= -\frac{100}{(100)^2}$
$= -\frac{100}{25}$	$= -\frac{100}{2500}$	$= -\frac{1}{100}$
$= -4$	$= -0.04$	$= -0.01$

These are the rates of change of the percentage with respect to time at 5, 50, and 100 min. The percentage of carbon dioxide that is released per unit time from the pop is decreasing. The pop is getting flat.

2.4 The Quotient Rule, pp. 97–98

1. For x, a, b real numbers,
$x^a x^b = x^{a+b}$
For example,
$x^9 x^{-6} = x^3$
Also,
$(x^a)^b = x^{ab}$
For example,
$(x^2)^3 = x^6$
Also,
$\frac{x^a}{x^b} = x^{a-b}$, $x \neq 0$
For example,
$\frac{x^5}{x^3} = x^2$

2.

Function	Rewrite	Differentiate and Simplify, If Necessary
$f(x) = \frac{x^2 + 3x}{x}$, $x \neq 0$	$f(x) = x + 3$	$f'(x) = 1$
$g(x) = \frac{3x^{\frac{5}{3}}}{x}$, $x \neq 0$	$g(x) = 3x^{\frac{2}{3}}$	$g'(x) = 2x^{-\frac{1}{3}}$
$h(x) = \frac{1}{10x^5}$, $x \neq 0$	$h(x) = \frac{1}{10}x^{-5}$	$h'(x) = \frac{-1}{2}x^{-6}$
$y = \frac{8x^3 + 6x}{2x}$, $x \neq 0$	$y = 4x^2 + 3$	$\frac{dy}{dx} = 8x$
$s = \frac{t^2 - 9}{t - 3}$, $t \neq 3$	$s = t + 3$	$\frac{ds}{dt} = 1$

3. In the previous problem, all of these rational examples could be differentiated via the power rule after a minor algebraic simplification.
A second approach would be to rewrite a rational example

$h(x) = \frac{f(x)}{g(x)}$

using the exponent rules as

$h(x) = f(x)(g(x))^{-1}$,

and then apply the product rule for differentiation (together with the power of a function rule to find $h'(x)$.
A third (and perhaps easiest) approach would be to just apply the quotient rule to find $h'(x)$.

4. a. $h'(x) = \frac{(x+1)(1) - x(1)}{(x+1)^2}$

$= \frac{1}{(x+1)^2}$

b. $h'(t) = \frac{(t+5)(2) - (2t-3)(1)}{(t+5)^2}$

$= \frac{13}{(t+5)^2}$

c. $h'(x) = \frac{(2x^2-1)(3x^2) - x^3(4x)}{(2x^2-1)^2}$

$= \frac{2x^4 - 3x^2}{(2x^2-1)^2}$

d. $h'(x) = \frac{(x^2+3)(0) - 1(2x)}{(x^2+3)^2}$

$= \frac{-2x}{(x^2+3)^2}$

e. $y = \frac{x(3x+5)}{(1-x^2)} = \frac{3x^2+5x}{1-x^2}$

$\frac{dy}{dx} = \frac{(6x+5)(1-x^2) - (3x^2+5x)(-2x)}{(1-x^2)^2}$

$= \frac{6x + 5 - 6x^3 - 5x^2 + 6x^3 + 10x^2}{(1-x^2)^2}$

$= \frac{5x^2 + 6x + 5}{(1-x^2)^2}$

f. $\frac{dy}{dx} = \frac{(x^2+3)(2x-1) - (x^2-x+1)(2x)}{(x^2+3)^2}$

$= \frac{2x^3 + 6x - x^2 - 3 - 2x^3 + 2x^2 - 2x}{(x^2+3)^2}$

$= \frac{x^2 + 4x - 3}{(x^2+3)^2}$

5. a. $y = \frac{3x+2}{x+5}, x = -3$

$\frac{dy}{dx} = \frac{(x+5)(3) - (3x+2)(1)}{(x+5)^2}$

At $x = -3$:

$\frac{dy}{dx} = \frac{(2)(3) - (-7)(1)}{(2)^2}$

$= \frac{13}{4}$

b. $y = \frac{x^3}{x^2+9}, x = 1$

$\frac{dy}{dx} = \frac{(x^2+9)(3x^2) - (x^3)(2x)}{(x^2+9)^2}$

At $x = 1$:

$\frac{dy}{dx} = \frac{(10)(3) - (1)(2)}{(10)^2}$

$= \frac{28}{100}$

$= \frac{7}{25}$

c. $y = \frac{x^2-25}{x^2+25}, x = 2$

$\frac{dy}{dx} = \frac{2x(x^2+25) - (x^2-25)(2x)}{(x^2+25)^2}$

At $x = 2$:

$\frac{dy}{dx} = \frac{4(29) - (-21)(4)}{(29)^2}$

$= \frac{116 + 84}{29^2}$

$= \frac{200}{841}$

d. $y = \frac{(x+1)(x+2)}{(x-1)(x-2)}, x = 4$

$= \frac{x^2 + 3x + 2}{x^2 - 3x + 2}$

$\frac{dy}{dx} = \frac{(2x+3)(x^2-3x+2)}{(x-1)^2(x-2)^2}$

$- \frac{(x^2+3x+2)(2x-3)}{(x-1)^2(x-2)^2}$

At $x = 4$:

$\frac{dy}{dx} = \frac{(11)(6) - (30)(5)}{(9)(4)}$

$= -\frac{84}{36}$

$= -\frac{7}{3}$

6. $y = \dfrac{x^3}{x^2 - 6}$

$\dfrac{dy}{dx} = \dfrac{3x^2(x^2 - 6) - x^3(2x)}{(x^2 - 6)^2}$

At (3, 9):

$\dfrac{dy}{dx} = \dfrac{3(9)(3) - (27)(6)}{(3)^2}$

$= 9 - 18$

$= -9$

The slope of the tangent to the curve at (3, 9) is -9.

7. $y = \dfrac{3x}{x - 4}$

$\dfrac{dy}{dx} = \dfrac{3(x - 4) - 3x}{(x - 4)^2} = -\dfrac{12}{(x - 4)^2}$

Slope of the tangent is $-\frac{12}{25}$.

Therefore, $\dfrac{12}{(x - 4)^2} = \frac{12}{25}$

$x - 4 = 5$ or $x - 4 = -5$

$x = 9$ or $x = -1$

Points are $\left(9, \frac{27}{5}\right)$ and $\left(-1, \frac{3}{5}\right)$.

8. $f(x) = \dfrac{5x + 2}{x + 2}$

$f'(x) = \dfrac{(x + 2)(5) - (5x + 2)(1)}{(x + 2)^2}$

$f'(x) = \dfrac{8}{(x + 2)^2}$

Since $(x + 2)^2$ is positive or zero for all $x \in \mathbf{R}$, $\dfrac{8}{(x + 2)^2} > 0$ for $x \neq -2$. Therefore, tangents to the graph of $f(x) = \dfrac{5x + 2}{x + 2}$ do not have a negative slope.

9. a. $y = \dfrac{2x^2}{x - 4}, x \neq 4$

$\dfrac{dy}{dx} = \dfrac{(x - 4)(4x) - (2x^2)(1)}{(x - 4)^2}$

$= \dfrac{4x^2 - 16x - 2x^2}{(x - 4)^2}$

$= \dfrac{2x^2 - 16x}{(x - 4)^2}$

$= \dfrac{2x(x - 8)}{(x - 4)^2}$

Curve has horizontal tangents when $\dfrac{dy}{dx} = 0$, or when $x = 0$ or 8. At $x = 0$:

$y = \dfrac{0}{-4}$

$= 0$

At $x = 8$:

$y = \dfrac{2(8)^2}{4}$

$= 32$

So the curve has horizontal tangents at the points (0, 0) and (8, 32).

b. $y = \dfrac{x^2 - 1}{x^2 + x - 2}$

$= \dfrac{(x - 1)(x + 1)}{(x + 2)(x - 1)}$

$= \dfrac{x + 1}{x + 2}, x \neq 1$

$\dfrac{dy}{dx} = \dfrac{(x + 2) - (x + 1)}{(x + 2)^2}$

$= \dfrac{1}{(x + 2)^2}$

Curve has horizontal tangents when $\dfrac{dy}{dx} = 0$.

No value of x will produce a slope of 0, so there are no horizontal tangents.

10. $p(t) = 1000\left(1 + \dfrac{4t}{t^2 + 50}\right)$

$p'(t) = 1000\left(\dfrac{4(t^2 + 50) - 4t(2t)}{(t^2 + 50)^2}\right)$

$= \dfrac{1000(200 - 4t^2)}{(t^2 + 50)^2}$

$p'(1) = \dfrac{1000(196)}{(51)^2} = 75.36$

$p'(2) = \dfrac{1000(184)}{(54)^2} = 63.10$

Population is growing at a rate of 75.4 bacteria per hour at $t = 1$ and at 63.1 bacteria per hour at $t = 2$.

11. $y = \dfrac{x^2 - 1}{3x}$

$= \dfrac{1}{3}x - \dfrac{1}{3}x^{-1}$

$\dfrac{dy}{dx} = \dfrac{1}{3} + \dfrac{1}{3}x^{-2}$

$= \dfrac{1}{3} + \dfrac{1}{3x^2}$

At $x = 2$:

$y = \dfrac{(2)^2 - 1}{3(2)}$

$= \dfrac{1}{2}$

and

$\dfrac{dy}{dx} = \dfrac{1}{3} + \dfrac{1}{3(2)^2}$

$= \frac{1}{3} + \frac{1}{12}$

$= \frac{5}{12}$

So the equation of the tangent to the curve at $x = 2$ is:

$y - \frac{1}{2} = \frac{5}{12}(x - 2)$, or $5x - 12y - 4 = 0$.

12. a. $s(t) = \frac{10(6 - t)}{t + 3}$, $0 \le t \le 6$, $t = 0$.

$s(0) = 20$

The boat is initially 20 m from the dock.

b. $v(t) = s'(t) = 10\left[\frac{(t + 3)(-1) - (6 - t)(1)}{(t + 3)^2}\right]$

$v(t) = \frac{-90}{(t + 3)^2}$

At $t = 0$, $v(0) = -10$, the boat is moving towards the dock at a speed of 10 m/s. When $s(t) = 0$, the boat will be at the dock.

$\frac{10(6 - t)}{t + 3} = 0$, $t = 6$.

$v(6) = \frac{-90}{9^2} = -\frac{10}{9}$

The speed of the boat when it bumps into the dock is $\frac{10}{9}$ m/s.

13. a. i. $t = 0$

$r(0) = \frac{1 + 2(0)}{1 + 0}$

$= 1$ cm

ii. $\frac{1 + 2t}{1 + t} = 1.5$

$1 + 2t = 1.5(1 + t)$

$1 + 2t = 1.5 + 1.5t$

$0.5t = 0.5$

$t = 1s$

iii. $r'(t) = \frac{(1 + t)(2) - (1 + 2t)(1)}{(1 + t)^2}$

$= \frac{2 + 2t - 1 - 2t}{(1 + t)^2}$

$= \frac{1}{(1 + t)^2}$

$r'(1.5) = \frac{1}{(1 + 1)^2}$

$= \frac{1}{4}$

$= 0.25$ cm/s

b. No, the radius will never reach 2 cm, because $y = 2$ is a horizontal asymptote of the graph of the function. Therefore, the radius approaches but never equals 2 cm.

14. $f(x) = \frac{ax + b}{(x - 1)(x - 4)}$

$f'(x) = \frac{(x - 1)(x - 4)(a)}{(x - 1)^2(x - 4)^2}$

$- \frac{(ax + b)\frac{d}{dx}[(x - 1)(x - 4)]}{(x - 1)^2(x - 4)^2}$

$= \frac{(x - 1)(x - 4)(a)}{(x - 1)^2(x - 4)^2}$

$- \frac{(ax + b)[(x - 1) + (x - 4)]}{(x - 1)^2(x - 4)^2}$

$= \frac{(x^2 - 5x + 4)(a) - (ax + b)(2x - 5)}{(x - 1)^2(x - 4)^2}$

$= \frac{-ax^2 - 2bx + 4a + 5b}{(x - 1)^2(x - 4)^2}$

Since the point $(2, -1)$ is on the graph (as it's on the tangent line) we know that

$-1 = f(2)$

$= \frac{2a + b}{(1)(-2)}$

$2 = 2a + b$

$b = 2 - 2a$

Also, since the tangent line is horizontal at $(2, -1)$, we know that

$0 = f'(2)$

$= \frac{-a(2)^2 - 2b(2) + 4a + 5b}{(1)^2(-2)^2}$

$b = 0$

$0 = 2 - 2a$

$a = 1$

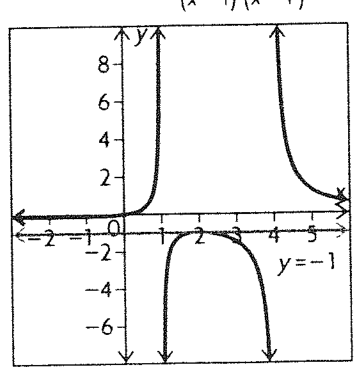
So we get

$f(x) = \frac{x}{(x - 1)(x - 4)}$

Since the tangent line is horizontal at the point $(2, -1)$, the equation of this tangent line is $y - (-1) = 0(x - 2)$, or $y = -1$

Here are the graphs of both $f(x)$ and this horizontal tangent line:

$f(x) = \frac{x}{(x-1)(x-4)}$



15. $c'(t) = \frac{(2t^2 + 7)(5) - (5t)(4t)}{(2t^2 + 7)^2}$

$= \frac{10t^2 + 35 - 20t^2}{(2t^2 + 7)^2}$

$= \frac{-10t^2 + 35}{(2t^2 + 7)^2}$

Set $c'(t) = 0$ and solve for t.

$\frac{-10t^2 + 35}{(2t^2 + 7)^2} = 0$

$-10t^2 + 35 = 0$

$10t^2 = 35$

$t^2 = 3.5$

$t = \pm\sqrt{3.5}$

$t \doteq \pm 1.87$

To two decimal places, $t = -1.87$ or $t = 1.87$, because $s'(t) = 0$ for these values. Reject the negative root in this case because time is positive ($t \geq 0$). Therefore, the concentration reaches its maximum value at $t = 1.87$ hours.

16. When the object changes direction, its velocity changes sign.

$s'(t) = \frac{(t^2 + 8)(1) - t(2t)}{(t^2 + 8)^2}$

$= \frac{t^2 + 8 - 2t^2}{(t^2 + 8)^2}$

$= \frac{-t^2 + 8}{(t^2 + 8)^2}$

solve for t when $s'(t) = 0$.

$\frac{-t^2 + 8}{(t^2 + 8)^2} = 0$

$-t^2 + 8 = 0$

$t^2 = 8$

$t = \pm\sqrt{8}$

$t \doteq \pm 2.83$

To two decimal places, $t = 2.83$ or $t = -2.83$, because $s'(t) = 0$ for these values. Reject the negative root because time is positive ($t \geq 0$). The object changes direction when $t = 2.83$ s.

17. $f(x) = \frac{ax + b}{cx + d}, x \neq -\frac{d}{c}$

$f'(x) = \frac{(cx + d)(a) - (ax + b)(c)}{(cx + d)^2}$

$f'(x) = \frac{ad - bc}{(cx + d)^2}$

For the tangents to the graph of $y = f(x)$ to have positive slopes, $f'(x) > 0$. $(cx + d)^2$ is positive for all $x \in \mathbf{R}$. $ad - bc > 0$ will ensure each tangent has a positive slope.

2.5 The Derivatives of Composite Functions, pp. 105–106

1. $f(x) = \sqrt{x}, g(x) = x^2 - 1$

a. $f(g(1)) = f(1 - 1)$

$= f(0)$

$= 0$

b. $g(f(1)) = g(1)$

$= 0$

c. $g(f(0)) = g(0)$

$= 0 - 1$

$= -1$

d. $f(g(-4)) = f(16 - 1)$

$= f(15)$

$= \sqrt{15}$

e. $f(g(x)) = f(x^2 - 1)$

$= \sqrt{x^2 - 1}$

f. $g(f(x)) = g(\sqrt{x})$

$= (\sqrt{x})^2 - 1$

$= x - 1$

2. a. $f(x) = x^2, g(x) = \sqrt{x}$

$(f \circ g)(x) = f(g(x))$

$= f(\sqrt{x})$

$= (\sqrt{x})^2$

$= x$

Domain $= \{x \geq 0\}$

$(g \circ f)(x) = g(f(x))$

$= g(x^2)$

$= \sqrt{x^2}$

$= |x|$

Domain $= \{x \in \mathbf{R}\}$

The composite functions are not equal for negative x-values (as $(f \circ g)$ is not defined for these x), but are equal for non-negative x-values.

b. $f(x) = \frac{1}{x}, g(x) = x^2 + 1$

$(f \circ g)(x) = f(g(x))$

$= f(x^2 + 1)$

$= \frac{1}{x^2 + 1}$

Domain $= \{x \in \mathbf{R}\}$

$(g \circ f)(x) = g(f(x))$

$= g\left(\frac{1}{x}\right)$

$= \left(\frac{1}{x}\right)^2 + 1$

$= \frac{1}{x^2} + 1$

Domain $= \{x \neq 0\}$

The composite functions are not equal here. For instance, $(f \circ g)(1) = \frac{1}{2}$ and $(g \circ f)(1) = 2$.

c. $f(x) = \frac{1}{x}$, $g(x) = \sqrt{x + 2}$

$(f \circ g)(x) = f(g(x))$

$= f(\sqrt{x + 2})$

$= \frac{1}{\sqrt{x + 2}}$

Domain $= \{x > -2\}$

$(g \circ f)(x) = g(f(x))$

$= g\left(\frac{1}{x}\right)$

$= \sqrt{\frac{1}{x} + 2}$

The domain is all x such that

$\frac{1}{x} + 2 \geq 0$ and $x \neq 0$, or equivalently

Domain $= \{x \leq -\frac{1}{2}$ or $x > 0\}$

The composite functions are not equal here. For instance, $(f \circ g)(2) = \frac{1}{2}$ and $(g \circ f)(2) = \sqrt{\frac{5}{2}}$.

3. If $f(x)$ and $g(x)$ are two differentiable functions of x, and

$h(x) = (f \circ g)(x)$

$= f(g(x))$

is the composition of these two functions, then

$h'(x) = f'(g(x)) \cdot g'(x)$

This is known as the "chain rule" for differentiation of composite functions. For example, if $f(x) = x^{10}$ and $g(x) = x^2 + 3x + 5$, then $h(x) = (x^2 + 3x + 5)^{10}$, and so

$h'(x) = f'(g(x)) \cdot g'(x)$

$= 10(x^2 + 3x + 5)^9(2x + 3)$

As another example, if $f(x) = x^{\frac{2}{3}}$ and $g(x) = x^2 + 1$, then $h(x) = (x^2 + 1)^{\frac{2}{3}}$.

and so

$h'(x) = \frac{2}{3}(x^2 + 1)^{-\frac{1}{3}}(2x)$

4. a. $f(x) = (2x + 3)^4$

$f'(x) = 4(2x + 3)^3(2)$

$= 8(2x + 3)^3$

b. $g(x) = (x^2 - 4)^3$

$g'(x) = 3(x^2 - 4)^2(2x)$

$= 6x(x^2 - 4)^2$

c. $h(x) = (2x^2 + 3x - 5)^4$

$h'(x) = 4(2x^2 + 3x - 5)^3(4x + 3)$

d. $f(x) = (\pi^2 - x^2)^3$

$f'(x) = 3(\pi^2 - x^2)^2(-2x)$

$= -6x(\pi^2 - x^2)^2$

e. $y = \sqrt{x^2 - 3}$

$= (x^2 - 3)^{\frac{1}{2}}$

$y' = \frac{1}{2}(x^2 - 3)^{\frac{1}{2}}(2x)$

$= \frac{x}{\sqrt{x^2 - 3}}$

f. $f(x) = \frac{1}{(x^2 - 16)^5}$

$= (x^2 - 16)^{-5}$

$f'(x) = -5(x^2 - 16)^{-6}(2x)$

$= \frac{-10x}{(x^2 - 16)^6}$

5. a. $y = -\frac{2}{x^3}$

$= -2x^{-3}$

$\frac{dy}{dx} = (-2)(-3)x^{-4}$

$= \frac{6}{x^4}$

b. $y = \frac{1}{x + 1}$

$= (x + 1)^{-1}$

$\frac{dy}{dx} = (-1)(x + 1)^{-2}(1)$

$= \frac{-1}{(x + 1)^2}$

c. $y = \frac{1}{x^2 - 4}$

$= (x^2 - 4)^{-1}$

$\frac{dy}{dx} = (-1)(x^2 - 4)^{-2}(2x)$

$= \frac{-2x}{(x^2 - 4)^2}$

d. $y = \frac{3}{9 - x^2} = 3(9 - x^2)^{-1}$

$\frac{dy}{dx} = \frac{6x}{(9 - x^2)^2}$

e. $y = \dfrac{1}{5x^2 + x}$

$= (5x^2 + x)^{-1}$

$\dfrac{dy}{dx} = (-1)(5x^2 + x)^{-2}(10x + 1)$

$= -\dfrac{10x + 1}{(5x^2 + x)^2}$

f. $y = \dfrac{1}{(x^2 + x + 1)^4}$

$= (x^2 + x + 1)^{-4}$

$\dfrac{dy}{dx} = (-4)(x^2 + x + 1)^{-5}(2x + 1)$

$= -\dfrac{8x + 4}{(x^2 + x + 1)^5}$

6. $h = g \circ f$

$= g(f(x))$

$h(-1) = g(f(-1))$

$= g(1)$

$= -4$

$h(x) = g(f(x))$

$h'(x) = g'(f(x))f'(x)$

$h'(-1) = g'(f(-1))f'(-1)$

$= g'(1)(-5)$

$= (-7)(-5)$

$= 35$

7. $f(x) = (x - 3)^2, g(x) = \dfrac{1}{x}, h(x) = f(g(x)),$

$f'(x) = 2(x - 3), g'(x) = -\dfrac{1}{x^2}$

$h'(x) = f'(g(x))g'(x)$

$= f'\left(\dfrac{1}{x}\right)\left(-\dfrac{1}{x^2}\right)$

$= 2\left(\dfrac{1}{x} - 3\right)\left(-\dfrac{1}{x^2}\right)$

$= -\dfrac{2}{x^2}\left(\dfrac{1}{x} - 3\right)$

8. a. $f(x) = (x + 4)^3(x - 3)^6$

$f'(x) = \dfrac{d}{dx}[(x + 4)^3] \cdot (x - 3)^6$

$+ (x + 4)^3\dfrac{d}{dx}[(x - 3)^6]$

$= 3(x + 4)^2(x - 3)^6$

$+ (x + 4)^3(6)(x - 3)^5$

$= (x + 4)^2(x - 3)^5$

$\times [3(x - 3) + 6(x + 4)]$

$= (x + 4)^2(x - 3)^5(9x + 15)$

b. $y = (x^2 + 3)^3(x^3 + 3)^2$

$\dfrac{dy}{dx} = \dfrac{d}{dx}[(x^2 + 3)^3] \cdot (x^3 + 3)^2$

$+ (x^2 + 3)^3 \cdot \dfrac{d}{dx}[(x^3 + 3)^2]$

$= 3(x^2 + 3)^2(2x)(x^3 + 3)^2$

$+ (x^2 + 3)^3(2)(x^3 + 3)(3x^2)$

$= 6x(x^2 + 3)^2(x^3 + 3)[(x^3 + 3) + x(x^2 + 3)]$

$= 6x(x^2 + 3)^2(x^3 + 3)(2x^3 + 3x + 3)$

c. $y = \dfrac{3x^2 + 2x}{x^2 + 1}$

$\dfrac{dy}{dx} = \dfrac{(6x + 2)(x^2 + 1) - (3x^2 + 2x)(2x)}{(x^2 + 1)^2}$

$= \dfrac{6x^3 + 2x^2 + 6x + 2 - 6x^3 - 4x^2}{(x^2 + 1)^2}$

$= \dfrac{-2x^2 + 6x + 2}{(x^2 + 1)^2}$

d. $h(x) = x^3(3x - 5)^2$

$h'(x) = \dfrac{d}{dx}[x^3] \cdot (3x - 5)^2 + x^3\dfrac{d}{dx}[(3x - 5)^2]$

$= 3x^2(3x - 5)^2 + x^3(2)(3x - 5)(3)$

$= 3x^2(3x - 5)[(3x - 5) + 2x]$

$= 3x^2(3x - 5)(5x - 5)$

$= 15x^2(3x - 5)(x - 1)$

e. $y = x^4(1 - 4x^2)^3$

$\dfrac{dy}{dx} = \dfrac{d}{dx}[x^4](1 - 4x^2)^3 + x^4 \cdot \dfrac{d}{dx}[(1 - 4x^2)^3]$

$= 4x^3(1 - 4x^2)^3 + x^4(3)(1 - 4x^2)^2(-8x)$

$= 4x^3(1 - 4x^2)^2[(1 - 4x^2) - 6x^2]$

$= 4x^3(1 - 4x^2)^2(1 - 10x^2)$

f. $y = \left(\dfrac{x^2 - 3}{x^2 + 3}\right)^4$

$\dfrac{dy}{dx} = 4\left(\dfrac{x^2 - 3}{x^2 + 2}\right)^3 \dfrac{d}{dx}\left[\dfrac{x^2 - 3}{x^2 + 3}\right]$

$= 4\left(\dfrac{x^2 - 3}{x^2 + 3}\right)^3 \cdot \dfrac{(x^2 + 3)(2x) - (x^2 - 3)(2x)}{(x^2 + 3)^2}$

$= 4\left(\dfrac{x^2 - 3}{x^2 + 3}\right)^3 \cdot \dfrac{12x}{(x^2 + 3)^2}$

$= \dfrac{48x(x^2 - 3)^3}{(x^2 + 3)^5}$

9. a. $s(t) = t^{\frac{1}{3}}(4t - 5)^{\frac{2}{3}}$

$= t^{\frac{1}{3}}[(4t - 5)^2]^{\frac{1}{3}}$

$= [t(4t - 5)^2]^{\frac{1}{3}}$

$= [t(16t^2 - 40t + 25)]^{\frac{1}{3}}$

$= (16t^3 - 40t^2 + 25t)^{\frac{1}{3}}, t = 8$

$$s'(t) = \frac{1}{3}(16t^3 - 40t^2 + 25t)^{-\frac{2}{3}}$$
$$\times (48t^2 - 80t + 25)$$
$$= \frac{(48t^2 - 80t + 25)}{3(16t^3 - 40t^2 + 25t)^{\frac{2}{3}}}$$

Rate of change at $t = 8$:

$$s'(8) = \frac{(48(8)^2 - 80(8) + 25)}{3(16(8)^3 - 40(8)^2 + 25(8))^{\frac{2}{3}}}$$
$$= \frac{2457}{972}$$
$$= \frac{91}{36}$$

b. $s(t) = \left(\frac{t - \pi}{t - 6\pi}\right)^{\frac{1}{3}}, t = 2\pi$

$$s'(t) = \frac{1}{3}\left(\frac{t - \pi}{t - 6\pi}\right)^{-\frac{2}{3}} \cdot \frac{d}{dt}\left[\frac{t - \pi}{t - 6\pi}\right]$$
$$= \frac{1}{3}\left(\frac{t - 6\pi}{t - \pi}\right)^{\frac{2}{3}} \cdot \frac{(t - 6\pi) - (t - \pi)}{(t - 6\pi)^2}$$
$$= \frac{1}{3}\left(\frac{t - 6\pi}{t - \pi}\right)^{\frac{2}{3}} \cdot \frac{-5\pi}{(t - 6\pi)^2}$$

Rate of change at $t = 2\pi$:

$$s'(2\pi) = \frac{1}{3}(-4)^{\frac{2}{3}} \cdot \frac{-5\pi}{16\pi^2}$$
$$= -\frac{5\sqrt[3]{2}}{24\pi}$$

10. $y = (1 + x^3)^2$ $\qquad y = 2x^6$

$\frac{dy}{dx} = 2(1 + x^3)(3x^2)$ $\qquad \frac{dy}{dx} = 12x^5$

For the same slope,

$6x^2(1 + x^3) = 12x^5$

$6x^2 + 6x^5 = 12x^5$

$6x^2 - 6x^5 = 0$

$6x^2(x^3 - 1) = 0$

$x = 0$ or $x = 1$.

Curves have the same slope at $x = 0$ and $x = 1$.

11. $y = (3x - x^2)^{-2}$

$$\frac{dy}{dx} = -2(3x - x^2)^{-3}(3 - 2x)$$

At $x = 2$,

$$\frac{dy}{dx} = -2[6 - 4]^{-3}(3 - 4)$$
$$= 2(2)^{-3}$$
$$= \frac{1}{4}$$

The slope of the tangent line at $x = 2$ is $\frac{1}{4}$.

12. $y = (x^3 - 7)^5$ at $x = 2$

$$\frac{dy}{dx} = 5(x^3 - 7)^4(3x^2)$$

When $x = 2$,

$$\frac{dy}{dx} = 5(1)^4(12)$$
$$= 60$$

Slope of the tangent is 60.

Equation of the tangent at (2, 1) is

$$y - 1 = 60(x - 2)$$

$60x - y - 119 = 0$.

13. a. $y = 3u^2 - 5u + 2$

$u = x^2 - 1, x = 2$

$u = 3$

$$\frac{dy}{du} = 6u - 5, \frac{du}{dx} = 2x$$
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= (6u - 5)(2x)$$
$$= (18 - 5)(4)$$
$$= 13(4)$$
$$= 52$$

b. $y = 2u^3 + 3u^2, u = x + x^{\frac{1}{2}}, x = 1$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
$$= (6u^2 + 6u)\left(1 + \frac{1}{2\sqrt{x}}\right)$$

At $x = 1$:

$u = 1 + 1^{\frac{1}{2}}$

$= 2$

$$\frac{dy}{dx} = (6(2)^2 + 6(2))\left(1 + \frac{1}{2\sqrt{1}}\right)$$
$$= 36 \times \frac{3}{2}$$
$$= 54$$

c. $y = u(u^2 + 3)^3, u = (x + 3)^2, x = -2$

$$\frac{dy}{du} = (u^2 + 3)^3 + 6u^2(u^2 + 3)^2, \frac{du}{dx} = 2(x + 3)$$

wrong!

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = [7^3 + 6(4)^2][2(1)]$$

4^3

$$= 439 \times 2$$
$$= ~~878~~ \quad 320$$

d. $y = u^3 - 5(u^3 - 7u)^2$,

$u = \sqrt{x}$

$= x^{\frac{1}{2}}, x = 4$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$= 3[3u^2 - 10(u^3 - 7u)(3u^2 - 7)] \cdot \left(\frac{1}{2}x^{\frac{1}{2}}\right)$

$= [3u^2 - 10(u^3 - 7u)(3u^2 - 7)] \cdot \frac{1}{2\sqrt{x}}$

At $x = 4$:

$u = \sqrt{4}$

$= 2$

$\frac{dy}{dx} = [3(2)^2 - 10((2)^3 - 7(2))(3(2)^2 - 7)]\frac{1}{2(2)}$

$= 78$

14. $h(x) = f(g(x))$, therefore

$h'(x) = f'(g(x)) \times g'(x)$

$f(u) = u^2 - 1, g(2) = 3, g'(2) = -1$

Now, $h'(2) = f'(g(2)) \times g'(2)$

$= f'(3) \times g'(2).$

Since $f(u) = u^2 - 1, f'(u) = 2u$, and $f'(3) = 6$,

$h'(2) = 6(-1)$

$= -6.$

15. $V(t) = 50\,000\left(1 - \frac{t}{30}\right)^2$

$V'(t) = 50\,000\left[2\left(1 - \frac{t}{30}\right)\left(-\frac{1}{30}\right)\right]$

$V'(10) = 50\,000\left[2\left(1 - \frac{10}{30}\right)\left(-\frac{1}{30}\right)\right]$

$= 50\,000\left[2\left(\frac{2}{3}\right)\left(-\frac{1}{30}\right)\right]$

$\doteq 2222$

At $t = 10$ minutes, the water is flowing out of the tank at a rate of 2222 L/min.

16. The velocity function is the derivative of the position function.

$s(t) = (t^3 + t^2)^{\frac{1}{2}}$

$v(t) = s'(t) = \frac{1}{2}(t^3 + t^2)^{-\frac{1}{2}}(3t^2 + 2t)$

$= \frac{3t^2 + 2t}{2\sqrt{t^3 + t^2}}$

$v(3) = \frac{3(3)^2 + 2(3)}{2\sqrt{3^3 + 3^2}}$

$= \frac{27 + 6}{2\sqrt{36}}$

$= \frac{33}{12}$

$= 2.75$

The particle is moving at 2.75 m/s.

17. a. $h(x) = p(x)q(x)r(x)$

$h'(x) = p'(x)q(x)r(x) + p(x)q'(x)r(x) + p(x)q(x)r'(x)$

b. $h(x) = x(2x + 7)^4(x - 1)^2$

Using the result from part a.,

$h'(x) = (1)(2x + 7)^4(x - 1)^2 + x[4(2x + 7)^3(2)](x - 1)^2 + x(2x + 7)^4[2(x - 1)]$

$h'(-3) = 1(16) + (-3)[4(1)(2)](16) + (-3)(1)[2(-4)]$

$= 16 - 384 + 24$

$= -344$

18. $y = (x^2 + x - 2)^3 + 3$

$\frac{dy}{dx} = 3(x^2 + x - 2)^2(2x + 1)$

At the point (1, 3), $x = 1$ and the slope of the tangent will be $3(1 + 1 - 2)^2(2 + 1) = 0$.

Equation of the tangent at (1, 3) is $y - 3 = 0$.

Solving this equation with the function, we have

$(x^2 + x - 2)^3 + 3 = 3$

$(x + 2)^3(x - 1)^3 = 0$

$x = -2$ or $x = 1$

Since -2 and 1 are both triple roots, the line with equation $y - 3 = 0$ will be a tangent at both $x = 1$ and $x = -2$. Therefore, $y - 3 = 0$ is also a tangent at $(-2, 3)$.

19. $y = \frac{x^2(1 - x)^3}{(1 + x)^3}$

$= x^2\left[\left(\frac{1 - x}{1 + x}\right)\right]^3$

$\frac{dy}{dx} = 2x\left(\frac{1 - x}{1 + x}\right)^3 + 3x^2\left(\frac{1 - x}{1 + x}\right)^2 \times \left[\frac{-(1 + x) - (1 - x)(1)}{(1 + x)^2}\right]$

$= 2x\left(\frac{1 - x}{1 + x}\right)^3 + 3x^2\left(\frac{1 - x}{1 + x}\right)^2\left[\frac{-2}{(1 + x)^2}\right]$

$= 2x\left(\frac{1 - x}{1 + x}\right)^2\left[\frac{1 - x}{1 + x} - \frac{3x}{(1 + x)^2}\right]$

$= 2x\left(\frac{1 - x}{1 + x}\right)^2\left[\frac{1 - x^2 - 3x}{(1 + x)^2}\right]$

$= -\frac{2x(x^2 + 3x - 1)(1 - x)^2}{(1 + x)^4}$

Review Exercise, pp. 110–113

1. To find the derivative $f'(x)$, the limit

$f'(x) = \lim_{h \to 0}\frac{f(x + h) - f(x)}{h}$

must be computed, provided it exists. If this limit does not exist, then the derivative of $f(x)$ does not exist at this particular value of x. As an alternative to this limit, we could also find $f'(x)$ from the definition by computing the equivalent limit

$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$.

These two limits are seen to be equivalent by substituting $z = x + h$.

2. a. $y = 2x^2 - 5x$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{(2(x+h)^2 - 5(x+h)) - (2x^2 - 5x)}{h}$$
$$= \lim_{h \to 0} \frac{2((x+h)^2 - x^2) - 5h}{h}$$
$$= \lim_{h \to 0} \frac{2((x+h) - x)((x+h) + x) - 5h}{h}$$
$$= \lim_{h \to 0} \frac{2h(2x+h) - 5h}{h}$$
$$= \lim_{h \to 0} (2(2x+h) - 5)$$
$$= 4x - 5$$

b. $y = \sqrt{x - 6}$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{\sqrt{(x+h) - 6} - \sqrt{x - 6}}{h}$$
$$= \lim_{h \to 0} \left[\frac{\sqrt{(x+h) - 6} - \sqrt{x - 6}}{h} \times \frac{\sqrt{(x+h) - 6} + \sqrt{x - 6}}{\sqrt{(x+h) - 6} + \sqrt{x - 6}} \right]$$
$$= \lim_{h \to 0} \frac{((x+h) - 6) - (x - 6)}{h(\sqrt{(x+h) - 6} + \sqrt{x - 6})}$$
$$= \lim_{h \to 0} \frac{1}{\sqrt{(x+h) - 6} + \sqrt{x - 6}}$$
$$= \frac{1}{2\sqrt{x - 6}}$$

c. $y = \frac{x}{4 - x}$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{\frac{x+h}{4 - (x+h)} - \frac{x}{4 - x}}{h}$$
$$= \lim_{h \to 0} \frac{\frac{(x+h)(4-x) - x(4 - (x+h))}{(4 - (x+h))(4 - x)}}{h}$$
$$= \lim_{h \to 0} \frac{4h}{h(4 - (x+h))(4 - x)}$$
$$= \lim_{h \to 0} \frac{4}{(4 - (x+h))(4 - x)}$$
$$= \frac{4}{(4 - x)^2}$$

3. a. $y = x^2 - 5x + 4$

$$\frac{dy}{dx} = 2x - 5$$

b. $f(x) = x^{\frac{3}{4}}$

$$f'(x) = \frac{3}{4}x^{-\frac{1}{4}}$$
$$= \frac{3}{4x^{\frac{1}{4}}}$$

c. $y = \frac{7}{3x^4}$

$$= \frac{7}{3}x^{-4}$$
$$\frac{dy}{dx} = \frac{-28}{3}x^{-5}$$
$$= -\frac{28}{3x^5}$$

d. $y = \frac{1}{x^2 + 5}$

$$= (x^2 + 5)^{-1}$$
$$\frac{dy}{dx} = (-1)(x^2 + 5)^{-2} \cdot (2x)$$
$$= -\frac{2x}{(x^2 + 5)^2}$$

e. $y = \frac{3}{(3 - x^2)^2}$

$$= 3(3 - x^2)^{-2}$$
$$\frac{dy}{dx} = (-6)(3 - x^2)^{-3} \cdot (-2x)$$
$$= \frac{12x}{(3 - x^2)^3}$$

f. $y = \sqrt{7x^2 + 4x + 1}$

$$= (7x^2 + 4x + 1)^{\frac{1}{2}}$$
$$\frac{dy}{dx} = \frac{1}{2}(7x^2 + 4x + 1)^{-\frac{1}{2}}(14x + 4)$$
$$= \frac{7x + 2}{\sqrt{7x^2 + 4x + 1}}$$

4. a. $f(x) = \frac{2x^3 - 1}{x^2}$

$$= 2x - \frac{1}{x^2}$$
$$= 2x - x^{-2}$$
$$f'(x) = 2 + 2x^{-3}$$
$$= 2 + \frac{2}{x^3}$$

b. $g(x) = \sqrt{x}(x^3 - x)$

$= x^{\frac{1}{2}}(x^3 - x)$

$= x^{\frac{7}{2}} - x^{\frac{3}{2}}$

$g'(x) = \frac{7}{2}x^{\frac{5}{2}} - \frac{3}{2}x^{\frac{1}{2}}$

$= \frac{\sqrt{x}}{2}(7x^2 - 3)$

c. $y = \frac{x}{3x - 5}$

$\frac{dy}{dx} = \frac{(3x - 5)(1) - (x)(3)}{(3x - 5)^2}$

$= -\frac{5}{(3x - 5)^2}$

d. $y = (x - 1)^{\frac{1}{2}}(x + 1)$

$y' = (x - 1)^{\frac{1}{2}} + (x + 1)\left(\frac{1}{2}\right)(x - 1)^{-\frac{1}{2}}$

$= \sqrt{x - 1} + \frac{x + 1}{2\sqrt{x - 1}}$

$= \frac{2x - 2 + x + 1}{2\sqrt{x - 1}}$

$= \frac{3x - 1}{2\sqrt{x - 1}}$

e. $f(x) = (\sqrt{x} + 2)^{-\frac{2}{3}}$

$= (x^{\frac{1}{2}} + 2)^{-\frac{2}{3}}$

$f'(x) = \frac{-2}{3}(x^{\frac{1}{2}} + 2)^{-\frac{5}{3}} \cdot \frac{1}{2}x^{-\frac{1}{2}}$

$= -\frac{1}{3\sqrt{x}(\sqrt{x} + 2)^{\frac{5}{3}}}$

f. $y = \frac{x^2 + 5x + 4}{x + 4}$

$= \frac{(x + 4)(x + 1)}{x + 4}$

$= x + 1, x \neq -4$

$\frac{dy}{dx} = 1$

5. a. $y = x^4(2x - 5)^6$

$y' = x^4[6(2x - 5)^5(2)] + 4x^3(2x - 5)^6$

$= 4x^3(2x - 5)^5[3x + (2x - 5)]$

$= 4x^3(2x - 5)^5(5x - 5)$

$= 20x^3(2x - 5)^5(x - 1)$

b. $y = x\sqrt{x^2 + 1}$

$y' = x\left[\frac{1}{2}(x^2 + 1)^{-\frac{1}{2}}(2x)\right] + (1)\sqrt{x^2 + 1}$

$= \frac{x^2}{\sqrt{x^2 + 1}} + \sqrt{x^2 + 1}$

c. $y = \frac{(2x - 5)^4}{(x + 1)^3}$

$y' = \frac{(x + 1)^3 4(2x - 5)^3(2)}{(x + 1)^6}$

$- \frac{3(2x - 5)^4(x + 1)^2}{(x + 1)^6}$

$= \frac{(x + 1)^2(2x - 5)^3[8x + 8 - 6x + 15]}{(x + 1)^6}$

$y' = \frac{(2x - 5)^3(2x + 23)}{(x + 1)^4}$

d. $y = \left(\frac{10x - 1}{3x + 5}\right)^6 = (10x - 1)^6(3x + 5)^{-6}$

$y' = (10x - 1)^6[-6(3x + 5)^{-7}(3)]$

$+ 6(10x - 1)^5(10)(3x + 5)^{-6}$

$= (10x - 1)^5(3x + 5)^{-7}[x - 18(10x - 1)]$

$+ 60(3x + 5)$

$= (10x - 1)^5(3x + 5)^{-7}$

$\times (-180x + 18 + 180x + 300)$

$= \frac{318(10x - 1)^5}{(3x + 5)^7}$

e. $y = (x - 2)^3(x^2 + 9)^4$

$y' = (x - 2)^3[4(x^2 + 9)^3(2x)]$

$+ 3(x - 2)^2(1)(x^2 + 9)^4$

$= (x - 2)^2(x^2 + 9)^3[8x(x - 2) + 3(x^2 + 9)]$

$= (x - 2)^2(x^2 + 9)^3(11x^2 - 16x + 27)$

f. $y = (1 - x^2)^3(6 + 2x)^{-3}$

$= \left(\frac{1 - x^2}{6 + 2x}\right)^3$

$y' = 3\left(\frac{1 - x^2}{6 + 2x}\right)^2$

$\times \left[\frac{(6 + 2x)(-2x) - (1 - x^2)(2)}{(6 + 2x)^2}\right]$

$= \frac{3(1 - x^2)^2(-12x - 4x^2 - 2 + 2x^2)}{(6 + 2x)^4}$

$= -\frac{3(1 - x^2)^2(2x^2 + 12x + 2)}{(6 + 2x)^4}$

$= -\frac{3(1 - x^2)^2(x^2 + 6x + 1)}{8(3 - x)^4}$

6. a. $g(x) = f(x^2)$

$g'(x) = f(x^2) \times 2x$

b. $h(x) = 2xf(x)$

$h'(x) = 2xf'(x) + 2f(x)$

7. a. $y = 5u^2 + 3u - 1, u = \frac{18}{x^2 + 5}$

$x = 2$

$u = 2$

$$\frac{dy}{du} = 10u + 3$$

$$\frac{du}{dx} = -\frac{36x}{(x^2 + 5)^2}$$

When $x = 2$,

$$\frac{du}{dx} = -\frac{72}{81} = -\frac{8}{9}$$

When $u = 2$,

$$\frac{dy}{du} = 20 + 3$$

$$= 23$$

$$\frac{dy}{dx} = 23\left(-\frac{8}{9}\right)$$

$$= -\frac{184}{9}$$

b. $y = \frac{u + 4}{u - 4}, u = \frac{\sqrt{x} + x}{10}$,

$x = 4$

$$u = \frac{3}{5}$$

$$\frac{dy}{du} = \frac{(u - 4) - (u + 4)}{(u - 4)^2}$$

$$\frac{du}{dx} = \frac{1}{10}\left(\frac{1}{2}x^{-\frac{1}{2}} + 1\right)$$

When $x = 4$,

$$= -\frac{8}{(u - 4)^2}\ \frac{du}{dx} = \frac{1}{10}\left(\frac{5}{4}\right)$$

$$= \frac{1}{8}$$

When $u = \frac{3}{5}$,

$$\frac{dy}{du} = -\frac{8}{\left(\frac{3}{5} - \frac{20}{5}\right)^2}$$

$$= -\frac{8(25)}{(-17)^2}$$

When $x = 4$,

$$\frac{dy}{dx} = \frac{8(25)}{17^2} \times \frac{1}{8}$$

$$= \frac{25}{289}$$

c. $y = f(\sqrt{x^2 + 9}), f'(5) = -2, x = 4$

$$\frac{dy}{dx} = f'(\sqrt{x^2 + 9}) \times \frac{1}{2}(x^2 + 9)^{-\frac{1}{2}}(2x)$$

$$\frac{dy}{dx} = f'(5) \cdot \frac{1}{2} \cdot \frac{1}{5} \cdot 8$$

$$= -2 \cdot \frac{4}{5}$$

$$= -\frac{8}{5}$$

8. $f(x) = (9 - x^2)^{\frac{2}{3}}$

$$f'(x) = \frac{2}{3}(9 - x^2)^{-\frac{1}{3}}(-2x)$$

$$= \frac{-4x}{3(9 - x^2)^{\frac{1}{3}}}$$

$$f'(1) = -\frac{2}{3}$$

The slope of the tangent line at $(1, 4)$ is $-\frac{2}{3}$.

9. $y = -x^3 + 6x^2$

$y' = -3x^2 + 12x$

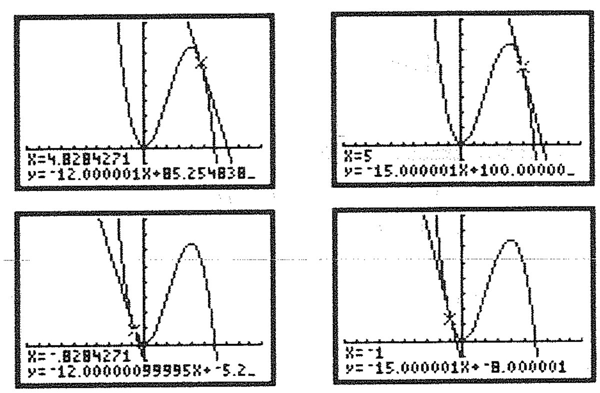
$-3x^2 + 12x = -12$ | $-3x^2 + 12x = -15$

$x^2 - 4x - 4 = 0$ | $x^2 - 4x - 5 = 0$

$x = \frac{4 \pm \sqrt{16 + 16}}{2}$ | $(x - 5)(x + 1) = 0$

$= \frac{4 \pm 4\sqrt{2}}{2}$ | $x = 5, x = -1$

$x = 2 \pm 2\sqrt{2}$



10. a. i. $y = (x^2 - 4)^5$

$y' = 5(x^2 - 4)^4(2x)$

Horizontal tangent,

$10x(x^2 - 4)^4 = 0$

$x = 0, x = \pm 2$

ii. $y = (x^3 - x)^2$

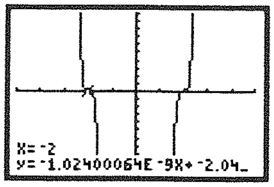
$y' = 2(x^3 - x)(3x^2 - 1)$

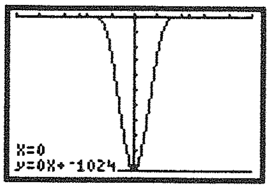
Horizontal tangent,

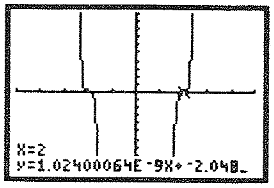
$2x(x^2 - 1)(3x^2 - 1) = 0$

$x = 0, x = \pm 1, x = \pm\frac{\sqrt{3}}{3}$.

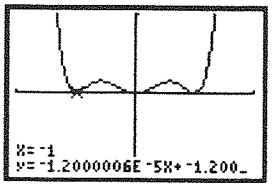
b. i.

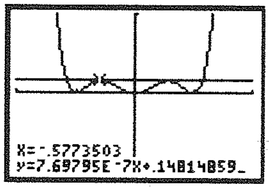


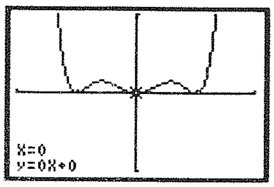


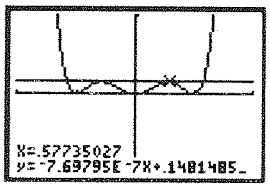


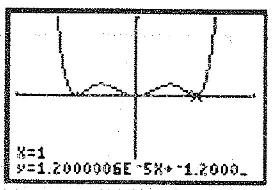
ii.











11. a. $y = (x^2 + 5x + 2)^4$ at $(0, 16)$

$y' = 4(x^2 + 5x + 2)^3(2x + 5)$

At $x = 0$,

$y' = 4(2)^3(5)$

$= 160$

Equation of the tangent at $(0, 16)$ is

$y - 16 = 160(x - 0)$

$y = 160x + 16$

or $160x - y + 16 = 0$

b. $y = (3x^{-2} - 2x^3)^5$ at $(1, 1)$

$y' = 5(3x^{-2} - 2x^3)^4(-6x^{-3} - 6x^2)$

At $x = 1$,

$y' = 5(1)^4(-6 - 6)$

$= -60$

Equation of the tangent at $(1, 1)$ is

$y - 1 = -60(x - 1)$

$60x + y - 61 = 0.$

12. $y = 3x^2 - 7x + 5$

$$\frac{dy}{dx} = 6x - 7$$

Slope of $x + 5y - 10 = 0$ is $-\frac{1}{5}$.

Since perpendicular, $6x - 7 = 5$

$x = 2$

$y = 3(4) - 14 + 5$

$= 3.$

Equation of the tangent at $(2, 3)$ is

$y - 3 = 5(x - 2)$

$5x - y - 7 = 0.$

13. $y = 8x + b$ is tangent to $y = 2x^2$

$$\frac{dy}{dx} = 4x$$

Slope of the tangent is 8, therefore $4x = 8, x = 2$.

Point of tangency is $(2, 8)$.

Therefore, $8 = 16 + b, b = -8$.

Or $8x + b = 2x^2$

$2x^2 - 8x - b = 0$

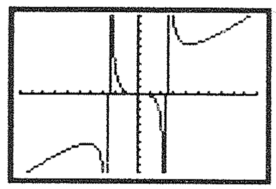
$$x = \frac{8 \pm \sqrt{64 + 8b}}{2(2)}.$$

For tangents, the roots are equal, therefore

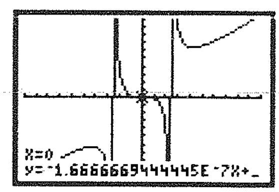
$64 + 8b = 0, b = -8.$

Point of tangency is $(2, 8)$, $b = -8$.

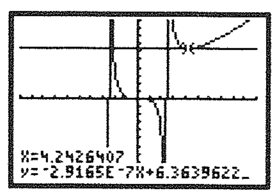
14. a.



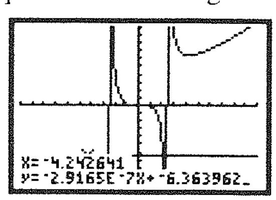
b.



The equation of the tangent is $y = 0$.



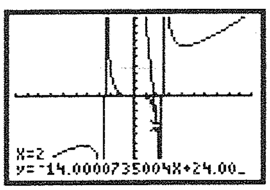
The equation of the tangent is $y = 6.36$.



The equation of the tangent is $y = -6.36$.

c. $$f'(x) = \frac{(x^2 - 6)(3x^2) - x^3(2x)}{(x^2 - 6)^2}$$
$$= \frac{x^4 - 18x^2}{(x^2 - 6)^2}$$
$$\frac{x^4 - 18x^2}{(x^2 - 6)^2} = 0$$
$$x^2(x^2 - 18) = 0$$
$$x^2 = 0 \text{ or } x^2 - 18 = 0$$
$$x = 0 \qquad x = \pm 3\sqrt{2}$$
The coordinates of the points where the slope is 0 are (0, 0), $\left(3\sqrt{2}, \frac{9\sqrt{2}}{2}\right)$, and $\left(-3\sqrt{2}, -\frac{9\sqrt{2}}{2}\right)$.

d. Substitute into the expression for $f'(x)$ from part b.
$$f'(2) = \frac{16 - 72}{(-2)^2}$$
$$= \frac{-56}{4}$$
$$= -14$$



15. a. $f(x) = 2x^{\frac{5}{3}} - 5x^{\frac{2}{3}}$
$$f'(x) = 2 \times \frac{5}{3}x^{\frac{2}{3}} - 5 \times \frac{2}{3}x^{-\frac{1}{3}}$$
$$= \frac{10}{3}x^{\frac{2}{3}} - \frac{10}{3x^{\frac{1}{3}}}$$
$$f(x) = 0 \quad \therefore x^{\frac{2}{3}}[2x - 5] = 0$$
$$x = 0 \text{ or } x = \frac{5}{2}$$
$y = f(x)$ crosses the x-axis at $x = \frac{5}{2}$, and
$$f'(x) = \frac{10}{3}\left(\frac{x - 1}{x^{\frac{1}{3}}}\right)$$
$$f'\left(\frac{5}{2}\right) = \frac{10}{3} \times \frac{3}{2} \times \frac{1}{\left(\frac{5}{2}\right)^{\frac{1}{3}}}$$
$$= 5 \times \frac{\sqrt[3]{2}}{\sqrt[3]{5}} = 5^{\frac{2}{3}} \times 2^{\frac{1}{3}}$$
$$= (25 \times 2)^{\frac{1}{3}}$$
$$= \sqrt[3]{50}$$
b. To find a, let $f(x) = 0$.
$$\frac{10}{3}x^{\frac{2}{3}} - \frac{10}{3x^{\frac{1}{3}}} = 0$$
$$30x = 30$$
$$x = 1$$
Therefore $a = 1$.

16. $M = 0.1t^2 - 0.001t^3$

a. When $t = 10$,
$$M = 0.1(100) - 0.001(1000)$$
$$= 9$$
When $t = 15$,
$$M = 0.1(225) - 0.001(3375)$$
$$= 19.125$$
One cannot memorize partial words, so 19 words are memorized after 15 minutes.

b. $M' = 0.2t - 0.003t^2$
When $t = 10$,
$$M' = 0.2(10) - 0.003(100)$$
$$= 1.7$$
The number of words memorized is increasing by 1.7 words/min.
When $t = 15$,
$$M' = 0.2(15) - 0.003(225)$$
$$= 2.325$$
The number of words memorized is increasing by 2.325 words/min.

17. a. $N(t) = 20 - \dfrac{30}{\sqrt{9 + t^2}}$
$$N'(t) = \frac{30t}{(9 + t^2)^{\frac{3}{2}}}$$
b. No, according to this model, the cashier never stops improving. Since $t > 0$, the derivative is always positive, meaning that the rate of change in the cashier's productivity is always increasing. However, these increases must be small, since, according to the model, the cashier's productivity can never exceed 20.

18. $C(x) = \frac{1}{3}x^3 + 40x + 700$

a. $C'(x) = x^2 + 40$

b. $$C'(x) = 76$$
$$x^2 + 40 = 76$$
$$x^2 = 36$$
$$x = 6$$
Production level is 6 gloves/week.

19. $R(x) = 750x - \dfrac{x^2}{6} - \dfrac{2}{3}x^3$

a. Marginal Revenue
$$R'(x) = 750 - \frac{x}{3} - 2x^2$$

b. $R'(10) = 750 - \frac{10}{3} - 2(100)$
$= \$546.67$

20. $D(p) = \frac{20}{\sqrt{p-1}}, p > 1$

$D'(p) = 20\left(-\frac{1}{2}\right)(p-1)^{-\frac{3}{2}}$

$= -\frac{10}{(p-1)^{\frac{3}{2}}}$

$D'(5) = \frac{10}{\sqrt{4^3}} = -\frac{10}{8}$

$= -\frac{5}{4}$

Slope of demand curve at (5, 10) is $-\frac{5}{4}$.

21. $B(x) = -0.2x^2 + 500, 0 \leq x \leq 40$

a. $B(0) = -0.2(0)^2 + 500 = 500$
$B(30) = -0.2(30)^2 + 500 = 320$

b. $B'(x) = -0.4x$
$B'(0) = -0.4(0) = 0$
$B'(30) = -0.4(30) = -12$

c. $B(0)$ = blood sugar level with no insulin
$B(30)$ = blood sugar level with 30mg of insulin
$B'(0)$ = rate of change in blood sugar level with no insulin
$B'(30)$ = rate of change in blood sugar level with 30 mg of insulin

d. $B'(50) = -0.4(50) = -20$
$B(50) = -0.2(50)^2 + 500 = 0$

$B'(50) = -20$ means that the patient's blood sugar level is decreasing at 20 units per mg of insulin 1 h after 50 mg of insulin is injected.

$B(50) = 0$ means that the patient's blood sugar level is *zero* 1 h after 50 mg of insulin is injected. These values are not logical because a person's blood sugar level can never reach zero and continue to decrease.

22. a. $f(x) = \frac{3x}{1-x^2}$

$= \frac{3x}{(1-x)(1+x)}$

$f(x)$ is not differentiable at $x = 1$ because it is not defined there (vertical asymptote at $x = 1$).

b. $g(x) = \frac{x-1}{x^2+5x-6}$

$= \frac{x-1}{(x+6)(x-1)}$

$= \frac{1}{(x+6)}$ for $x \neq 1$

$g(x)$ is not differentiable at $x = 1$ because it is not defined there (hole at $x = 1$).

c. $h(x) = \sqrt[3]{(x-2)^2}$
The graph has a cusp at (2, 0) but it is differentiable at $x = 1$.

d. $m(x) = |3x - 3| - 1$.
The graph has a corner at $x = 1$, so $m(x)$ is not differentiable at $x = 1$.

23. a. $f(x) = \frac{3}{4x^2 - x}$

$= \frac{3}{x(4x-1)}$

$f(x)$ is not defined at $x = 0$ and $x = 0.25$. The graph has vertical asymptotes at $x = 0$ and $x = 0.25$. Therefore, $f(x)$ is not differentiable at $x = 0$ and $x = 0.25$.

b. $f(x) = \frac{x^2 - x - 6}{x^2 - 9}$

$= \frac{(x-3)(x+2)}{(x-3)(x+3)}$

$= \frac{(x+2)}{(x+3)}$ for $x \neq 3$

$f(x)$ is not defined at $x = 3$ and $x = -3$. At $x = -3$, the graph as a vertical symptote and at $x = 3$ it has a hole. Therefore, $f(x)$ is not differentiable at $x = 3$ and $x = -3$.

c. $f(x) = \sqrt{x^2 - 7x + 6}$

$= \sqrt{(x-6)(x-1)}$

$f(x)$ is not defined for $1 < x < 6$. Therefore, $f(x)$ is not differentiable for $1 < x < 6$.

24. $p'(t) = \frac{(t+1)(25) - (25t)(t)}{(t+1)^2}$

$= \frac{25t + 25 - 25t}{(t+1)^2}$

$= \frac{25}{(t+1)^2}$

25. Answers may vary. For example,

$f(x) = 2x + 3$

$y = \frac{1}{2x+3}$

$y' = \frac{(2x+3)(0) - (1)(2)}{(2x+3)^2}$

$= -\frac{2}{(2x+3)^2}$

$f(x) = 5x + 10$

$y = \frac{1}{5x+10}$

$y' = \frac{(5x+10)(0) - (1)(5)}{(5x+10)^2}$

$$= -\frac{5}{(5x + 10)^2}$$

Rule: If $f(x) = ax + b$ and $y = \frac{1}{f(x)}$, then

$$y' = \frac{-a}{(ax + b)^2}$$

$$y' = \lim_{h\to 0}\frac{1}{h}\left[\frac{1}{a(x + h) + b} - \frac{1}{ax + b}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{ax + b - [a(x + h)b]}{[a(x + h) + b](ax + h)}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{ax + b - ax - ah - b}{[a(x + h) + b](ax + b)}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{-ah}{[a(x + h) + b](ax + b)}\right]$$

$$= \lim_{h\to 0}\frac{-a}{[a(x + h) + b](ax + b)}$$

$$= \frac{-a}{(ax + b)^2}$$

26. a. Let $y = f(x)$.

$$y = \frac{(2x - 3)^2 + 5}{2x - 3}$$

Let $u = 2x - 3$.

Then $y = \frac{u^2 + 5}{u}$.

$y = u + 5u^{-1}$

b. $f'(x) = \frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= (1 - 5u^{-2})(2)$$

$$= 2(1 - 5(2x - 3)^{-2})$$

27. $g(x) = \sqrt{2x - 3} + 5(2x - 3)$

a. Let $y = g(x)$.

$$y = \sqrt{2x - 3} + 5(2x - 3)$$

Let $u = 2x - 3$.

Then $y = \sqrt{u} + 5u$.

b. $g'(x) = \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$= \left(\frac{1}{2}u^{-\frac{1}{2}} + 5\right)(2)$$

$$= u^{-\frac{1}{2}} + 10$$

$$= (2x - 3)^{-\frac{1}{2}} + 10$$

28. a. $f(x) = (2x - 5)^3(3x^2 + 4)^5$

$$f'(x) = (2x - 5)^3(5)(3x^2 + 4)^4(6x) + (3x^2 + 4)^5(3)(2x - 5)^2(2)$$

$$= 30x(2x - 5)^3(3x^2 + 4)^4 + 6(3x^2 + 4)^5(2x - 5)^2$$

$$= 6(2x - 5)^2(3x^2 + 4)^4 \times [5x(2x - 5) + (3x^2 + 4)]$$

$$= 6(2x - 5)^2(3x^2 + 4)^4 \times (10x^2 - 25x + 3x^2 + 4)$$

$$= 6(2x - 5)^2(3x^2 + 4)^4 \times (13x^2 - 25x + 4)$$

b. $g(x) = (8x^3)(4x^2 + 2x - 3)^5$

$$g'(x) = (8x^3)(5)(4x^2 + 2x - 3)^4(8x + 2) + (4x^2 + 2x - 3)^5(24x^2)$$

$$= 40x^3(4x^2 + 2x - 3)^4(8x + 2) + 24x^2(4x^2 + 2x - 3)^5$$

$$= 8x^2(4x^2 + 2x - 3)^4[5x(8x + 2) + 3(4x^2 + 2x - 3)]$$

$$= 8x^2(4x^2 + 2x - 3)^4 (40x^2 + 10x + 12x^2 + 6x - 9)$$

$$= 8x^2(4x^2 + 2x - 3)^4(52x^2 + 16x - 9)$$

c. $y = (5 + x)^2(4 - 7x^3)^6$

$$y' = (5 + x)^2(6)(4 - 7x^3)^5(-21x^2) + (4 - 7x^3)^6(2)(5 + x)$$

$$= -126x^2(5 + x)^2(4 - 7x^3)^5 + 2(5 + x)(4 - 7x^3)^6$$

$$= 2(5 + x)(4 - 7x^3)^5[-63x^2(5 + x) + 4 - 7x^3]$$

$$= 2(5 + x)(4 - 7x^3)^5(4 - 315x^2 - 70x^3)$$

d. $h(x) = \frac{6x - 1}{(3x + 5)^4}$

$$h'(x) = \frac{(3x + 5)^4(6) - (6x - 1)(4)(3x + 5)^3(3)}{((3x + 5)^4)^2}$$

$$= \frac{6(3x + 5)^3[(3x + 5) - 2(6x - 1)]}{(3x + 5)^8}$$

$$= \frac{6(-9x + 7)}{(3x + 5)^5}$$

e. $y = \frac{(2x^2 - 5)^3}{(x + 8)^2}$

$$\frac{dy}{dx} = \frac{(x + 8)^2(3)(2x^2 - 5)^2(4x)}{((x + 8)^2)^2} - \frac{(2x^2 - 5)^3(2)(x + 8)}{((x + 8)^2)^2}$$

$$= \frac{2(x + 8)(2x^2 - 5)^2[6x(x + 8) - (2x^2 - 5)]}{(x + 8)^4}$$

$$= \frac{2(2x^2 - 5)^2(4x^2 + 48x + 5)}{(x + 8)^3}$$

f. $f(x) = \dfrac{-3x^4}{\sqrt{4x - 8}}$

$= \dfrac{-3x^4}{(4x - 8)^{\frac{1}{2}}}$

$f'(x) = \dfrac{(4x - 8)^{\frac{1}{2}}(-12x^3)}{((4x - 8)^{\frac{1}{2}})^2}$

$- \dfrac{(-3x^4)\left(\frac{1}{2}\right)(4x - 8)^{-\frac{1}{2}}(4)}{((4x - 8)^{\frac{1}{2}})^2}$

$= \dfrac{-6x^3(4x - 8)^{-\frac{1}{2}}[2(4x - 8) - x]}{4x - 8}$

$= \dfrac{-6x^3(7x - 16)}{(4x - 8)^{\frac{3}{2}}}$

$= \dfrac{-3x^3(7x - 16)}{(4x - 8)^{\frac{3}{2}}}$

g. $g(x) = \left(\dfrac{2x + 5}{6 - x^2}\right)^4$

$g'(x) = 4\left(\dfrac{2x + 5}{6 - x^2}\right)^3$

$\times \left(\dfrac{(6 - x^2)(2) - (2x + 5)(-2x)}{(6 - x^2)^2}\right)$

$= 4\left(\dfrac{2x + 5}{6 - x^2}\right)^3\left(\dfrac{2(6 + x^2 + 5x)}{(6 - x^2)^2}\right)$

$= 8\left(\dfrac{2x + 5}{6 - x^2}\right)^3\left(\dfrac{(x + 2)(x + 3)}{(6 - x^2)^2}\right)$

h. $y = \left[\dfrac{1}{(4x + x^2)^3}\right]^3$

$= (4x + x^2)^{-9}$

$\dfrac{dy}{dx} = -9(4x + x^2)^{-10}(4 + 2x)$

29. $f(x) = ax^2 + bx + c,$

It is given that $(0, 0)$ and $(8, 0)$ are on the curve, and $f'(2) = 16$.

Calculate $f'(x) = 2ax + b$.

Then,

$16 = 2a(2) + b$

$4a + b = 16$ (1)

Since $(0, 0)$ is on the curve,

$0 = a(0)^2 + b(0) + c$

$c = 0$

Since $(8, 0)$ is on the curve,

$0 = a(8)^2 + b(8) + c$

$0 = 64a + 8b + 0$

$8a + b = 0$ (2)

Solve (1) and (2):

From (2), $b = -8a$ (1)

In (1),

$4a - 8a = 16$

$-4a = 16$

$a = -4$

Using (1),

$b = -8(-4) = 32$

$a = -4, b = 32, c = 0, f(x) = -4x^2 + 32x$

30. a. $A(t) = -t^3 + 5t + 750$

$A'(t) = -3t^2 + 5$

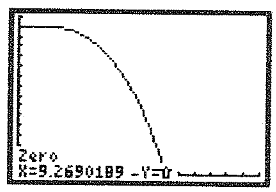
b. $A'(5) = -3(25) + 5$

$= -70$

At 5 h, the number of ants living in the colony is decreasing by 7000 ants/h.

c. $A(0) = 750$, so there were $750(100)$ or 75 000 ants living in the colony before it was treated with insecticide.

d. Determine t so that $A(t) = 0$. $-t^3 + 5t + 750$ cannot easily be factored, so find the zeros by using a graphing calculator.



All of the ants have been killed after about 9.27 h.

Chapter 2 Test, p. 114

1. You need to use the chain rule when the derivative for a given function cannot be found using the sum, difference, product, or quotient rules or when writing the function in a form that would allow the use of these rules is tedious. The chain rule is used when a given function is a composition of two or more functions.

2. f is the blue graph (it's a cubic). f' is the red graph (it is quadratic). The derivative of a polynomial function has degree one less than the derivative of the function. Since the red graph is a quadratic (degree 2) and the blue graph is cubic (degree 3), the blue graph is f and the red graph is f'.

3. $f(x) = \lim_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$

$= \lim_{h \to 0} \dfrac{x + h - (x + h)^2 - (x - x^2)}{h}$

$= \lim_{h \to 0} \dfrac{x + h - (x^2 + 2hx + h^2) - x + x^2}{h}$

$= \lim_{h \to 0} \dfrac{h - 2hx - h^2}{h}$

$$= \lim_{h \to 0} \frac{h(1 - 2x - h)}{h}$$
$$= \lim_{h \to 0} (1 - 2x - h)$$
$$= 1 - 2x$$

Therefore, $\frac{d}{dx}(x - x^2) = 1 - 2x.$

4. a. $y = \frac{1}{3}x^3 - 3x^{-5} + 4\pi$

$$\frac{dy}{dx} = x^2 + 15x^{-6}$$

b. $y = 6(2x - 9)^5$

$$\frac{dy}{dx} = 30(2x - 9)^4(2)$$
$$= 60(2x - 9)^4$$

c. $y = \frac{2}{\sqrt{x}} + \frac{x}{\sqrt{3}} + 6\sqrt[3]{x}$

$$= 2x^{-\frac{1}{2}} + \frac{1}{\sqrt{3}}x + 6x^{\frac{1}{3}}$$
$$\frac{dy}{dx} = -x^{-\frac{3}{2}} + \frac{1}{\sqrt{3}} + 2x^{-\frac{2}{3}}$$

d. $y = \left(\frac{x^2 + 6}{3x + 4}\right)^5$

$$\frac{dy}{dx} = 5\left(\frac{x^2 + 6}{3x + 4}\right)^4 \frac{2x(3x + 4) - (x^2 + 6)3}{(3x + 4)^2}$$
$$= \frac{5(x^2 + 6)^4(3x^2 + 8x - 18)}{(3x + 4)^6}$$

e. $y = x^2\sqrt[3]{6x^2 - 7}$

$$\frac{dy}{dx} = 2x(6x^2 - 7)^{\frac{1}{3}} + x^2\frac{1}{3}(6x^2 - 7)^{-\frac{2}{3}}(12x)$$
$$= 2x(6x^2 - 7)^{-\frac{2}{3}}((6x^2 - 7) + 2x^2)$$
$$= 2x(6x^2 - 7)^{-\frac{2}{3}}(8x^2 - 7)$$

f. $y = \frac{4x^5 - 5x^4 + 6x - 2}{x^4}$

$$= 4x - 5 + 6x^{-3} - 2x^{-4}$$
$$\frac{dy}{dx} = 4 - 18x^{-4} + 8x^{-5}$$
$$= \frac{4x^5 - 18x + 8}{x^5}$$

5. $y = (x^2 + 3x - 2)(7 - 3x)$

$$\frac{dy}{dx} = (2x + 3)(7 - 3x) + (x^2 + 3x - 2)(-3)$$

At $(1, 8)$,

$$\frac{dy}{dx} = (5)(4) + (2)(-3)$$
$$= 14.$$

The slope of the tangent to $y = (x^2 + 3x - 2)(7 - 3x)$ at $(1, 8)$ is 14.

6. $y = 3u^2 + 2u$

$$\frac{dy}{du} = 6u + 2$$
$$u = \sqrt{x^2 + 5}$$
$$\frac{du}{dy} = \frac{1}{2}(x^2 + 5)^{-\frac{1}{2}}2x$$
$$\frac{dy}{dx} = (6u + 2)\left(\frac{x}{\sqrt{x^2 + 5}}\right)$$

At $x = -2, u = 3$.

$$\frac{dy}{dx} = (20)\left(-\frac{2}{3}\right)$$
$$= -\frac{40}{3}$$

7. $y = (3x^{-2} - 2x^3)^5$

$$\frac{dy}{dx} = 5(3x^{-2} - 2x^3)^4(-6x^{-3} - 6x^2)$$

At $(1, 1)$,

$$\frac{dy}{dx} = 5(1)^4(-6 - 6)$$
$$= -60.$$

Equation of tangent line at $(1, 1)$ is $y - 1 = 60(x - 1)$

$$y - 1 = -60x + 60$$
$$60x + y - 61 = 0.$$

8. $P(t) = (t^{\frac{1}{4}} + 3)^3$

$$P'(t) = 3(t^{\frac{1}{4}} + 3)^2\left(\frac{1}{4}t^{-\frac{3}{4}}\right)$$
$$P'(16) = 3(16^{\frac{1}{4}} + 3)^2\left(\frac{1}{4} \times 16^{-\frac{3}{4}}\right)$$
$$= 3(2 + 3)^2\left(\frac{1}{4} \times \frac{1}{8}\right)$$
$$= \frac{75}{32}$$

The amount of pollution is increasing at a rate of $\frac{75}{32}$ ppm/year.

9. $y = x^4$

$$\frac{dy}{dx} = 4x^3$$
$$-\frac{1}{16} = 4x^3$$

Normal line has a slope of 16. Therefore,

$$\frac{dy}{dx} = -\frac{1}{16}.$$
$$x^3 = -\frac{1}{64}$$

$x = -\frac{1}{4}$

$y = \frac{1}{256}$

Therefore, $y = x^4$ has a normal line with a slope of 16 at $\left(-\frac{1}{4}, \frac{1}{256}\right)$.

10. $y = x^3 - x^2 - x + 1$

$\frac{dy}{dx} = 3x^2 - 2x - 1$

For a horizontal tangent line, $\frac{dy}{dx} = 0$.

$3x^2 - 2x - 1 = 0$

$(3x + 1)(x - 1) = 0$

$x = -\frac{1}{3}$ or $x = 1$

$y = -\frac{1}{27} - \frac{1}{9} + \frac{1}{3} + 1$ $\quad$ $y = 1 - 1 - 1 + 1 = 0$

$= \frac{-1 - 3 + 9 + 27}{27}$

$= \frac{32}{27}$

The required points are $\left(-\frac{1}{3}, \frac{32}{27}\right)$, $(1, 0)$.

11. $y = x^2 + ax + b$

$\frac{dy}{dx} = 2x + a$

$y = x^3$

$\frac{dy}{dx} = 3x^2$

Since the parabola and cubic function are tangent at $(1, 1)$, then $2x + a = 3x^2$.

At $(1, 1)$ $2(1) + a = 3(1)^2$

$a = 1.$

Since $(1, 1)$ is on the graph of $y = x^2 + x + b$, $1 = 1^2 + 1 + b$

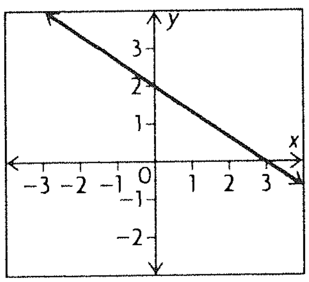
$b = -1.$

The required values are 1 and -1 for a and b, respectively.

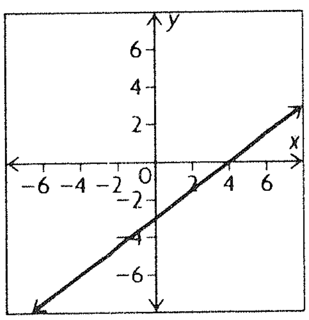
CHAPTER 3
Derivatives and Their Applications

Review of Prerequisite Skills, pp. 116–117

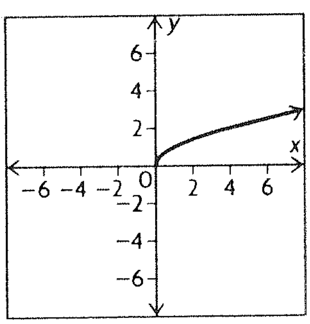
1. a.



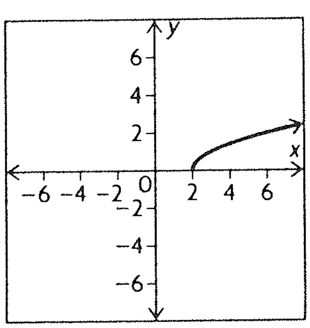
b.



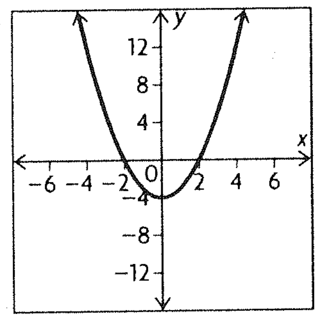
c.



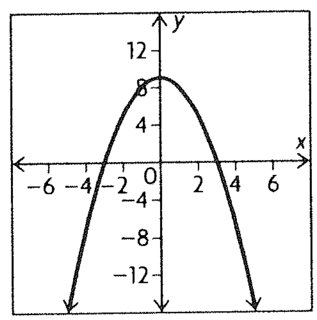
d.



e.



f.



2. a. $3(x-2)+2(x-1)-6=0$

$3x-6+2x-2-6=0$

$5x=14$

$x=\frac{14}{5}$

b. $\frac{1}{3}(x-2)+\frac{2}{5}(x+3)=\frac{x-5}{2}$

$10(x-2)+12(x+3)=15(x-5)$

$10x-20+12x+36=15x-75$

$22x+16=15x-75$

$7x=-91$

$x=-13$

c. $t^2-4t+3=0$

$(t-3)(t-1)=0$

$t=3$ or $t=1$

d. $2t^2-5t-3=0$

$(2t+1)(t-3)=0$

$t=-\frac{1}{2}$ or $t=3$

e. $\frac{6}{t}+\frac{t}{2}=4$

$12+t^2=8t$

$t^2-8t+12=0$

$(t-6)(t-2)=0$

$\therefore t=2$ or $t=6$

f. $x^3 + 2x^2 - 3x = 0$
$x(x^2 + 2x - 3) = 0$
$x(x + 3)(x - 1) = 0$
$x = 0$ or $x = -3$ or $x = 1$

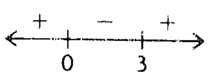
g. $x^3 - 8x^2 + 16x = 0$
$x(x^2 - 8x + 16) = 0$
$x(x - 4)^2 = 0$
$x = 0$ or $x = 4$

h. $4t^3 + 12t^2 - t - 3 = 0$
$4t^2(t + 3) - 1(t + 3) = 0$
$(t + 3)(4t^2 - 1) = 0$
$(t + 3)(2t - 1)(2t + 1) = 0$
$t = -3$ or $t = \frac{1}{2}$ or $t = -\frac{1}{2}$

i. $4t^4 - 13t^2 + 9 = 0$
$(4t^2 - 9)(t^2 - 1) = 0$
$t = \pm\frac{9}{4}$ or $t = \pm 1$

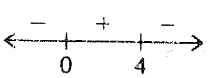
3. a. $3x - 2 > 7$
$3x > 9$
$x > 3$

b. $x(x - 3) > 0$



$x < 0$ or $x > 3$

c. $-x^2 + 4x > 0$



$x(x - 4) < 0$
$0 < x < 4$

4. a. $P = 4s$
$20 = 4s$
$5 = s$
$A = s^2$
$= 5^2$
$= 25 \text{ cm}^2$

b. $A = lw$
$= 8(6) = 48 \text{ cm}^2$

c. $A = \pi r^2$
$= \pi(7)^2$
$= 49\pi \text{ cm}^2$

d. $C = 2\pi r$
$12\pi = 2\pi r$
$6 = r$
$A = \pi r^2$
$= \pi(6)^2$
$= 36\pi \text{ cm}^2$

5. a. $SA = 2\pi rh + 2\pi r^2$
$= 2\pi(4)(3) + 2\pi(4)^2$
$= 24\pi + 32\pi$
$= 56\pi \text{ cm}^2$
$V = \pi r^2 h$
$= \pi(4)^2(3)$
$= 48\pi \text{ cm}^3$

b. $V = \pi r^2 h$
$96\pi = \pi(4)^2 h$
$h = 6 \text{ cm}$
$SA = 2\pi rh + 2\pi r^2$
$= 2\pi(4)(6) + 2\pi(4)^2$
$= 48\pi + 32\pi$
$= 80\pi \text{ cm}^2$

c. $V = \pi r^2 h$
$216\pi = \pi r^2(6)$
$r = 6 \text{ cm}$
$SA = 2\pi rh + 2\pi r^2$
$= 2\pi(6)(6) + 2\pi(6)^2$
$= 72\pi + 72\pi$
$= 144\pi \text{ cm}^2$

d. $SA = 2\pi rh + 2\pi r^2$
$120\pi = 2\pi(5)h + 2\pi(5)^2$
$120\pi = 10\pi h + 50\pi$
$70\pi = 10\pi h$
$h = 7 \text{ cm}$
$V = \pi r^2 h$
$= \pi(5)^2(7)$
$= 175\pi \text{ cm}^3$

6. For a cube, $SA = 6s^2$ and $V = s^3$, where s is the length of any edge of the cube.

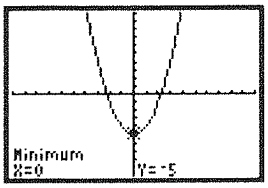
a. $SA = 6(3)^2$
$= 54 \text{ cm}^2$
$V = 3^3$
$= 27 \text{ cm}^3$

b. $SA = 6(\sqrt{5})^2$
$= 30 \text{ cm}^2$
$V = (\sqrt{5})^3$
$= 5\sqrt{5} \text{ cm}^3$

c. $SA = 6(2\sqrt{3})^2$
$= 72 \text{ cm}^2$
$V = (2\sqrt{3})^3$
$= 24\sqrt{3} \text{ cm}^3$

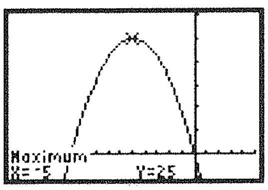
d. $SA = 6(2k)^2$
$= 24k^2 \text{ cm}^2$
$V = (2k)^3$
$= 8k^3 \text{ cm}^3$

7. a. $(3, \infty)$
b. $(-\infty, -2]$
c. $(-\infty, 0)$
d. $[-5, \infty)$
e. $(-2, 8]$
f. $(-4, 4)$
8. a. $\{x \in \mathbf{R} | x > 5\}$
b. $\{x \in \mathbf{R} | x \leq -1\}$
c. $\{x \in \mathbf{R}\}$
d. $\{x \in \mathbf{R} | -10 \leq x \leq 12\}$
e. $\{x \in \mathbf{R} | -1 < x < 3\}$
f. $\{x \in \mathbf{R} | 2 \leq x < 20\}$
9. a.



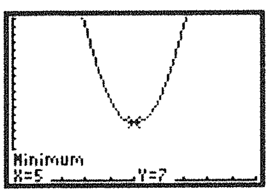
The function has a minimum value of -5 and no maximum value.

b.



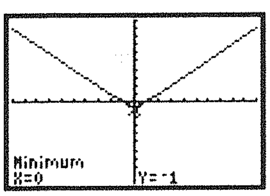
The function has a maximum value of 25 and no minimum value.

c.



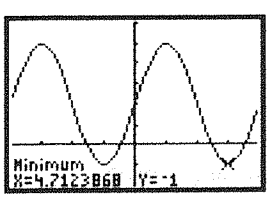
The function has a minimum value of 7 and no maximum value.

d.

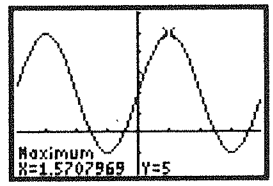


The function has a minimum value of -1 and no maximum value.

e.

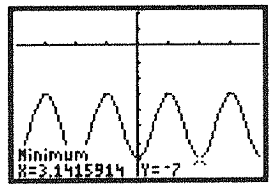


The function has a minimum value of -1.

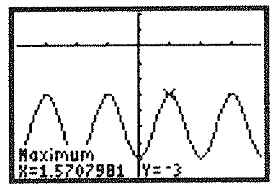


The function has a maximum value of 5.

f.



The function has a minimum value of -7.



The function has a maximum value of -3.

3.1 Higher-Order Derivatives, Velocity, and Acceleration, pp. 127–129

1. $v(1) = 2 - 1 = 1$
$v(5) = 10 - 25 = -15$
At $t = 1$, the velocity is positive; this means that the object is moving in whatever is the positive direction for the scenario. At $t = 5$, the velocity is negative; this means that the object is moving in whatever is the negative direction for the scenario.

2. a. $y = x^{10} + 3x^6$
$y' = 10x^9 + 18x^5$
$y'' = 90x^8 + 90x^4$

b. $f(x) = \sqrt{x} = x^{\frac{1}{2}}$
$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$
$f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}$

c. $y = (1 - x)^2$
$y' = 2(1 - x)(-1)$
$= -2 + 2x$
$y'' = 2$

d. $h(x) = 3x^4 - 4x^3 - 3x^2 - 5$
$h'(x) = 12x^3 - 12x^2 - 6x$
$h''(x) = 36x^2 - 24x - 6$

e. $y = 4x^{\frac{3}{2}} - x^{-2}$
$y' = 6x^{\frac{1}{2}} + 2x^{-3}$

$y'' = 3x^{-\frac{1}{2}} - 6x^{-4}$

$= \dfrac{3}{\sqrt{x}} - \dfrac{6}{x^4}$

f. $f(x) = \dfrac{2x}{x+1}$

$f'(x) = \dfrac{(x+1)(2) - (2x)(1)}{(x+1)^2}$

$= \dfrac{2x + 2 - 2x}{(x+1)^2}$

$= \dfrac{2}{(x+1)^2}$

$f''(x) = \dfrac{(x+1)^2(0) - (2)(2(x+1))}{(x+1)^4}$

$= \dfrac{-4x - 4}{(x+1)^4}$

g. $y = x^2 + x^{-2}$

$y' = 2x - 2x^{-3}$

$y'' = 2 + 6x^{-4}$

$= 2 + \dfrac{6}{x^4}$

h. $g(x) = (3x - 6)^{\frac{1}{2}}$

$g'(x) = \dfrac{3}{2}(3x - 6)^{-\frac{1}{2}}$

$g''(x) = -\dfrac{9}{4}(3x - 6)^{-\frac{3}{2}}$

$= -\dfrac{9}{4(3x - 6)^{\frac{3}{2}}}$

i. $y = (2x + 4)^3$

$y' = 6(2x + 4)^2$

$y'' = 24(2x + 4)$

$= 48x + 96$

j. $h(x) = x^{\frac{5}{3}}$

$h'(x) = \dfrac{5}{3}x^{\frac{2}{3}}$

$h''(x) = \dfrac{10}{9}x^{-\frac{1}{3}}$

$= \dfrac{10}{9x^{\frac{1}{3}}}$

3. a. $s(t) = 5t^2 - 3t + 15$

$v(t) = 10t - 3$

$a(t) = 10$

b. $s(t) = 2t^3 + 36t - 10$

$v(t) = 6t^2 + 36$

$a(t) = 12t$

c. $s(t) = t - 8 + \dfrac{6}{t}$

$= t - 8 + 6t^{-1}$

$v(t) = 1 - 6t^{-2}$

$a(t) = 12t^{-3}$

d. $s(t) = (t - 3)^2$

$v(t) = 2(t - 3)$

$a(t) = 2$

e. $s(t) = \sqrt{t + 1}$

$v(t) = \dfrac{1}{2}(t + 1)^{-\frac{1}{2}}$

$a(t) = -\dfrac{1}{4}(t + 1)^{-\frac{3}{2}}$

f. $s(t) = \dfrac{9t}{t + 3}$

$v(t) = \dfrac{9(t + 3) - 9t}{(t + 3)^2}$

$= \dfrac{27}{(t + 3)^2}$

$a(t) = -54(t + 3)^{-3}$

4. a. i. $t = 3$

ii. $1 < t < 3$

iii. $3 < t < 5$

b. i. $t = 3, t = 7$

ii. $1 < t < 3, 7 < t < 9$

iii. $3 < t < 7$

5. a. $s = \dfrac{1}{3}t^3 - 2t^2 + 3t$

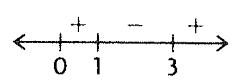
$v = t^2 - 4t + 3$

$a = 2t - 4$

b. For $v = 0$,

$(t - 3)(t - 1) = 0$

$t = 3$ or $t = 1$.



The direction of the motion of the object changes at $t = 1$ and $t = 3$.

c. Initial position is $s(0) = 0$.

Solving,

$0 = \dfrac{1}{3}t^3 - 2t^2 + 3t$

$= t^3 - 6t^2 + 9t$

$= t(t^2 - 6t + 9)$

$= t(t - 3)^2$

$t = 0$ or $t = 3$

$s = 0$ or $s = 0$.

The object returns to its initial position after 3 s.

6. a. $s = -\frac{1}{3}t^2 + t + 4$

$v = -\frac{2}{3}t + 1$

$v(1) = -\frac{2}{3} + 1$

$= \frac{1}{3}$

$v(4) = -\frac{2}{3}(4) + 1$

$= -\frac{5}{3}$

For $t = 1$, moving in a positive direction.
For $t = 4$, moving in a negative direction.

b. $s(t) = t(t-3)^2$

$v(t) = (t-3)^2 + 2t(t-3)$

$= (t-3)(t-3+2t)$

$= (t-3)(3t-3)$

$= 3(t-1)(t-3)$

$v(1) = 0$

$v(4) = 9$

For $t = 1$, the object is stationary.
$t = 4$, the object is moving in a positive direction.

c. $s(t) = t^3 - 7t^2 + 10t$

$v(t) = 3t^2 - 14t + 10$

$v(1) = -1$

$v(4) = 2$

For $t = 1$, the object is moving in a negative direction.
For $t = 4$, the object is moving in a positive direction.

7. a. $s(t) = t^2 - 6t + 8$

$v(t) = 2t - 6$

b. $2t - 6 = 0$

$t = 3$ s

8. $s(t) = 40t - 5t^2$

$v(t) = 40 - 10t$

a. When $v = 0$, the object stops rising.

$t = 4$ s

b. Since $s(t)$ represents a quadratic function that opens down because $a = -5 < 0$, a maximum height is attained. It occurs when $v = 0$. Height is a maximum for

$s(4) = 160 - 5(16)$

$= 80$ m.

9. $s(t) = 8 - 7t + t^2$

$v(t) = -7 + 2t$

$a(t) = 2$

a. $v(5) = -7 + 10$

$= 3$ m/s

b. $a(5) = 2$ m/s^2

10. $s(t) = t^{\frac{5}{2}}(7 - t)$

a. $v(t) = \frac{5}{2}t^{\frac{3}{2}}(7 - t) - t^{\frac{5}{2}}$

$= \frac{35}{2}t^{\frac{3}{2}} - \frac{5}{2}t^{\frac{5}{2}} - t^{\frac{5}{2}}$

$= \frac{35}{2}t^{\frac{3}{2}} - \frac{7}{2}t^{\frac{5}{2}}$

$a(t) = \frac{105}{2}t^{\frac{1}{2}} - \frac{35}{4}t^{\frac{3}{2}}$

b. The object stops when its velocity is 0.

$v(t) = \frac{35}{2}t^{\frac{3}{2}} - \frac{7}{2}t^{\frac{5}{2}}$

$= \frac{7}{2}t^{\frac{3}{2}}(5 - t)$

$v(t) = 0$ for $t = 0$ (when it starts moving) and $t = 5$.
So the object stops after 5 s.

c. The direction of the motion changes when its velocity changes from a positive to a negative value or visa versa.

t	$0 \le t < 5$	$t = 5$	$t > 5$
$v(t)$	$(+)(+) = +$	0	$(+)(-) = -$

$v(t) = \frac{7}{2}t^{\frac{3}{2}}(5 - t)$ $\quad$ $v(t) = 0$ for $t = 5$

Therefore, the object changes direction at 5 s.

d. $a(t) = 0$ for $\frac{35}{4}t^{\frac{1}{2}}(6 - t) = 0$.

$t = 0$ or $t = 6$ s.

t	$0 < t < 6$	$t = 6$	$t > 6$
$a(t)$	$(+)(+) = +$	0	$(+)(-) = -$

Therefore, the acceleration is positive for $0 < t < 6$ s.
Note: $t = 0$ yields $a = 0$.

e. At $t = 0$, $s(0) = 0$. Therefore, the object's original position is at 0, the origin.
When $s(t) = 0$,

$t^{\frac{5}{2}}(7 - t) = 0$

$t = 0$ or $t = 7$.

Therefore, the object is back to its original position after 7 s.

11. a. $h(t) = -5t^2 + 25t$

$v(t) = -10t + 25$

$v(0) = 25$ m/s

b. The maximum height occurs when $v(t) = 0$.

$$-10t + 25 = 0$$
$$t = 2.5 \text{ s}$$
$$h(2.5) = -5(2.5)^2 + 25(2.5)$$
$$= 31.25 \text{ m}$$

c. The ball strikes the ground when $h(t) = 0$.

$$-5t^2 + 25t = 0$$
$$-5t(t - 5) = 0$$
$$t = 0 \text{ or } t = 5$$

The ball strikes the ground at $t = 5$ s.

$$v(5) = -50 + 25$$
$$= -25 \text{ m/s}$$

12. $s(t) = 6t^2 + 2t$

$$v(t) = 12t + 2$$
$$a(t) = 12$$

a. $v(8) = 96 + 2 = 98$ m/s

Thus, as the dragster crosses the finish line at $t = 8$ s, the velocity is 98 m/s. Its acceleration is constant throughout the run and equals 12 m/s^2.

b. $s = 60$

$$6t^2 + 2t - 60 = 0$$
$$2(3t^2 + t - 30) = 0$$
$$2(3t + 10)(t - 3) = 0$$
$$t = \frac{-10}{3} \quad \text{or} \quad t = 3$$

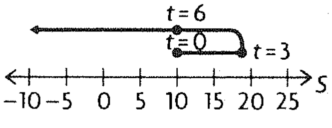
inadmissible $0 \le t \le 8$ $\qquad v(3) = 36 + 2 = 38$

Therefore, the dragster was moving at 38 m/s when it was 60 m down the strip.

13. a. $s = 10 + 6t - t^2$

$$v = 6 - 2t$$
$$= 2(3 - t)$$
$$a = -2$$

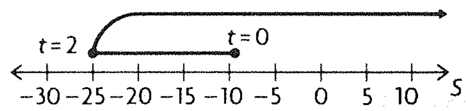
The object moves to the right from its initial position of 10 m from the origin, 0, to the 19 m mark, slowing down at a rate of 2 m/s^2. It stops at the 19 m mark then moves to the left accelerating at 2 m/s^2 as it goes on its journey into the universe. It passes the origin after $(3 + \sqrt{19})$ s.



b. $s = t^3 - 12t - 9$

$$v = 3t^2 - 12$$
$$= 3(t^2 - 4)$$
$$= 3(t - 2)(t + 2)$$
$$a = 6t$$

The object begins at 9 m to the left of the origin, 0, and slows down to a stop after 2 s when it is 25 m to the left of the origin. Then, the object moves to the right accelerating at faster rates as time increases. It passes the origin just before 4 s (approximately 3.7915) and continues to accelerate as time goes by on its journey into space.



14. $s(t) = t^5 - 10t^2$

$$v(t) = 5t^4 - 20t$$
$$a(t) = 20t^3 - 20$$

For $a(t) = 0$,

$$20t^3 - 20 = 0$$
$$20(t^3 - 1) = 0$$
$$t = 1.$$

Therefore, the acceleration will be zero at 1 s.

$$s(1) = 1 - 10$$
$$= -9$$
$$< 0$$
$$v(1) = 5 - 20$$
$$= -15$$
$$< 0$$

Since the signs of both s and v are the same at $t = 1$, the object is moving away from the origin at that time.

15. a. $s(t) = kt^2 + (6k^2 - 10k)t + 2k$

$$v(t) = 2kt + (6k^2 - 10k)$$
$$a(t) = 2k + 0$$
$$= 2k$$

Since $k \ne 0$ and $k \in \mathbf{R}$, then $a(t) = 2k \ne 0$ and an element of the Real numbers. Therefore, the acceleration is constant.

b. For $v(t) = 0$

$$2kt + 6k^2 - 10k = 0$$
$$2kt = 10k - 6k^2$$
$$t = 5 - 3k$$
$$k \ne 0$$

$$s(5 - 3k)$$
$$= k(5 - 3k)^2 + (6k^2 - 10k)(5 - 3k) + 2k$$
$$= k(25 - 30k + 9k^2) + 30k^2 - 18k^3 - 50k + 30k^2 + 2k$$
$$= 25k - 30k^2 + 9k^3 + 30k^2 - 18k^3 - 50k + 30k^2 + 2k$$
$$= -9k^3 + 30k^2 - 23k$$

Therefore, the velocity is 0 at $t = 5 - 3k$, and its position at that time is $-9k^3 + 30k^2 - 23k$.

16. a. The acceleration is continuous at $t = 0$ if $\lim_{t \to 0} a(t) = a(0)$.

For $t \geq 0$,

$$s(t) = \frac{t^3}{t^2 + 1}$$

$$\text{and } v(t) = \frac{3t^2(t^2 + 1) - 2t(t^3)}{(t^2 + 1)^2}$$

$$= \frac{t^4 + 3t^2}{(t^2 + 1)^2}$$

$$\text{and } a(t) = \frac{(4t^3 + 6t)(t^2 + 1)^2}{(t^2 + 1)^2}$$

$$- \frac{2(t^2 + 1)(2t)(t^4 + 3t^2)}{(t^2 + 1)^2}$$

$$= \frac{(4t^3 + 6t)(t^2 + 1) - 4t(t^4 + 3t^2)}{(t^2 + 1)^3}$$

$$= \frac{4t^5 + 6t^3 + 4t^3 + 6t - 4t^5 - 12t^3}{(t^2 + 1)^3}$$

$$= \frac{-2t^3 + 6t}{(t^2 + 1)^3}$$

$$\text{Therefore, } a(t) = \begin{cases} 0, \text{ if } t < 0 \\ \dfrac{-2t^3 + 6t}{(t^2 + 1)^3}, \text{ if } t \geq 0 \end{cases}$$

$$\text{and } v(t) = \begin{cases} 0, \text{ if } t < 0 \\ \dfrac{t^4 + 3t^2}{(t^2 + 1)^2}, \text{ if } t \geq 0 \end{cases}$$

$$\lim_{t \to 0^-} a(t) = 0, \ \lim_{t \to 0^+} a(t) = \frac{0}{1}$$
$$= 0.$$

Thus, $\lim_{t \to 0} a(t) = 0$.

$$\text{Also, } a(0) = \frac{0}{1}$$
$$= 0.$$

Therefore, $\lim_{t \to 0} a(t) = a(0)$.

Thus, the acceleration is continuous at $t = 0$.

b. $$\lim_{t \to +\infty} v(t) = \lim_{t \to +\infty} \frac{t^4 + 3t^2}{t^4 + 2t^2 + 1}$$

$$= \lim_{t \to +\infty} \frac{1 + \frac{3}{t^2}}{1 + \frac{2}{t^2} + \frac{1}{t^4}}$$

$$= 1$$

$$\lim_{t \to +\infty} a(t) = \lim_{t \to \infty} \frac{\frac{-2}{t^3} + \frac{6}{t^4}}{1 + \frac{3}{t^2} + \frac{3}{t^4} + \frac{1}{t^6}}$$

$$= \frac{0}{1}$$
$$= 0$$

17. $v = \sqrt{b^2 + 2gs}$

$$v = (b^2 + 2gs)^{\frac{1}{2}}$$

$$\frac{dv}{dt} = \frac{1}{2}(b^2 + 2gs)^{-\frac{1}{2}} \cdot \left(0 + 2g\frac{ds}{dt}\right)$$

$$a = \frac{1}{2v} \cdot 2gv$$

$$a = g$$

Since g is a constant, a is a constant, as required.

Note: $\dfrac{ds}{dt} = v$

$$\frac{dv}{dt} = a$$

18. $F = m_0 \dfrac{d}{dt}\left(\dfrac{v}{\sqrt{1 - (\frac{v}{c})^2}}\right)$

Using the quotient rule,

$$= \frac{m_0 \frac{dv}{dt}(1 - \frac{v^2}{c^2})^{\frac{1}{2}} - \frac{1}{2}(1 - \frac{v^2}{c^2})^{-\frac{1}{2}}\left(-\frac{2v\frac{dv}{dt}}{c^2}\right) \cdot v}{1 - \frac{v^2}{c^2}}$$

Since $\dfrac{dv}{dt} = a$,

$$= \frac{m_0\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}\left[a\left(1 - \frac{v^2}{c^2}\right) + \frac{v^2 a}{c^2}\right]}{1 - \frac{v^2}{c^2}}$$

$$= \frac{m_0\left[\frac{ac^2 - av^2}{c^2} + \frac{v^2 a}{c^2}\right]}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}}$$

$$= \frac{m_0 ac^2}{c^2\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}}$$

$$= \frac{m_0 a}{c^2\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}}, \text{ as required.}$$

3.2 Maximum and Minimum on an Interval (Extreme Values), pp. 135–138

1. a. The algorithm can be used; the function is continuous.

b. The algorithm cannot be used; the function is discontinuous at $x = 2$.

c. The algorithm cannot be used; the function is discontinuous at $x = 2$.

d. The algorithm can be used; the function is continuous on the given domain.

2. a. max 8; min -12

b. max 30; min -5

c. max 100; min -100

d. max 30; min -20

3. a. $f(x) = x^2 - 4x + 3, 0 \le x \le 3$

$f'(x) = 2x - 4$

Let $2x - 4 = 0$ for max or min

$x = 2$

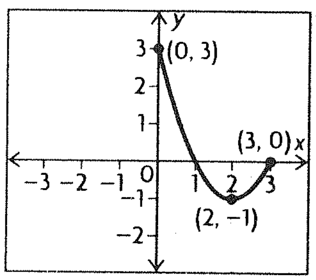
$f(0) = 3$

$f(2) = 4 - 8 + 3 = -1$

$f(3) = 9 - 12 + 3 = 0$

max is 3 at $x = 0$

min is -1 at $x = 2$



b. $f(x) = (x - 2)^2, 0 \le x \le 2$

$f'(x) = 2x - 4$

Let $f'(x) = 0$ for max or min

$2x - 4 = 0$

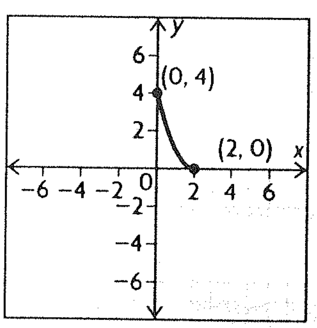
$x = 2$

$f(0) = 4$

$f(2) = 0$

max is 4 at $x = 0$

min is 2 at $x = 2$



c. $f(x) = x^3 - 3x^2, -1 \le x \le 3$

$f'(x) = 3x^2 - 6x$

Let $f'(x) = 0$ for max or min

$3x^2 - 6x = 0$

$3x(x - 2) = 0$

$x = 0$ or $x = 2$

$f(-1) = -1 - 3$

$= -4$

$f(0) = 0$

$f(2) = 8 - 12$

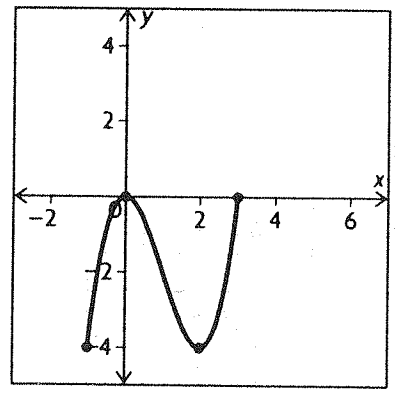
$= -4$

$f(3) = 27 - 27$

$= 0$

min is -4 at $x = -1, 2$

max is 0 at $x = 0, 3$



d. $f(x) = x^3 - 3x^2, x \in [-2, 1]$

$f'(x) = 3x^2 - 6x$

Let $f'(x) = 0$ for max or min

$3x^2 - 6x = 0$

$3x(x - 2) = 0$

$x = 0$ or $x = 2$

$x = 2$ is outside the given interval.

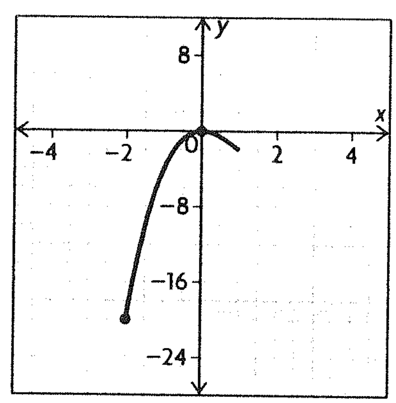
$f(-2) = -20$

$f(0) = 0$

$f(1) = -2$

max is 0 at $x = 0$

min is -20 at $x = -2$



e. $f(x) = 2x^3 - 3x^2 - 12x + 1, x \in [-2, 0]$

$f'(x) = 6x^2 - 6x - 12$

Let $f'(x) = 0$ for max or min

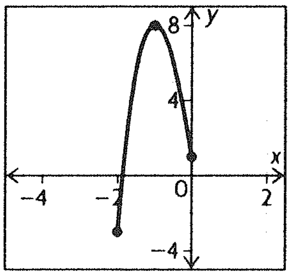
$6x^2 - 6x - 12 = 0$

$x^2 - x - 2 = 0$

$(x - 2)(x + 1) = 0$

$x = 2$ or $x = -1$

$f(-2) = -16 - 12 + 24 + 1$
$= -3$
$f(-1) = 8$
$f(0) = 1$
$f(2) = \text{not in region}$
max of 8 at $x = -1$
min of -3 at $x = -2$



f. $f(x) = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 6x, x \in [0, 4]$

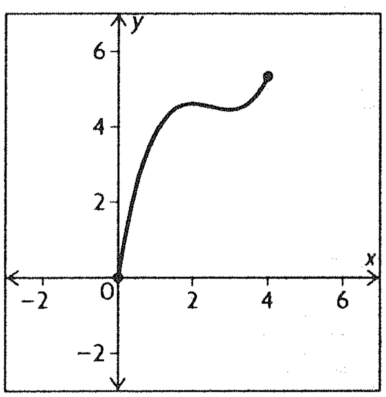
$f'(x) = x^2 - 5x + 6$
Let $f'(x) = 0$ for max or min
$x^2 - 5x + 6 = 0$
$(x - 2)(x - 3) = 0$
$x = 2$ or $x = 3$
$f(0) = 0$

$f(2) = \frac{14}{3}$

$f(3) = \frac{9}{2}$

$f(4) = \frac{16}{3}$

max is $\frac{16}{3}$ at $x = 4$
min is 0 at $x = 0$



4. a. $f(x) = x + \frac{4}{x}$

$f'(x) = 1 - \frac{4}{x^2}$

$= \frac{x^2 - 4}{x^2}$

Set $f'(x) = 0$ to solve for the critical values.

$\frac{x^2 - 4}{x^2} = 0$

$x^2 - 4 = 0$
$(x - 2)(x + 2) = 0$
$x = 2, x = -2$

Now, evaluate the function, $f(x)$, at the critical values and the endpoints. Note, however, that -2 is not in the domain of the function.

$f(1) = 1 + \frac{4}{1} = 1 + 4 = 5$

$f(2) = 2 + \frac{4}{2} = 2 + 2 = 4$

$f(10) = 10 + \frac{4}{10} = \frac{50}{5} + \frac{2}{5} = \frac{52}{5} = 10.4$

So, the minimum value in the interval is 4 when $x = 2$ and the maximum value is 10.4 when $x = 10$.

b. $f(x) = 4\sqrt{x} - x, 2 \leq x \leq 9$

$f'(x) = 2x^{-\frac{1}{2}} - 1$
Let $f'(x) = 0$ for max or min

$\frac{2}{\sqrt{x}} - 1 = 0$

$\sqrt{x} = 2$
$x = 4$

$f(2) = 4\sqrt{2} - 2 \doteq 3.6$
$f(4) = 4\sqrt{4} - 4 = 4$
$f(9) = 4\sqrt{9} - 9 = 3$
min value of 3 when $x = 9$
max value of 4 when $x = 4$

c. $f(x) = \frac{1}{x^2 - 2x + 2}, 0 \leq x \leq 2$

$f'(x) = -(x^2 - 2x + 2)^{-2}(2x - 2)$

$= -\frac{2x - 2}{(x^2 - 2x + 2)^2}$

Let $f'(x) = 0$ for max or min.

$-\frac{2x - 2}{(x^2 - 2x + 2)} = 0$

$2x - 2 = 0$
$x = 1$

$f(0) = \frac{1}{2}, f(1) = 1, f(2) = \frac{1}{2}$

max value of 1 when $x = 1$
min value of $\frac{1}{2}$ when $x = 0, 2$

d. $f(x) = 3x^4 - 4x^3 - 36x^2 + 20$

$f'(x) = 12x^3 - 12x^2 - 72x$
Set $f'(x) = 0$ to solve for the critical values.
$12x^3 - 12x^2 - 72x = 0$
$12x(x^2 - x - 6) = 0$

$12x(x-3)(x+2)=0$

$x=0, x=3, x=-2$

Now, evaluate the function, $f(x)$, at the critical values and the endpoints.

$f(-3)=3(-3)^4-4(-3)^3-36(-3)^2+20=47$

$f(-2)=3(-2)^4-4(-2)^3-36(-2)^2+20=-44$

$f(0)=3(0)^4-4(0)^3-36(0)^2+20=20$

$f(3)=3(3)^4-4(3)^3-36(3)^2+20=-169$

$f(4)=3(4)^4-4(4)^3-36(4)^2+20=-44$

So, the minimum value in the interval is -169 when $x=3$ and the maximum value is 47 when $x=-3$.

e. $f(x)=\dfrac{4x}{x^2+1}, -2\le x\le 4$

$$f'(x)=\frac{4(x^2+1)-2x(4x)}{(x^2+1)^2}$$

$$=\frac{-4x^2+4}{x^2+1}$$

Let $f'(x)=0$ for max or min:

$-4x^2+4=0$

$x^2=1$

$x=\pm 1$

$f(-2)=\dfrac{-8}{5}$

$f(-1)=\dfrac{-4}{2}$

$=-2$

$f(1)=\dfrac{4}{2}$

$=2$

$f(4)=\dfrac{16}{17}$

max value of 2 when $x=1$

min value of -2 when $x=-1$

f. Note that part e. is the same function but restricted to a different domain. So, from e. it is seen that the critical points are $x=1$ and $x=-1$.

Now, evaluate the function, $f(x)$, at the critical values and the endpoints. Note, however, that -1 and 1 are not in the domain of the function. Therefore, the only points that need to be checked are the endpoints.

$$f(2)=\frac{4(2)}{(2)^2+1}=\frac{8}{5}=1.6$$

$$f(4)=\frac{4(4)}{(4)^2+1}=\frac{16}{17}\doteq 0.94$$

So, the minimum value in the interval is 0.94 when $x=4$ and the maximum value is 1.6 when $x=2$.

5. a. $v(t)=\dfrac{4t^2}{4+t^3}, t\ge 0$

Interval $1\le t\le 4$

$v(1)=\dfrac{4}{5}$

$v(4)=\dfrac{16}{17}$

$$v'(t)=\frac{(4+t^3)(8t)-4t^2(3t^2)}{(4+t^3)^2}=0$$

$32t+8t^4-12t^4=0$

$-4t(t^3-8)=0$

$t=0, t=2$

$v(2)=\dfrac{16}{12}=\dfrac{4}{3}$

max velocity is $\frac{4}{3}$ m/s

min velocity is $\frac{4}{5}$ m/s

b. $v(t)=\dfrac{4t^2}{1+t^2}$

$$v'(t)=\frac{(1+t^2)(8t)-(4t^2)(2t)}{(1+t^2)^2}$$

$$=\frac{8t+8t^3-8t^3}{(1+t^2)^2}$$

$$=\frac{8t}{(1+t^2)^2}$$

$$\frac{8t}{(1+t^2)^2}=0$$

$8t=0$

$t=0$

$f(0)=0$ is the minimum value that occurs at $x=0$. There is no maximum value on the interval. As x approaches infinity, $f(x)$ approaches the horizontal asymptote $y=4$.

6. $N(t)=30t^2-240t+500$

$N'(t)=60t-240$

$60t-240=0$

$t=4$

$N(0)=500$

$N(4)=30(16)-240(4)+500=20$

$N(7)=30(49)-240(7)+500=290$

The lowest number is 20 bacteria/cm^3.

7. a. $E(v)=\dfrac{1600v}{v^2+6400} 0\le v\le 100$

$$E'(v)=\frac{1600(v^2+6400)-1600v(2v)}{(v^2+6400)^2}$$

Let $E'(N)=0$ for max or min

$1600v^2+6400\times 1600-3200v^2=0$

$1600v^2=6400\times 1600$

$v=\pm 80$

$E(0) = 0$
$E(80) = 10$
$E(100) = 9.756$

The legal speed that maximizes fuel efficiency is 80 km/h.

b. $E(v) = \dfrac{1600v}{v^2 + 6400}$ $0 \le v \le 50$

$$E'(v) = \frac{1600(v^2 + 6400) - 1600v(2v)}{(v^2 + 6400)^2}$$

Let $E'(N) = 0$ for max or min

$1600v^2 + 6400 \times 1600 - 3200v^2 = 0$
$1600v^2 = 6400 \times 1600$
$v = \pm 80$
$E(0) = 0$
$E(50) = 9$

The legal speed that maximizes fuel efficiency is 50 km/h.

c. The fuel efficiency will be increasing when $E'(v) > 0$. This will show when the slopes of the values of $E(v)$ are positive, and hence increasing. From part a. it is seen that there is one critical value for $v > 0$. This is $v = 80$.

v	slope of E(v)
$0 \le v < 80$	+
$80 < v \le 100$	−

Therefore, within the legal speed limit of 100 km/h, the fuel efficiency E is increasing in the speed interval $0 \le v < 80$.

d. The fuel efficiency will be decreasing when $E'(v) < 0$. This will show when the slopes of the values of $E(v)$ are negative, and hence decreasing. From part a. it is seen that there is one critical value for $v > 0$. This is $v = 80$.

v	slope of E(v)
$0 \le v < 80$	+
$80 < v \le 100$	−

Therefore, within the legal speed limit of 100 km/h, the fuel efficiency E is decreasing in the speed interval $80 < v \le 100$.

8. $C(t) = \dfrac{0.1t}{(t + 3)^2}$, $1 \le t \le 6$

$$C'(t) = \frac{0.1(t + 3)^2 - 0.2t(t + 3)}{(t + 3)^4} = 0$$

$(t + 3)(0.1t + 0.3 - 0.2t) = 0$
$t = 3$

$C(1) \doteq 0.006\,25$
$C(3) = 0.0083$, $C(6) \doteq 0.0074$

The min concentration is at $t = 1$ and the max concentration is at $t = 3$.

9. $P(t) = 2t + \dfrac{1}{162t + 1}$, $0 \le t \le 1$

$P'(t) = 2 - (162t + 1)^{-2}(162) = 0$

$$\frac{162}{(162t + 1)^2} = 2$$

$81 = 162^2 + t^2 + 324t + 1$
$162^2t^2 + 324t - 80 = 0$
$81^2t^2 + 81t - 20 = 0$
$(81t + 5)(81t - 4) = 0$

$t > 0$ $\quad t = \dfrac{4}{81}$
$= 0.05$
$P(0) = 1$
$P(0.05) = 0.21$
$P(1) = 2.01$

Pollution is at its lowest level in 0.05 years or approximately 18 days.

10. $r(x) = \dfrac{1}{400}\left(\dfrac{4900}{x} + x\right)$

$$r'(x) = \frac{1}{400}\left(\frac{-4900}{x^2} + 1\right) = 0$$

Let $r'(x) = 0$
$x^2 = 4900$,
$x = 70$, $x > 0$
$r(30) = 0.4833$
$r(70) = 0.35$
$r(120) = 0.402$

A speed of 70 km/h uses fuel at a rate of 0.35 L/km. Cost of trip is $0.35 \times 200 \times 0.45 = \$31.50$.

11. $f(x) = 0.001x^3 - 0.12x^2 + 3.6x + 10$,
$0 \le x \le 75$

$f'(x) = 0.003x^2 - 0.24x + 3.6$
Set $0 = 0.003x^2 - 0.24x + 3.6$

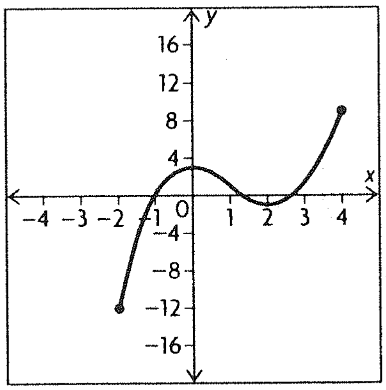
$$x = \frac{0.24 \pm \sqrt{(-0.24)^2 - 4(0.003)(3.6)}}{2(0.003)}$$

$$x = \frac{0.24 \pm 0.12}{0.006}$$

$x = 60$ or $x = 20$
$f(0) = 10$
$f(20) = 42$
$f(60) = 10$
$f(75) = 26.875$

Absolute max. value $= 42$ at $(20, 42)$ and absolute min. value $= 10$ at $(0, 10)$ and $(60, 10)$.

12. a.



b. D: $-2 \le x \le 4$

c. increasing: $-2 \le x < 0$

$2 < x \le 4$

decreasing: $0 < x < 2$

13. Absolute max.: Compare all local maxima and values of $f(a)$ and $f(b)$ when domain of $f(x)$ is $a \le x \le b$. The one with highest value is the absolute maximum.

Absolute min.: We need to consider all local minima and the value of $f(a)$ and $f(b)$ when the domain of $f(x)$ is $a \le x \le b$. Compare them and the one with the lowest value is the absolute minimum.

You need to check the endpoints because they are not necessarily critical points.

14. $C(x) = 3000 + 9x + 0.05x^2,\ 1 \le x \le 300$

$$\text{Unit cost } u(x) = \frac{C(x)}{x}$$

$$= \frac{3000 + 9x + 0.05x^2}{x}$$

$$= \frac{3000}{x} + 9 + 0.05x$$

$$U'(x) = \frac{-3000}{x^2} + 0.05$$

For max or min, let $U'(x) = 0$:

$0.05x^2 = 3000$

$x^2 = 60\,000$

$x \doteq 244.9$

$U(1) = 3009.05$

$U(244) = 33.4950$

$U(245) = 33.4948$

$U(300) = 34.$

Production level of 245 units will minimize the unit cost to \$33.49.

15. $C(x) = 6000 + 9x + 0.05x^2$

$$U(x) = \frac{C(x)}{x}$$

$$= \frac{6000 + 9x + 0.05x^2}{x}$$

$$= \frac{6000}{x} + 9 + 0.05x$$

$$U'(x) = -\frac{6000}{x^2} + 0.05$$

Set $U'(x) = 0$ and solve for x.

$$-\frac{6000}{x^2} + 0.05 = 0$$

$$0.05 = \frac{6000}{x^2}$$

$0.05x^2 = 6000$

$x^2 = 120\,000$

$x \doteq 346.41$

However, 346.41 is not in the given domain of $1 \le x \le 300$.

Therefore, the only points that need to be checked are the endpoints.

$f(1) = 6009.05$

$f(300) = 44$

Therefore, a production level of 300 units will minimize the unit cost to \$44.

Mid-Chapter Review, pp. 139–140

1. a. $h(x) = 3x^4 - 4x^3 - 3x^2 - 5$

$h'(x) = 12x^3 - 12x^2 - 6x$

$h''(x) = 36x^2 - 24x - 6$

b. $f(x) = (2x - 5)^3$

$f'(x) = 6(2x - 5)^2$

$f''(x) = 24(2x - 5)$

$= 48x - 120$

c. $y = 15(x + 3)^{-1}$

$y' = -15(x + 3)^{-2}$

$y'' = 30(x + 3)^{-3}$

$$= \frac{30}{(x + 3)^3}$$

d. $g(x) = (x^2 + 1)^{\frac{1}{2}}$

$g'(x) = x(x^2 + 1)^{-\frac{1}{2}}$

$g''(x) = -x^2(x^2 + 1)^{-\frac{3}{2}} + (x^2 + 1)^{-\frac{1}{2}}$

$$= -\frac{x^2}{(x^2 + 1)^{\frac{3}{2}}} + \frac{1}{(x^2 + 1)^{\frac{1}{2}}}$$

2. a. $s(3) = (3)^3 - 21(3)^2 + 90(3)$

$= 27 - 189 + 270$

$= 108$

b. $v(t) = s'(t) = 3t^2 - 42t + 90$

$v(5) = 3(5)^2 - 42(5) + 90$

$= 75 - 210 + 90$
$= -45$

c. $a(t) = v'(t) = 6t - 42$
$a(4) = 6(4) - 42$
$= 24 - 42$
$= -18$

3. a. $v(t) = h'(t) = -9.8t + 6$
The initial velocity occurs when time $t = 0$.
$v(0) = -9.8(0) + 6$
$= 6$
So, the initial velocity is 6 m/s.

b. The ball reaches its maximum height when $v(t) = 0$. So set $v(t) = 0$ and solve for t.
$v(t) = 0 = -9.8t + 6$
$9.8t = 6$
$t \doteq 0.61$
Therefore, the ball reaches its maximum height at time $t \doteq 0.61$ s.

c. The ball hits the ground when the height, h, is 0.
$h(t) = 0 = -4.9t^2 + 6t + 2$

$$t = \frac{-6 \pm \sqrt{36 + 39.2}}{-9.8}$$

Taking the negative square root because the value t needs to be positive,

$$t = \frac{-6 - 8.67}{-9.8}$$

$t \doteq 1.50$
So, the ball hits the ground at time $t = 1.50$ s.

d. The question asks for the velocity, $v(t)$, when $t = 1.50$.
$v(1.50) = -9.8(1.50) + 6$
$\doteq -8.67$
Therefore, when the ball hits the ground, the velocity is -8.67 m/s.

e. The acceleration, $a(t)$, is the derivative of the velocity.
$a(t) = v'(t) = -9.8$
This is a constant function. So, the acceleration of the ball at any point in time is -9.8 m/s^2.

4. a. $v(t) = s'(t) = 4 - 14t + 6t^2$
$v(2) = 4 - 14(2) + 6(2)^2$
$= 4 - 28 + 24$
$= 0$
So, the velocity at time $t = 2$ is 0 m/s.
$a(t) = v'(t) = -14 + 12t$
$a(2) = -14 + 12(2)$
$= 10$
So, the acceleration at time $t = 2$ is 10 m/s.

b. The object is stationary when $v(t) = 0$.
$v(t) = 0 = 4 - 14t + 6t^2$
$0 = (6t - 2)(t - 2)$

$$t = \frac{1}{3}, t = 2$$

Therefore, the object is stationary at time $t = \frac{1}{3}$ s and $t = 2$ s.
Before $t = \frac{1}{3}$, $v(t)$ is positive and therefore the object is moving to the right.
Between $t = \frac{1}{3}$ and $t = 2$, $v(t)$ is negative and therefore the object is moving to the left.
After $t = 2$, $v(t)$ is positive and therefore the object is moving to the right.

c. Set $a(t) = 0$ and solve for t.
$a(t) = 0 = -14 + 12t$
$14 = 12t$

$$\frac{7}{6} = t$$

$t \doteq 1.2$
So, at time $t \doteq 1.2$ s the acceleration is equal to 0. At that time, the object is neither accelerating nor decelerating.

5. a. $f(x) = x^3 + 3x^2 + 1$
$f'(x) = 3x^2 + 6x$
Set $f'(x) = 0$ to solve for the critical values.
$3x^2 + 6x = 0$
$3x(x + 2) = 0$
$x = 0, x = -2$
Now, evaluate the function, $f(x)$, at the critical values and the endpoints.
$f(-2) = (-2)^3 + 3(-2)^2 + 1 = 5$
$f(0) = (0)^3 + 3(0)^2 + 1 = 1$
$f(2) = (2)^3 + 3(2)^2 + 1 = 21$
So, the minimum value in the interval is 1 when $x = 0$ and the maximum value is 21 when $x = 2$.

b. $f(x) = (x + 2)^2$
$f'(x) = 2(x + 2)$
$= 2x + 4$
Set $f'(x) = 0$ to solve for the critical values.
$2x + 4 = 0$
$2x = -4$
$x = -2$
Now, evaluate the function, $f(x)$, at the critical values and the endpoints.
$f(-3) = (-3 + 2)^2 = (-1)^2 = 1$
$f(-2) = (-2 + 2)^2 = 0$
$f(3) = (3 + 2)^2 = (5)^2 = 25$

So, the minimum value in the interval is 0 when $x = -2$ and the maximum value is 25 when $x = 3$.

c. $f(x) = \frac{1}{x} - \frac{1}{x^3}$

$$f'(x) = -\frac{1}{x^2} + \frac{3}{x^4} = \frac{-x^4 + 3x^2}{x^6}$$

Set $f'(x) = 0$ to solve for the critical values.

$$\frac{-x^4 + 3x^2}{x^6} = 0$$
$$-x^4 + 3x^2 = 0$$
$$x^2(-x^2 + 3) = 0$$
$$x = 0$$
$$x^2 = 3$$
$$x = \pm\sqrt{3}$$

Note, however, that $-\sqrt{3}$ and 0 are not in the given domain of the function.
Now, evaluate the function, $f(x)$, at the critical values and the endpoints.

$$f(1) = \frac{1}{1} - \frac{1}{(1)^3} = 1 - 1 = 0$$
$$f(\sqrt{3}) = \frac{1}{\sqrt{3}} - \frac{1}{(\sqrt{3})^3} \doteq 0.38$$
$$f(5) = \frac{1}{5} - \frac{1}{(5)^3} = \frac{24}{125}$$

So, the minimum value in the interval is 0 when $x = 1$ and the maximum value is 0.38 when $x = \sqrt{3}$.

6. The question asks for the maximum temperature of V.

$$V(t) = -0.000\,067t^3 + 0.008\,504\,3t^2 - 0.064\,26t + 999.87$$
$$V'(t) = -0.000\,201t^2 + 0.017\,008\,6t - 0.064\,26$$

Set $V'(t) = 0$ to solve for the critical values.

$$-0.000\,201t^2 + 0.017\,008\,6t - 0.064\,26 = 0$$
$$t^2 - 84.619\,900\,5t + 319.701\,492\,5 = 0$$

Using the quadratic formula,
$t \doteq 3.96$ and $t \doteq 80.66$.
However, 80.66 is not in the domain of the function.
Now, evaluate the function, $V(t)$, at the critical values and the endpoints.

$$V(0) = 999.87$$
$$V(3.96) \doteq 999.74$$
$$V(30) = 1003.79$$

So, the minimum value in the interval is 999.74 when temperature $t = 3.96$.
Therefore, at a temperature of $t = 3.96$°C the volume of water is the greatest in the interval.

7. a. $f(x) = x^4 - 3x$

$$f'(x) = 4x^3 - 3$$
$$f'(3) = 4(3)^3 - 3 = 105$$

b. $f(x) = 2x^3 + 4x^2 - 5x + 8$

$$f'(x) = 6x^2 + 8x - 5$$
$$f'(-2) = 6(-2)^2 + 8(-2) - 5 = 3$$

c. $f(x) = -3x^2 - 5x + 7$

$$f'(x) = -6x - 5$$
$$f''(x) = -6$$
$$f''(1) = -6$$

d. $f(x) = 4x^3 - 3x^2 + 2x - 6$

$$f'(x) = 12x^2 - 6x + 2$$
$$f''(x) = 24x - 6$$
$$f''(-3) = 24(-3) - 6 = -78$$

e. $f(x) = 14x^2 + 3x - 6$

$$f'(x) = 28x + 3$$
$$f'(0) = 28(0) + 3 = 3$$

f. $f(x) = x^4 + x^5 - x^3$

$$f'(x) = 4x^3 + 5x^4 - 3x^2$$
$$f''(x) = 12x^2 + 20x^3 - 6x$$
$$f''(4) = 12(4)^2 + 20(4)^3 - 6(4) = 1448$$

g. $f(x) = -2x^5 + 2x - 6 - 3x^3$

$$f'(x) = -10x^4 + 2 - 9x^2$$
$$f''(x) = -40x^3 - 18x$$
$$f''\left(\frac{1}{3}\right) = -40\left(\frac{1}{3}\right)^3 - 18\left(\frac{1}{3}\right) = -\frac{40}{27} - 6 = -\frac{202}{27}$$

h. $f(x) = -3x^3 - 7x^2 + 4x - 11$

$$f'(x) = -9x^2 - 14x + 4$$
$$f'\left(\frac{3}{4}\right) = -9\left(\frac{3}{4}\right)^2 - 14\left(\frac{3}{4}\right) + 4 = -\frac{81}{16} - \frac{21}{2} + 4 = -\frac{185}{16}$$

8. $s(t) = t\left(-\frac{5}{6}t + 1\right)$

$= -\frac{5}{6}t^2 + t$

$s'(t) = -\frac{5}{3}t + 1$

$s''(t) = -\frac{5}{3}$

$\doteq -1.7 \text{ m/s}^2$

9. $s(t) = 189t - t^{\frac{7}{3}}$

a. $s'(t) = 189 - \frac{7}{3}t^{\frac{4}{3}}$

$s'(0) = 189 - \frac{7}{3}(0)^{\frac{4}{3}}$

$= 189 \text{ m/s}$

b. $s'(t) = 0$

$189 - \frac{7}{3}t^{\frac{4}{3}} = 0$

$\frac{7}{3}t^{\frac{4}{3}} = 189$

$t^{\frac{4}{3}} = 81$

$t = \left(81^{\frac{3}{4}}\right)$

$t = 3^3$

$t = 27 \text{ s}$

c. $s(27) = 189(27) - (27)^{\frac{7}{3}}$

$= 5103 - 2187$

$= 2916 \text{ m}$

d. $s''(t) = -\frac{28}{9}t^{\frac{1}{3}}$

$s''(8) = -\frac{28}{9}(8)^{\frac{1}{3}}$

$= -\frac{56}{9}$

$\doteq -6.2 \text{ m/s}^2$

It is decelerating at 6.2 m/s^2.

10. $s(t) = 12t - 4t^{\frac{3}{2}}$

$s'(t) = 12t - 6t^{\frac{1}{2}}$

To find when the stone stops, set $s'(t) = 0$:

$12 - 6t^{\frac{1}{2}} = 0$

$6t^{\frac{1}{2}} = 12$

$t^{\frac{1}{2}} = 2$

$t = (2)^2$

$= 4$

$s(4) = 12(4) - 4(4)^{\frac{3}{2}}$

$= 48 - 32$

$= 16 \text{ m}$

The stone travels 16 m before its stops after 4 s.

11. a. $h(t) = -4.9t^2 + 21t + 0.45$

$0 = -4.9t^2 + 21t + 0.45$

$t = \frac{-21 \pm \sqrt{(21)^2 - 4(-4.9)(0.45)}}{2(-4.9)}$

$t = \frac{-21 \pm \sqrt{449.82}}{-9.8}$

$t \doteq 4.31$ or $t \doteq -0.021$ (rejected since $t \geq 0$)

Note that $h(0) = 0.45 > 0$ because the football is punted from that height. The function is only valid after this point.

Domain: $0 \leq t \leq 4.31$

b. $h(t) = -4.9t^2 + 21t + 0.45$

To determine the domain, find when $h'(t) = 0$.

$h'(t) = -9.8t + 21$

Set $h'(t) = 0$

$0 = -9.8t + 21$

$t \doteq 2.14$

For $0 < t < 2.14$, the height is increasing.

For $2.14 < t < 4.31$, the height is decreasing.

The football will reach its maximum height at 2.14 s.

c. $h(2.14) = -4.9(2.14)^2 + 21(2.14) + 0.45$

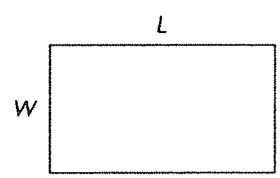
$h(2.14) \doteq -22.44 + 44.94 + 0.45$

$h(2.14) \doteq 22.95$

The football will reach a maximum height of 22.95 m.

3.3 Optimization Problems, pp. 145–147

1.



Let the length be L cm and the width be W cm.

$2(L + W) = 100$

$L + W = 50$

$L = 50 - W$

$A = L \cdot W$

$= (50 - W)(W)$

$A(W) = -W^2 + 50W$ for $0 \leq W \leq 50$

$A'(W) = -2W + 50$

Let $A'(W) = 0$:

$-2W + 50 = 0$

$W = 25$

$A(0) = 0$

$A(25) = 25 \times 25$

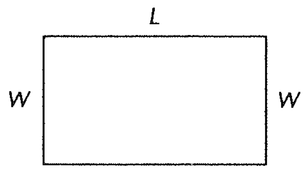
$= 625$

$A(50) = 0.$

The largest area is 625 cm^2 and occurs when $W = 25$ cm and $L = 25$ cm.

2. If the perimeter is fixed, then the figure will be a square.

3.



Let the length of L m and the width W m.

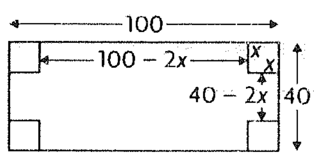
$$2W + L = 600$$
$$L = 600 - 2W$$
$$A = L \cdot W$$
$$= W(600 - 2W)$$
$$A(W) = -2w^2 + 600W, 0 \le W \le 300$$
$$A'(W) = -4w + 600$$

For max or min, let $\frac{dA}{dW} = 0$:

$$W = 50$$
$$A(0) = 0$$
$$A(150) = -2(150)^2 + 600 \times 150$$
$$= 45\,000$$
$$A(300) = 0$$

The largest area of 45 000 m^2 occurs when $W = 150$ m and $L = 300$ m.

4. Let dimensions of cut be x cm by x cm. Therefore, the height is x cm.



Length of the box is $100 - 2x$.
Width of the box is $40 - 2x$.
$V = (100 - 2x)(40 - 2x)(x)$ for domain $0 \le x \le 20$

Using Algorithm for Extreme Value,

$$\frac{dV}{dx} = (100 - 2x)(40 - 4x) + (40x - 2x^2)(-2)$$
$$= 4000 - 480x + 8x^2 - 80x + 4x^2$$
$$= 12x^2 - 560x + 4000$$

Set $\frac{dV}{dx} = 0$

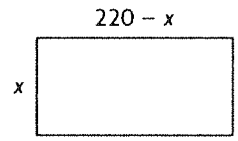
$$3x^2 - 140x + 1000 = 0$$
$$x = \frac{140 \pm \sqrt{7600}}{6}$$
$$x = \frac{140 \pm 128.8}{6}$$
$$x = 8.8 \text{ or } x = 37.9$$

Reject $x = 37.9$ since $0 \le x \le 20$

When $x = 0, V = 0$
$x = 8.8, V = 28\,850$ cm^2
$x = 20, V = 0$.

Therefore, the box has a height of 8.8 cm, a length of $100 - 2 \times 8.8 = 82.4$ cm, and a width of $40 - 3 \times 8.8 = 22.4$ cm.

5.



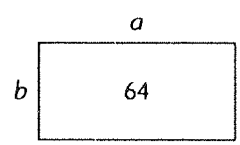
$$A(x) = x(220 - x)$$
$$A(x) = 220x - x^2$$
$$A'(x) = 220 - 2x$$

Set $A'(x) = 0$.

$$0 = 220 - 2x$$
$$x = 110$$
$$220 - 110 = 110$$
$$A'(220) = -220 < 0$$
$$A'(0) = 220 > 0$$

maximum: The dimensions that will maximize the rectangles' area are 110 cm by 110 cm.

6.



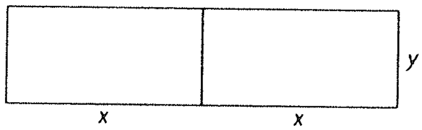
$$ab = 64$$
$$P = 2a + 2b$$
$$P = 2a + 2\left(\frac{64}{a}\right)$$
$$P = 2a + \frac{128}{a}$$
$$P = 2a + 128a^{-1}$$
$$P' = 2 - \frac{128}{a^2}$$

Set $P' = 0$

$$0 = 2 - \frac{128}{a^2}$$
$$2 = \frac{128}{a^2}$$
$$a^2 = 64$$
$$a = 8 \ (-8 \text{ is inadmissible})$$
$$b = \frac{64}{8}$$
$$b = 8$$
$$P'(1) = -126 < 0$$
$$P'(9) \doteq 1.65 > 0$$

maximum: The rectangle should have dimensions 8 m by 8 m.

7.



Given:

$4x + 3y = 1000$

$y = \frac{1000 - 4x}{3}$

$A = 2xy$

$A = 2x\left(\frac{1000 - 4x}{3}\right)$

$A = \frac{2000}{3}x - \frac{8}{3}x^2$

$A' = \frac{2000}{3} - \frac{16}{3}x$

Set $A' = 0$

$0 = \frac{2000}{3} - \frac{16}{3}x$

$\frac{16}{3}x = \frac{2000}{3}$

$x = 125$

$y = \frac{1000 - 4(125)}{3}$

$y \doteq 166.67$

$A'(250) = -\frac{2000}{3} < 0$

$A'(0) = \frac{2000}{3} > 0$

maximum: The ranger should build the corrals with the dimensions 125 m by 166.67 m to maximize the enclosed area.

8. Netting refers to the area of the rectangular prism. Minimize area while holding the volume constant.

$V = lwh$

$V = x^2y$

$144 = x^2y$

$y = \frac{144}{x^2}$

$A_{Total} = A_{Side} + A_{Top} + A_{Side} + A_{End}$

$A = xy + xy + xy + x^2$

$A = 3xy + x^2$

$A = 3x\left(\frac{144}{x^2}\right) + x^2$

$A = \frac{432}{x} + x^2$

$A = x^2 + 432x^{-1}$

$A' = 2x - 432x^{-2}$

Set $A' = 0$

$0 = 2x - 432x^{-2}$

$2x = 432x^{-2}$

$x^3 = 216$

$x = 6$

$y = \frac{144}{6^2}$

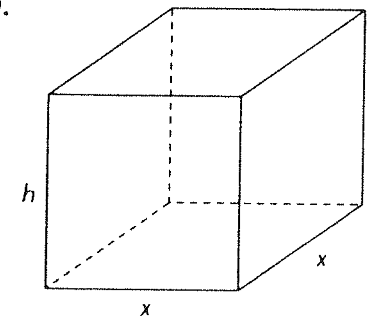
$y = 4$

$A'(4) = -19 < 0$

$A'(8) = 9.25 > 0$

minimum: The enclosure should have dimensions 4 m × 6 m × 6 m.

9.



Let the base be x by x and the height be h

$x^2h = 1000$

$h = \frac{1000}{x^2}$ ①

Surface area $= 2x^2 + 4xh$

$A = 2x^2 + 4xh$ ②

$= 2x^2 + 4x\left(\frac{1000}{x^2}\right)$

$= 2x^2 + \frac{4000}{x}$ for domain $0 \le x \le 10\sqrt{2}$

Using the max min Algorithm,

$\frac{dA}{dx} = 4x - \frac{4000}{x^2} = 0$

$x \neq 0,\ 4x^3 = 4000$

$x^3 = 1000$

$x = 10$

$A = 200 + 400 = 600\ \text{cm}^2$

Step 2: At $x \to 0$, $A \to \infty$

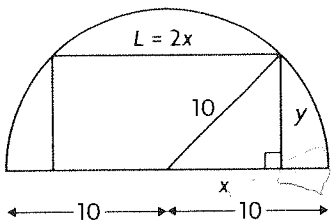
Step 3: At $x = 10\sqrt{10}$,

$A = 2000 + \frac{4000}{10\sqrt{10}} \times \frac{\sqrt{10}}{\sqrt{10}}$

$= 2000 + 40\sqrt{10}$

Minimum area is 600 cm² when the base of the box is 10 cm by 10 cm and height is 10 cm.

10.



Let the length be $2x$ and the height be y. We know $x^2 + y^2 = 100$.

$$y = \pm\sqrt{100 - x^2}$$

Omit negative area $= 2xy$

$$= 2x\sqrt{100 - x^2}$$

for domain $0 \le x \le 10$

Using the max min Algorithm,

$$\frac{dA}{dx} = 2\sqrt{100 - x^2} + 2y \cdot \frac{1}{2}(100 - x^2)^{-\frac{1}{2}}(-2x).$$

Let $\frac{dA}{dx} = 0$.

$$2\sqrt{100 - x^2} - \frac{2x^2}{\sqrt{100 - x^2}} = 0$$

$$2(100 - x^2) - 2x^2 = 0$$

$$100 = 2x^2$$

$$x^2 = 50$$

$x = 5\sqrt{2}, x > 0$. Thus, $y = 5\sqrt{2}$, $L = 10\sqrt{2}$

Part 2: If $x = 0$, $A = 0$

Part 3: If $x = 10$, $A = 0$

The largest area occurs when $x = 5\sqrt{2}$ and the area is $10\sqrt{2}\sqrt{100 - 50}$

$= 10\sqrt{2}\sqrt{50}$

$= 100$ square units.

11. a. Let the radius be r cm and the height be h cm.

Then $\pi r^2 h = 1000$

$$h = \frac{1000}{\pi r^2}$$

Surface Area: $A = 2\pi r^2 + 2\pi rh$

$$= 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right)$$

$$= 2\pi r^2 + \frac{2000}{4}, 0 \le r \le \infty$$

$$\frac{dA}{dr} = 4\pi r - \frac{2000}{r^2}$$

For max or min, let $\frac{dA}{dr} = 0$.

$$4\pi r - \frac{2000}{r^2} = 0$$

$$r^3 = \frac{500}{\pi}$$

$$r = \sqrt[3]{\frac{500}{\pi}} \doteq 5.42$$

When $r = 0$, $A \to \infty$

$r = 5.42$ $A \doteq 660.8$

$r \to \infty$, $A \to \infty$

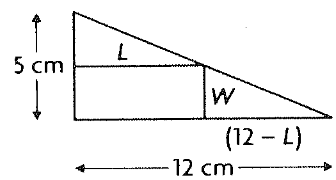
The minimum surface area is approximately $661\ \text{cm}^3$ when $r = 5.42$.

b. $r = 5.42$, $h = \frac{1000}{\pi(5.42)^2} \doteq 10.84$

$$\frac{h}{d} = \frac{10.84}{2 \times 5.42} = \frac{1}{1}$$

Yes, the can has dimensions that are larger than the smallest that the market will accept.

12. a.



Let the rectangle have length L cm on the 12 cm leg and width W cm on the 5 cm leg.

$A = LW$

By similar triangles, $\frac{12 - L}{12} = \frac{W}{5}$

$$60 - 5L = 12W$$

$$L = \frac{60 - 12W}{5}$$

$A = \frac{(60 - 12W)W}{5}$ for domain $0 \le W \le 5$

Using the max min Algorithm.

$\frac{dA}{dW} = \frac{1}{5}[60 - 24W] = 0$, $W = \frac{60}{24} = 2.5$ cm.

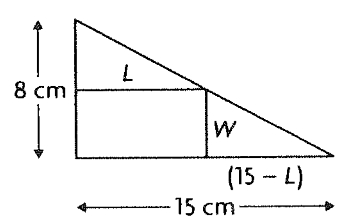
When $W = 2.5$ cm, $A = \frac{(60 - 30) \times 2.5}{5} = 15\ \text{cm}^2$.

Step 2: If $W = 0$, $A = 0$

Step 3: If $W = 5$, $A = 0$

The largest possible area is $15\ \text{cm}^2$ and occurs when $W = 2.5$ cm and $L = 6$ cm.

b.



Let the rectangle have length L cm on the 15 cm leg and width W cm on the 8 cm leg.

$A = LW$ ①

By similar triangles, $\frac{15 - L}{15} = \frac{W}{8}$

$$120 - 8L = 15W$$

$$L = \frac{120 - 15W}{8} \quad ②$$

$A = \frac{(120 - 15W)W}{8}$ for domain $0 \leq W \leq 8$

Using the max min Algorithm,

$\frac{dA}{dW} = \frac{1}{8}[120 - 30W] = 0, W = \frac{120}{30} = 4$ cm.

When $W = 4$ cm, $A = \frac{(120 - 60) \times 4}{8} = 30$ cm^2.

Step 2: If $W = 0, A = 0$

Step 3: If $W = 8, A = 0$

The largest possible area is 30 cm^2 and occurs when $W = 4$ cm and $L = 7.5$ cm.

c. The largest area occurs when the length and width are each equal to one-half of the sides adjacent to the right angle.

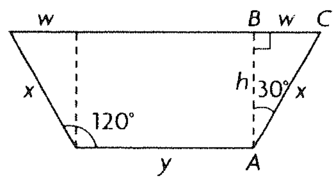
13. a. Let the base be y cm, each side x cm and the height h cm.

$2x + y = 60$

$y = 60 - 2x$

$A = yh + 2 \times \frac{1}{2}(wh)$

$= yh + wh$



From $\triangle ABC$

$\frac{h}{x} = \cos 30°$

$h = x \cos 30°$

$= \frac{\sqrt{3}}{2}x$

$\frac{w}{x} = \sin 30°$

$w = x \sin 30°$

$= \frac{1}{2}x$

Therefore, $A = (60 - 2x)\left(\frac{\sqrt{3}}{2}x\right) + \frac{x}{2} \times \frac{\sqrt{3}}{2}x$

$A(x) = 30\sqrt{3}x - \sqrt{3}x^2 + \frac{\sqrt{3}}{4}x^2, 0 \leq x \leq 30$

Apply the Algorithm for Extreme Values,

$A'(x) = 30\sqrt{3} - 2\sqrt{3}x + \frac{\sqrt{3}}{2}x$

Now, set $A'(x) = 0$

$30\sqrt{3} - 2\sqrt{3} + \frac{\sqrt{3}}{2}x = 0.$

Divide by $\sqrt{3}$:

$30 - 2x + \frac{x}{2} = 0$

$x = 20.$

To find the largest area, substitute $x = 0$, 20, and 30.

$A(0) = 0$

$A(20) = 30\sqrt{3}(20) - \sqrt{3}(20)^2 + \frac{\sqrt{3}}{4}(20)^2$

$= 520$

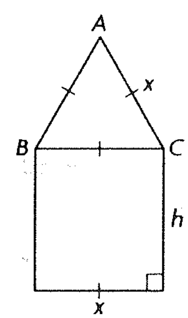
$A(30) = 30\sqrt{3}(30) - \sqrt{3}(30)^2 + \frac{\sqrt{3}}{4}(30)^2$

$\doteq 390$

The maximum area is 520 cm^2 when the base is 20 cm and each side is 20 cm.

b. Multiply the cross-sectional area by the length of the gutter, 500 cm. The maximum volume that can be held by this gutter is approximately 500(520) or 260 000 cm^3.

14. a.



$4x + 2h = 6$

$2x + h = 3$ or $h = 3 - 2x$

$\text{Area} = xh + \frac{1}{2} \times x \times \frac{\sqrt{3}}{2}x$

$= x(3 - 2x) + \frac{\sqrt{3}x^2}{4}$

$A(x) = 3x - 2x^2 + \frac{\sqrt{3}}{4}x^2$

$A'(x) = 3 - 4x + \frac{\sqrt{3}}{2}x, 0 \leq x \leq 1.5$

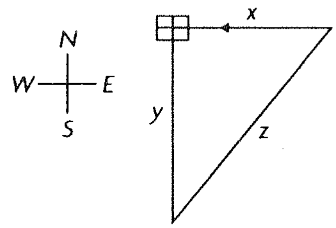
For max or min, let $A'(x) = 0, x \doteq 1.04.$

$A(0) = 0, A(1.04) \doteq 1.43, A(1.5) \doteq 1.42$

The maximum area is approximately 1.43 cm^2 and occurs when $x = 0.96$ cm and $h = 1.09$ cm.

b. Yes. All the wood would be used for the outer frame.

15.



Let z represent the distance between the two trains. After t hours, $y = 60t$, $x = 45(1 - t)$

$$z^2 = 3600t^2 + 45^2(1 - t)^2, 0 \le t \le 1$$

$$2z\frac{dz}{dt} = 7200t - 4050(1 - t)$$

$$\frac{dz}{dt} = \frac{7200t - 4050(1 - t)}{2z}$$

For max or min, let $\frac{dz}{dt} = 0$.

$$7200t - 4050(1 - t) = 0$$
$$t = 0.36$$

When $t = 0$, $z^2 = 45^2$, $z = 45$

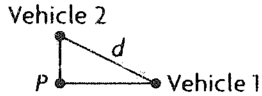
$t = 0.36$, $z^2 = 3600(0.36)^2 + 45^2(1 - 0.36)^2$

$$z^2 = 129$$
$$z = 36$$

$t = 1$, $z^2 = \sqrt{3600} = 60$

The closest distance between the trains is 36 km and occurs at 0.36 h after the first train left the station.

16.



At any time after 1:00 p.m., the distance between the first vehicle and the second vehicle is the hypotenuse of a right triangle, where one side of the triangle is the distance from the first vehicle to P and the other side is the distance from the second vehicle to P. The distance between them is therefore $d = \sqrt{(60t)^2 + (5 - 80t)^2}$ where t is the time in hours after 1:00. To find the time when they are closest together, d must be minimized.

$$d = \sqrt{(60t)^2 + (5 - 80t)^2}$$

$$d = \sqrt{3600t^2 + 25 - 800t + 6400t^2}$$

$$d = \sqrt{10\,000t^2 + 25 - 800t}$$

$$d' = \frac{20\,000t - 800}{2\sqrt{10\,000t^2 + 25 - 800t}}$$

Let $d' = 0$:

$$\frac{20\,000t - 800}{2\sqrt{10\,000t^2 + 25 - 800t}} = 0$$

Therefore $20\,000t - 800 = 0$

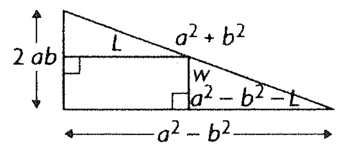
$$20\,000t = 800$$
$$t = .04 \text{ hours}$$

There is a critical number at $t = .04$ hours

v	$t < .04$	.04	$t > .04$
d′(t)	−	0	+
Graph	Dec.	Local Min	Inc.

There is a local minimum at $t = .04$, so the two vehicles are closest together .04 hours after 1:00, or 1:02. The distance between them at that time is 3 km.

17.



$$\frac{a^2 - b^2 - L}{a^2 - b^2} = \frac{W}{2ab}$$

$$W = \frac{2ab}{a^2 - b^2}(a^2 - b^2 - L)$$

$$A = LW = \frac{2ab}{a^2 - b^2}[a^2L - b^2L - L^2]$$

Let $\frac{dA}{dL} = a^2 - b^2 - 2L = 0$.

$$L = \frac{a^2 - b^2}{2}$$

and $W = \frac{2ab}{a^2 - b^2}\left[a^2 - b^2 - \quad - \frac{a^2 - b^2}{2}\right]$

$$= ab.$$

The hypothesis is proven.

18. Let the height be h and the radius r.

Then, $\pi r^2 h = k$, $h = \frac{k}{\pi r^2}$.

Let M represent the amount of material.

$$M = 2\pi r^2 + 2\pi rh$$

$$= 2\pi r^2 + 2\pi rh\left(\frac{k}{\pi r^2}\right)$$

$$= 2\pi r^2 + \frac{2k}{r}, 0 \le r \le \infty$$

Using the max min Algorithm,

$$\frac{dM}{dr} = 4\pi r - \frac{2k}{r^2}$$

Let $\frac{dM}{dr} = 0$, $r^3 = \frac{k}{2\pi}$, $r \neq 0$ or $r = \left(\frac{k}{2\pi}\right)^{\frac{1}{3}}$.

When $r \to 0$, $M \to \infty$

$r \to \infty$, $M \to \infty$

$$r = \left(\frac{k}{2\pi}\right)^{\frac{1}{3}}$$

$$d = 2\left(\frac{k}{2\pi}\right)^{\frac{1}{3}}$$

$$h = \frac{k}{\pi\left(\frac{k}{2\pi}\right)^{\frac{2}{3}}} = \frac{k}{\pi} \cdot \frac{(2\pi)^{\frac{2}{3}}}{\;} = \frac{k^{\frac{1}{3}}}{\;} \cdot 2^{\frac{2}{3}}$$

Min amount of material is

$$M = 2\pi\left(\frac{k}{2\pi}\right)^{\frac{2}{3}} + 2k\left(\frac{2\pi}{k}\right)^{\frac{1}{3}}.$$

$$\text{Ratio } \frac{h}{d} = \frac{\left(\frac{k}{\pi}\right)^{\frac{1}{3}} \cdot 2^{\frac{2}{3}}}{2\left(\frac{k}{2\pi}\right)^{\frac{1}{3}}} = \frac{\left(\frac{k}{\pi}\right)^{\frac{1}{3}} \cdot 2^{\frac{2}{3}}}{2^{\frac{2}{3}}\left(\frac{k}{\pi}\right)^{\frac{1}{3}}} = \frac{1}{1}$$

19. A —— x —— P —— $100 - x$ —— B

Cut the wire at P and label diagram as shown. Let AP form the circle and PB the square.

Then, $2\pi r = x$

$$r = \frac{x}{2\pi}$$

And the length of each side of the square is $\frac{100 - x}{4}$.

$$\text{Area of circle} = \pi\left(\frac{x}{2\pi}\right)^2 = \frac{x^2}{4\pi}$$

$$\text{Area of square} = \left(\frac{100 - x}{4}\right)^2$$

The total area is

$$A(x) = \frac{x^2}{4\pi} + \left(\frac{100 - x}{4}\right)^2, \text{ where } 0 \le x \le 100.$$

$$A'(x) = \frac{2x}{4\pi} + 2\left(\frac{100 - x}{4}\right)\left(-\frac{1}{4}\right) = \frac{x}{2\pi} - \frac{100 - x}{8}$$

For max or min, let $A'(x) = 0$.

$$\frac{x}{2\pi} - \frac{100 - x}{8} = 0$$

$$x = \frac{100\pi}{r} + \pi \doteq 44$$

$$A(0) = 625$$

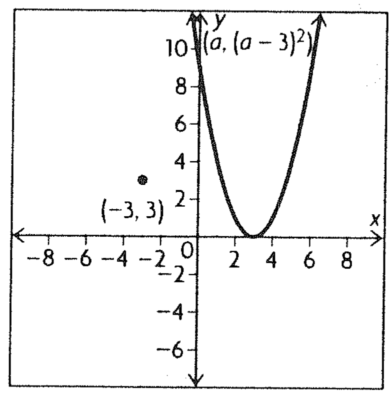
$$A(44) = \frac{44^2}{4\pi} + \left(\frac{100 - 44}{4}\right)^2 \doteq 350$$

$$A(100) = \frac{100^2}{4\pi} \doteq 796$$

a. The maximum area is 796 cm^2 and occurs when all of the wire is used to form a circle.

b. The minimum area is 350 cm^2 when a piece of wire of approximately 44 cm is bent into a circle.

20.



Any point on the curve can be represented by $(a, (a - 3)^2)$.

The distance from $(-3, 3)$ to a point on the curve is $d = \sqrt{(a + 3)^2 + ((a - 3)^2 - 3)^2}$.

To minimize the distance, we consider the function $d(a) = (a + 3)^2 + (a^2 - 6a + 6)^2$.

in minimizing $d(a)$, we minimize d since $d > 1$ always.

For critical points, set $d'(a) = 0$.

$d'(a) = 2(a + 3) + 2(a^2 - 6a + 6)(2a - 6)$

if $d'(a) = 0$,

$$a + 3 + (a^2 - 6a + 6)(2a - 6) = 0$$
$$2a^3 - 18a^2 + 49a - 33 = 0$$
$$(a - 1)(2a^2 - 16a + 33) = 0$$

$$a = 1, \text{ or } a = \frac{16 \pm \sqrt{-8}}{4}$$

There is only one critical value, $a = 1$.

To determine whether $a = 1$ gives a minimal value, we use the second derivative test:

$d'(a) = 6a^2 - 36a + 49$

$d''(1) = 6 - 36 + 49$

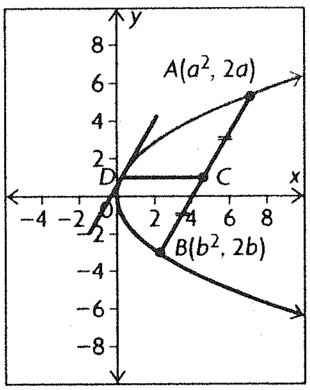
$\ge 0.$

Then, $d(1) = 4^2 + 1^2$

$= 17.$

The minimal distance is $d = \sqrt{17}$, and the point on the curve giving this result is $(1, 4)$.

21.



Let the point A have coordinates $(a^2, 2a)$. (Note that the x-coordinate of any point on the curve is positive, but that the y-coordinate can be positive or negative. By letting the x-coordinate be a^2, we eliminate this concern.) Similarly, let B have coordinates $(b^2, 2b)$.
The slope of AB is

$$\frac{2a - 2b}{a^2 - b^2} = \frac{2}{a + b}.$$

Using the mid-point property, C has coordinates

$$\left(\frac{a^2 + b^2}{2}, a + b\right).$$

Since CD is parallel to the x-axis, the y-coordinate of D is also $a + b$. The slope of the tangent at D is given by $\frac{dy}{dx}$ for the expression $y^2 = 4x$.
Differentiating,

$$2y\frac{dy}{dx} = 4$$

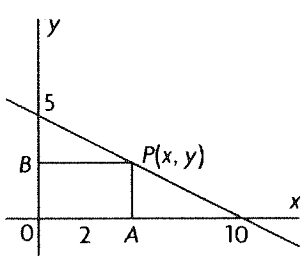
$$\frac{dy}{dx} = \frac{2}{y}$$

And since at point D, $y = a + b$,

$$\frac{dy}{dx} = \frac{2}{a + b}.$$

But this is the same as the slope of AB. Then, the tangent at D is parallel to the chord AB.

22.



Let the point $P(x, y)$ be on the line $x + 2y - 10 = 0$.
Area of $\triangle APB = xy$

$x + 2y = 10$ or $x = 10 - 2y$

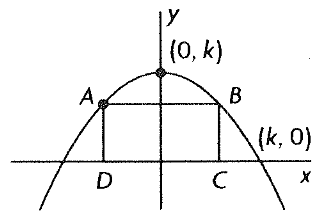
$$A(y) = (10 - 2y)y$$
$$= 10y - 2y^2, 0 \le y \le 5$$
$$A'(y) = 10.4y$$

For max or min, let $A'(y) = 0$ or $10 - 4y = 0$, $y = 2.5$,

$$A(0) = 0$$
$$A(2.5) = (10 - 5)(2.5) = 12.5$$
$$A(5) = 0.$$

The largest area is 12.5 units squared and occurs when P is at the point $(5, 2.5)$.

23.



A is $(-x, y)$ and $B(x, y)$

Area $= 2xy$ where $y = k^2 - x^2$

$$A(x) = 2x(k^2 - x^2)$$
$$= 2k^2x - 2x^3, -k \le x \le k$$
$$A'(x) = 2k^2 - 6x^2$$

For max or min, let $A'(x) = 0$,

$$6x^2 = 2k^2$$
$$x = \pm\frac{k}{\sqrt{3}}$$

When $x = \pm\frac{k}{\sqrt{3}}$, $y = k^2 - \left(\frac{k}{\sqrt{3}}\right)^2 = \frac{2}{3}k^2$

Max area is $A = \frac{2k}{\sqrt{3}} \times \frac{2}{3}k^2 = \frac{4k^3}{3\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$

$= \frac{4k^3}{9}$ square units.

3.4 Optimization Problems in Economics and Science, pp. 151–154

1. a. $C(625) = 75(\sqrt{625} - 10)$
$= 1125$

Average cost is $\frac{1125}{625} = \$1.80$.

b. $C(x) = 75(\sqrt{x} - 10)$
$= 75\sqrt{x} - 750$

$$C'(x) = \frac{75}{2\sqrt{x}}$$

$$C'(1225) = \frac{75}{2\sqrt{1225}} = \$1.07$$

c. For a marginal cost of \$0.50/L,

$$\frac{75}{2\sqrt{x}} = 0.5$$
$$75 = \sqrt{x}$$
$$x = 5625$$

The amount of product is 5625 L.

2. $N(t) = 20t - t^2$

a. $N(3) = 60 - 9$
$= 51$
$N(2) = 40 - 4$
$= 36$
$51 - 36 = 15$ terms

b. $N'(t) = 20 - 2t$
$N'(2) = 20 - 4$
$= 16$ terms/h

c. $t > 0$, so the maximum rate (maximum value of $N'(t)$) is 20. 20 terms/h

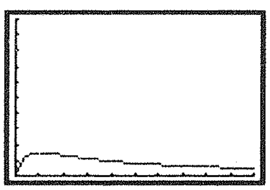
3. $L(t) = \dfrac{6t}{t^2 + 2t + 1}$

a. $L'(t) = \dfrac{6(t^2 + 2t + 1) - 6t(2t + 2)}{(t^2 + 2t + 1)^2}$
$= \dfrac{-6t^2 + 6}{(t^2 + 2t + 1)^2}$

Let $L'(t) = 0$, then $-6t^2 + 6 = 0$,
$t^2 = 1$
$t^2 = \pm 1$.

b. $L(1) = \dfrac{6}{1 + 2 + 1} = \dfrac{6}{4} = 1.5$

c.



d. The level will be a maximum.

e. The level is decreasing.

4. $C = 4000 + \dfrac{h}{15} + \dfrac{15\,000\,000}{h}$, $1000 \le h \le 20\,000$

$\dfrac{dC}{dh} = \dfrac{1}{15} - \dfrac{15\,000\,000}{h^2}$

Set $\dfrac{dC}{dh} = 0$, therefore, $\dfrac{1}{15} - \dfrac{15\,000\,000}{h^2} = 0$,
$h^2 = 225\,000\,000$
$h = 15\,000$, $h > 0$.

Using the max min Algorithm, $1000 \le h \le 20\,000$.

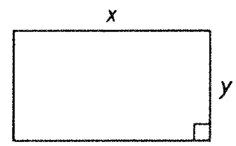
When $h = 1000$, $C = 4000 + \dfrac{1000}{15} + \dfrac{15\,000\,000}{1000}$,
$\doteq 19\,067$.

When $h = 15\,000$, $C = 4000 + \dfrac{15\,000}{15} + \dfrac{15\,000\,000}{15\,000}$
$= 6000$.

When $h = 20\,000$, $C \doteq 6083$.

The minimum operating cost of \$6000/h occurs when the plane is flying at 15 000 m.

5.



Label diagram as shown and let the side of length x cost \$6/m and the side of length y be \$9/m.

Therefore, $(2x)(6) + (2y)(9) = 9000$
$2x + 3y = 1500$.

Area $A = xy$

But $y = \dfrac{1500 - 2x}{3}$.

$A(x) = x\left(\dfrac{1500 - 2x}{3}\right)$
$= 500x - \dfrac{2}{3}x^2$ for domain $0 \le x \le 500$

$A'(x) = 500 - \dfrac{4}{3}x$

Let $A'(x) = 0$, $x = 375$.

Using max min Algorithm, $0 \le x \le 500$,

$A(0) = 0$, $A(375) = 500(375) - \dfrac{2}{3}(375)^2$
$= 93\,750$
$A(500) = 0$.

The largest area is 93 750 m^2 when the width is 250 m by 375 m.

6. Let x be the number of \$25 increases in rent.

$P(x) = (900 + 25x)(50 - x) - (50 - x)(75)$
$P(x) = (50 - x)(825 + 25x)$
$P(x) = 41\,250 + 1250x - 825x - 25x^2$
$P(x) = 41\,250 + 425x - 25x^2$
$P'(x) = 425 - 50x$

Set $P'(x) = 0$
$0 = 425 - 50x$
$50x = 425$
$x = 8.5$
$x = 8$ or $x = 9$
$P'(0) = 425 > 0$
$P'(10) = -75 < 0$

maximum: The real estate office should charge \$900 + \$25(8) = \$1100 or \$900 + \$25(9) = \$1125 rent to maximize profits. Both prices yield the same profit margin.

7. Let the number of fare changes be x. Now, ticket price is \$20 + \$0.5x. The number of passengers is $10\,000 - 200x$.

The revenue $R(x) = (10\,000 - 200x)(20 + 0.5x)$,
$R(x) = -200(20 + 0.5x) + 0.5(1000 - 200x)$
$= -4000 - 100x + 5000 - 100x$.

Let $R'(x) = 0$:

$$200x = 1000$$
$$x = 5.$$

The new fare is \$20 + \$0.5(5) = \$22.50 and the maximum revenue is \$202 500.

8. Cost $C - \left(\frac{v^3}{2} + 216\right) \times t$

Where $vt = 500$ or $t = \frac{500}{v}$.

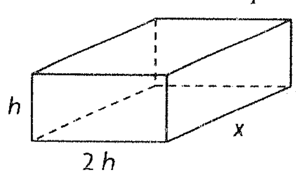
$$C(v) = \left(\frac{v^3}{2} + 216\right)\left(\frac{500}{v}\right)$$
$$= 250v^2 + \frac{108\ 000}{v}, \text{ where } v \geq 0.$$
$$C'(v) = 500v - \frac{108\ 000}{v^2}$$

Let $C'(v) = 0$, then $500v = \frac{108\ 000}{v^2}$

$$v^3 = \frac{108\ 000}{500}$$
$$v^3 = 216$$
$$v = 6.$$

The most economical speed is 6 nautical miles/h.

9.



Label diagram as shown.

We know that $(x)(2h)(h) = 20\ 000$

or $h^2x = 10\ 000$

$$x = \frac{10\ 000}{h^2}$$

Cost $C = 40(2hx) + 2xh(200)$
$+ 100(2)(2h^2 + xh)$
$= 80xh + 400xh + 400h^2 + 200xh$
$= 680xh + 400h^2$

Since $x = \frac{10\ 000}{h^2}$,

$$C(h) = 680h\left(\frac{10\ 000}{h^2}\right) + 400h^2, 0 \leq h \leq 100$$
$$C(h) = \frac{6\ 800\ 000}{h} + 400h^2$$
$$C'(h) = \frac{6\ 800\ 000}{h^2} + 800h.$$

Let $C'(h) = 0$,

$$800h^3 = 6\ 800\ 000$$
$$h^3 = 8500$$
$$h \doteq 20.4.$$

Apply max min Algorithm.

as $h \to 0$ $C(0) \to \infty$

$$C(20.4) = \frac{6\ 800\ 000}{20.4} + 400(20.4)^2$$
$$= 499\ 800$$
$$C(100) = 4\ 063\ 000.$$

Therefore, the dimensions that will keep the cost to a minimum are 20.4 m by 40.8 m by 24.0 m.

10. Let the height of the cylinder be h cm, the radius r cm. Let the cost for the walls be \$$k$ and for the top \$$2k$.

$$V = 1000 = \pi r^2 h \text{ or } h = \frac{1000}{\pi r^2}$$

The cost $C = (2\pi r^2)(2k) + (2\pi rh)k$

or $C = 4\pi kr^2 + 2\pi kr\left(\frac{1000}{\pi r^2}\right)$

$$C(r) = 4\pi kr^2 + \frac{2000k}{r}, r \geq 0$$
$$C'(r) = 8\pi kr - \frac{2000k}{r^2}$$

Let $C'(r) = 0$, then $8\pi kr = \frac{2000k}{r^2}$

or $r^3 = \frac{2000}{8\pi}$

$$r \doteq 4.3$$
$$h = \frac{1000}{\pi(4.3)^2} = 17.2.$$

Since $r \geq 0$, minimum cost occurs when $r = 4.3$ cm and $h = 17.2$ cm.

11. a. Let the number of \$0.50 increase be n.

New price $= 10 + 0.5n$.

Number sold $= 200 - 7n$.

Revenue $R(n) = (10 + 0.5n)(200 - 7n)$
$= 2000 + 30n - 3.5n^2$

Profit $P(n) = R(n) - C(n)$
$= 2000 + 30n + 3.5n^2 - 6(200 - 7n)$
$= 800 + 72n - 3.5n^2$

$P'(n) = 72 - 7n$

Let $P'(n) = 0$,

$72 - 7n = 0, n \doteq 10.$

Price per cake $= 10 + 5 = \$15$

Number sold $= 200 - 70 = 130$

b. Since $200 - 165 = 35$, it takes 5 price increases to reduce sales to 165 cakes.

New price is $10 + 0.5 \times 5 = \$12.50$.

The profit is $165 \times 5 = \$825$.

c. If you increase the price, the number sold will decrease. Profit in situation like this will increase for several price increases and then it will decrease because too many customers stop buying.

12. Let x be the base length and y be the height.

Top/bottom: $\$20/\text{m}^2$

Sides: $\$30/\text{m}^2$

$$4000\text{ cm}^3\left(\frac{1\text{ m}}{100\text{ cm}}\right)^3 = 0.004\text{ m}^3$$

$$0.004 = x^2y$$

$$y = \frac{0.004}{x^2}$$

$$A_{\text{Top}} + A_{\text{Bottom}} = x^2 + x^2$$

$$= 2x^2$$

$$4A_{\text{Side}} = 4xy$$

$$C = 20(2x^2) + 30(4xy)$$

$$C = 40x^2 + 120x\left(\frac{0.004}{x^2}\right)$$

$$C = 40x^2 + 0.48x^{-1}$$

$$C' = 80x - 0.48x^{-2}$$

Set $C' = 0$

$$0 = 80x - 0.48x^{-2}$$

$$80x^3 = 0.48$$

$$x^3 = 0.006$$

$$x \doteq 0.182$$

$$y = \frac{0.004}{0.182^2}$$

$$y \doteq 0.121$$

$$C'(1) = 79.52 > 0$$

$$C'(-1) = -80.48 < 0$$

maximum

The jewellery box should be

12.1 cm × 18.2 cm × 18.2 cm to minimize the cost of materials.

13. Let x be the number of price changes and R be the revenue.

$$R = (90 - x)(50 + 5x)$$

$$R' = 5(90 - x) - 1(50 + 5x)$$

Set $R' = 0$

$$0 = 5(90 - x) - 1(50 + 5x)$$

$$0 = 450 - 5x - 50 - 5x$$

$$0 = 400 - 10x$$

$$10x = 400$$

$$x = 40$$

$$\text{Price} = \$90 - \$40$$

$$\text{Price} = \$50$$

$$R'(0) = 400 > 0$$

$$R'(100) = -600 < 0$$

maximum: The price of the CD player should be \$50.

14. Let x be the number of price changes and R be the revenue.

$$R = (75 - 5x)(14\,000 + 800x),\ x \le 7.5$$

$$R' = 800(75 - 5x) + (-5)(14\,000 + 800x)$$

Set $R' = 0$

$$0 = 60\,000 - 4000x - 70\,000 - 4000x$$

$$10\,000 = -8000x$$

$$x = -1.25$$

$$\text{Price} = \$75 - \$5(-1.25)$$

$$\text{Price} = \$81.25$$

$$R'(-2) = 6000 > 0$$

$$R'(2) = -26\,000 < 0$$

maximum: The price of a ticket should be \$81.25.

15. $P(x) = (2000 - 5x)(1000x)$

$$- (15\,000\,000 + 1\,800\,000x + 75x^2)$$

$$P(x) = 2\,000\,000x - 5000x^2 - 15\,000\,000$$

$$- 1\,800\,000x - 75x^2$$

$$P(x) = -5075x^2 + 200\,000x - 15\,000\,000$$

$$P'(x) = -10\,150x + 200\,000$$

Set $P'(x) = 0$

$$0 = -10\,150x + 200\,000$$

$$10\,150x = 200\,000$$

$$x \doteq 19.704$$

$$P'(0) = 200\,000 > 0$$

$$P'(20) = -3000 < 0$$

maximum: The computer manufacturer should sell 19 704 units to maximize profit.

16. $P(x) = R(x) - C(x)$

Marginal Revenue $= R'(x)$.

Marginal Cost $= C'(x)$.

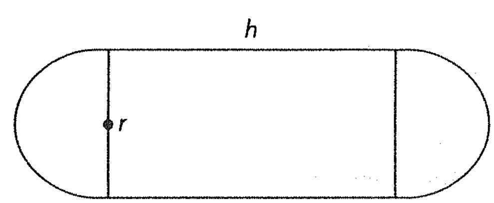
Now $P'(x) = R'(x) - C'(x)$.

The critical point occurs when $P'(x) = 0$.

If $R'(x) = C'(x)$, then $P'(x) = R'(x) - R'(x) = 0$.

Therefore, the instantaneous rate of change in profit is 0 when the marginal revenue equals the marginal cost.

17.



Label diagram as shown, Let cost of cylinder be $\$k/m^3$.

$V = 200$

$= \pi r^2 h + \frac{4}{3}\pi r^3$

Note: Surface Area = Total cost C

Cost $C = (2\pi rh)k + (4\pi r^2)2k$

But, $200 = \pi r^2 h + \frac{4}{8}\pi r^3$ or $600 = 3\pi r^2 h + 4\pi r^2$

Therefore, $h = \frac{600 - 4\pi r^3}{3\pi r^2}$.

$$C(r) = 2k\pi r\left(\frac{600 - 4\pi r^3}{3\pi r^2}\right)1 + 8k\pi r^2$$

$$= 2k\left(\frac{600 - 4\pi r^3}{3r}\right) + 8k\pi r^2$$

Since $h \le 16$, $r \le \left(\frac{600}{4\pi}\right)^{\frac{1}{3}}$ or $0 \le r \le 3.6$

$$C(r) = \frac{400k}{r} - \frac{8k\pi r^2}{3} + 3k\pi r^2$$

$$= \frac{400k}{r} + \frac{16k\pi r^2}{3}$$

$$C'(r) = -\frac{400k}{r^2} + \frac{32k\pi r}{3}$$

Let $C'(r) = 0$

$$\frac{400k}{r^2} = \frac{32k\pi r}{3}$$

$$\frac{50}{r^2} = \frac{4\pi r}{3}$$

$$4\pi r^3 = 150$$

$$r^3 = \frac{150}{4\pi}$$

$$r = 2.29$$

$$h \doteq 8.97 \text{ m}$$

Note: $C(0) \to \infty$

$C(2.3) \doteq 262.5k$

$C(3.6) \doteq 330.6k$

The minimum cost occurs when $r = 230$ cm and h is about 900 cm.

18. $C = 1.15 \times \frac{450}{8 - .1(s - 110)} + (35 + 15.5)\frac{450}{s}$

$$C = \frac{517.5}{-.1s + 19} + \frac{22\,725}{s}$$

$$C = \frac{517.5s - 2272.5s + 431\,775}{19s - .1s^2}$$

$$C = \frac{431\,775 - 1755s}{19s - .1s^2}$$

To find the value of s that minimizes C, we need to calculate the derivative of C.

$$C' = \frac{-1755(19s - .1s^2)}{(19s - .1s^2)^2} - \frac{(431\,775 - 1755s)(19 - .2s)}{(19s - .1s^2)^2}$$

$$C' = \frac{(-33\,345s + 175.5s^2)}{(19s - .1s^2)^2} - \frac{(8\,203\,725 - 119\,700s + 351s^2)}{(19s - .1s^2)^2}$$

$$C' = \frac{-175.5s^2 + 86\,355s - 8\,203\,725}{(19s - .1s^2)^2}$$

Let $C' = 0$:

$$\frac{-175.5s^2 + 86\,355s - 8\,203\,725}{(19s - .1s^2)^2} = 0$$

$$s = 128.4$$

There is a critical number at $s = 128.4$ km/h

s	$s < 128.4$	128.4	$s > 128.4$
$C'(s)$	−	0	+
Graph	Dec.	Local Min	Inc.

There is a local minimum for $s = 128.4$, so the cost is minimized for a speed of 128.4 km/h.

19. $v(r) = Ar^2(r_0 - r), 0 \le r \le r_0$

$v(r) = Ar_0r^2 - Ar^3$

$v'(r) = 2Ar_0r - 3Ar^2$

Let $v'(r) = 0$:

$2Ar_0r - 3Ar^2 = 0$

$2r_0r - 3r^2 = 0$

$r(2r_0 - 3r) = 0$

$r = 0$ or $r = \frac{2r_0}{3}$.

$v(0) = 0$

$$v\left(\frac{2r_0}{3}\right) = A\left(\frac{4}{9}r_0^2\right)\left(r_0 - \frac{2r_0}{3}\right)$$

$$= \frac{4}{27}r_0A$$

$A(r_0) = 0$

The maximum velocity of air occurs when radius is $\frac{2r_0}{3}$.

Review Exercise, pp. 156–159

1. $f(x) = x^4 - \frac{1}{x^4}$

$= x^4 - x^{-4}$

$f'(x) = 4x^3 + 4x^{-5}$

$f''(x) = 12x^2 - 20x^{-6}$

2. $y = x^9 - 7x^3 + 2$

$\frac{dy}{dx} = 9x^8 - 21x^2$

$\frac{d^2y}{dx^2} = 72x^7 - 42x$

3. $s(t) = t^2 + 2(2t - 3)^{\frac{1}{2}}$

$v = s'(t) = 2t + \frac{1}{2}(2t - 3)^{-\frac{1}{2}}(2)$

$= 2t + (2t - 3)^{-\frac{1}{2}}$

$a = s''(t) = 2 - \frac{1}{2}(2t - 3)^{-\frac{3}{2}}(2)$

$= 2 - (2t - 3)^{-\frac{3}{2}}$

4. $s(t) = t - 7 + \frac{5}{t}$

$= t - 7 + 5t^{-1}$

$v(t) = 1 - 5t^{-2}$

$a(t) = 10t^{-3}$

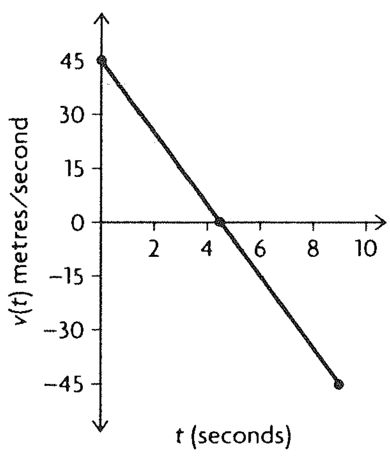
5. $s(t) = 45t - 5t^2$

$v(t) = 45 - 10t$

For $v(t) = 0$, $t = 4.5$.

t	$0 \le t < 4.5$	4.5	$t > 4.5$
$v(t)$	+	0	−

Therefore, the upward velocity is positive for $0 \le t < 4.5$ s, zero for $t = 4.5$ s, negative for $t > 4.5$ s.



6. a. $f(x) = 2x^3 - 9x^2$

$f'(x) = 6x^2 - 18x$

For max min, $f'(x) = 0$:

$6x(x - 3) = 0$

$x = 0$ or $x = 3$.

x	$f(x) = 2x^3 - 9x^2$	
−2	−52	min
0	0	max
3	−27	
4	−16	

The minimum value is −52.
The maximum value is 0.

b. $f(x) = 12x - x^3$

$f'(x) = 12 - 3x^2$

For max min, $f'(x) = 0$:

$3(4 - x^2) = 0$

$x = -2$ or $x = 2$

x	$f(x) = 12x - x^3$	
−3	−9	
−2	−16	
2	16	max
5	−65	min

c. $f(x) = 2x + \frac{18}{x}$

$f'(x) = 2 - 18x^{-2}$

For max min, $f'(x) = 0$:

$\frac{18}{x^2} = 2$

$x^2 = 9$

$x = \pm 3$.

x	$f(x) = 2x + \frac{18}{x}$
1	20
3	12
5	$10 + \frac{18}{5} = 13.6$

The minimum value is 12.
The maximum value is 20.

7. a. $s(t) = 62 - 16t + t^2$

$v(t) = -16 + 2t$

$s(0) = 62$

Therefore, the front of the car was 62 m from the stop sign.

b. When $v = 0$, $t = 8$,

$s(8) = 62 - 16(8) + (8)^2$

$= 62 - 128 + 64$

$= -2$

Yes, the car goes 2 m beyond the stop sign before stopping.

c. Stop signs are located two are more metres from an intersection. Since the car only went 2 m beyond the stop sign, it is unlikely the car would hit another vehicle travelling perpendicular.

8. $s(t) = 1 + 2t - \dfrac{8}{t^2 + 1}$

$$v(t) = 2 + 8(t^2 + 1)^{-2}(2t) = 2 + \frac{16t}{(t^2 + 1)^2}$$

$$a(t) = 16(t^2 + 1)^{-2} + 16t(-2)(t^2 + 1)^{-3}2t$$
$$= 16(t^2 + 1)^{-2} - 64t^2(t^2 + 1)^{-3}$$
$$= 16(t^2 + 1)^{-3}[t^2 + 1 - 4t^2]$$

For max min velocities, $a(t) = 0$:

$3t^2 = 1$

$$t = \pm\frac{1}{\sqrt{6}}.$$

t	$v(t) = 2 + \dfrac{16t}{(t^2 + 1)^2}$
0	2 min
$\dfrac{1}{\sqrt{3}}$	$2 + \dfrac{\frac{16}{\sqrt{3}}}{(\frac{1}{3} + 1)^2} = 2 + \dfrac{\frac{16\sqrt{3}}{3}}{\frac{16}{9}} = 2 + 3\sqrt{3}$ max
2	$2 + \dfrac{32}{25} = 3.28$

The minimum value is 2.

The maximum value is $2 + 3\sqrt{3}$.

9. $u(x) = 625x^{-1} + 15 + 0.01x$

$u'(x) = -625x^2 + 0.01$

For a minimum, $u'(x) = 0$

$$x^2 = 62\,500$$
$$x = 250$$

x	$u(x) = \dfrac{625}{x} + 0.01x$
1	625.01
250	2.5 + 2.5 = 5 min
500	$\dfrac{625}{500} + 5 = 6.25$

Therefore, 250 items should be manufactured to ensure unit waste is minimized.

10. a. $C(x) = 3x + 1000$

i. $C(400) = 1200 + 1000$
$= 2200$

ii. $\dfrac{2200}{400} = \$5.50$

iii. $C'(x) = 3$

The marginal cost when $x = 400$ and the cost of producing the 401st item are \$3.00.

b. $C(x) = 0.004x^2 + 40x + 8000$

i. $C(400) = 640 + 16\,000 + 8000$
$= 24\,640$

ii. $\dfrac{24\,640}{400} = \$61.60$

iii. $C'(x) = 0.008x + 40$

$C'(400) = 0.008(400) + 40$
$= 43.20$

$C'(401) = 0.008(401) + 40$
$= \$43.21$

The marginal cost when $x = 400$ is \$43.20, and the cost of producing the 401st item is \$43.21.

c. $C(x) = \sqrt{x} + 5000$

i. $C(400) = 20 + 5000$
$= \$5020$

ii. $C(400) = \dfrac{5020}{400}$
$= \$12.55$

iii. $C'(x) = \dfrac{1}{2}x^{-\frac{1}{2}}$
$= \dfrac{1}{2\sqrt{x}}$

$C'(400) = \dfrac{1}{40}$
$= \$0.025$
$\doteq \$0.03$

$C'(401) = \dfrac{1}{2\sqrt{401}}$
$= \$0.025$
$\doteq \$0.03$

The cost to produce the 401st item is \$0.03.

d. $C(x) = 100x^{\frac{1}{2}} + 5x + 700$

i. $C(400) = \dfrac{100}{20} + 2000 + 700$
$= \$2705$

ii. $C(400) = \dfrac{2750}{400}$
$= \$6.875$
$= \$6.88$

iii. $C'(x) = -50x^{-\frac{3}{2}} + 5$

$C'(400) = \dfrac{-50}{(20)^3} + 5$
$= 5.00625$
$= \$5.01$

$C'(401) = \$5.01$

The cost to produce the 401st item is \$5.01.

11. $C(x) = 0.004x^2 + 40x + 16\,000$

Average cost of producing x items is

$$C(x) = \frac{C(x)}{x}$$

$$C(x) = 0.004x + 40 + \frac{16\,000}{x}$$

To find the minimum average cost, we solve

$$C'(x) = 0$$

$$0.004 - \frac{16\,000}{x^2} = 0$$

$$4x^2 - 16\,000\,000 = 0$$

$$x^2 = 4\,000\,000$$

$$x = 2000, x > 0$$

From the graph, it can be seen that $x = 2000$ is a minimum. Therefore, a production level of 2000 items minimizes the average cost.

12. a. $s(t) = 3t^2 - 10$

$v(t) = 6t$

$v(3) = 18$

$v(3) > 0$, so the object is moving to the right.

$s(3) = 27 - 10 = 17$. The object is to the right of the starting point and moving to the right, so it is moving away from its starting point.

b. $s(t) = -t^3 + 4t^2 - 10$

$s(0) = -10$

Therefore, its starting position is at -10.

$s(3) = -27 + 36 - 10$

$= -1$

$v(t) = -3t^2 + 8t$

$v(3) = -27 + 24$

$= -3$

Since $s(3)$ and $v(3)$ are both negative, the object is moving away from the origin and towards its starting position.

13. $s = 27t^3 + \frac{16}{t} + 10, t > 0$

a. $v = 81t^2 - \frac{16}{t^2}$

$$81t^2 - \frac{16}{t^2} = 0$$

$$81t^4 = 16$$

$$t^4 = \frac{16}{81}$$

$$t = \pm\frac{2}{3}$$

$$t > 0$$

Therefore, $t = \frac{2}{3}$.

b. $a = \frac{dv}{dt} = 162t + \frac{32}{t^3}$

At $t = \frac{2}{3}$, $a = 162 \times \frac{2}{3} + \frac{32}{\frac{2}{3}}$

$= 216$

Since $a > 0$, the particle is accelerating.

14. Let the base be x cm by x cm and the height h cm.

Therefore, $x^2h = 10\,000$.

$A = x^2 + 4xh$

But $h = \frac{10\,000}{x^2}$.

$$A(x) = x^2 + 4x\left(\frac{10\,000}{x^2}\right)$$

$$= x^2 + \frac{400\,000}{x}, \text{ for } x \geq 5$$

$$A'(x) = 2x - \frac{400\,000}{x^2},$$

Let $A'(x) = 0$, then $2x = \frac{400\,000}{x^2}$

$x^3 = 200\,000$

$x = 27.14$.

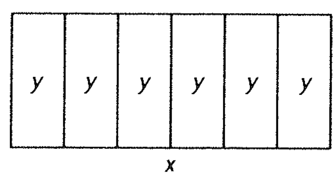
Using the max min Algorithm,

$A(5) = 25 + 80\,000 = 80\,025$

$A(27.14) \doteq 15\,475$

The dimensions of a box of minimum area is 27.14 cm for the base and height 13.57 cm.

15. Let the length be x and the width y.



$P = 2x + 6y$ and $xy = 12\,000$ or $y = \frac{12\,000}{x}$

$$P(x) = 2x + 6 \times \frac{12\,000}{x}$$

$$P(x) = 2x + \frac{72\,000}{x}, 10 \leq x \leq 1200(5 \times 240)$$

$$A'(x) = 2 - \frac{72\,000}{x^2}$$

Let $A'(x) = 0$,

$2x^2 = 72\,000$

$x^2 = 36\,000$

$x \doteq 190$.

Using max min Algorithm,

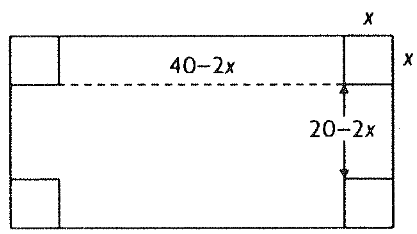
$A(10) = 20 + 7200 = 7220 \text{ m}^2$

$A(190) \doteq 759 \text{ m}^2$

$A(1200) = 1\,440\,060$

The dimensions for the minimum amount of fencing is a length of 190 m by a width of approximately 63 m.

16.



Let the width be w and the length $2w$.

Then, $2w^2 = 800$

$w^2 = 400$

$w = 20, \ w > 0.$

Let the corner cuts be x cm by x cm. The dimensions of the box are shown. The volume is

$$V(x) = x(40 - 2x)(20 - 2x)$$
$$= 4x^3 - 120x^2 - 800x, 0 \le x \le 10$$
$$V'(x) = 12x^2 - 240x - 800$$

Let $V'(x) = 0$:

$12x^2 - 240x - 800 = 0$

$3x^2 - 60x - 200 = 0$

$$x = \frac{60 \pm \sqrt{3600 - 2400}}{6}$$

$x \doteq 15.8$ or $x = 4.2$, but $x \le 10$.

Using max min Algorithm,

$V(0) = 0$

$V(4.2) = 1540 \text{ cm}^2$

$V(10) = 0.$

Therefore, the base is

$40 - 2 \times 4.2 = 31.6$

by $20 - 2 \times 4.2 = 11.6$

The dimensions are 31.6 cm by 11.6 cm by 4.2 cm.

17. Let the radius be r cm and the height h cm.

$V = \pi r^2 h = 500$

$A = 2\pi r^2 + 2\pi rh$

Since $h = \frac{500}{\pi r^2}, 6 \le h \le 15$

$$A(r) = 2\pi r^2 + 2\pi r\left(\frac{500}{\pi r^2}\right)$$
$$= 2\pi r^2 + \frac{1000}{r} \text{ for } 2 \le r \le 5$$
$$A'(r) = 4\pi r - \frac{1000}{r^2}.$$

Let $A'(r) = 0$, then $4\pi r^3 = 1000$,

$$r^3 = \frac{1000}{4\pi}$$

$r \doteq 4.3.$

Using max min Algorithm,

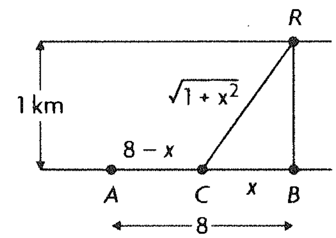
$A(2) \doteq 550$

$A(4.3) \doteq 349$

$A(5) \doteq 357$

For a minimum amount of material, the can should be constructed with a radius of 4.3 cm and a height of 8.6 cm.

18.



Let x be the distance CB, and $8 - x$ the distance AC.

Let the cost on land be \$$k$ and under water \$$1.6k$.

The cost $C(x) = k(8 - x) + 1.6k\sqrt{1 + x^2}$, $0 \le x \le 8$.

$$C'(x) = -k + 1.6k \times \frac{1}{2}(1 + x^2)^{-\frac{1}{2}}(2x)$$
$$= -k + \frac{1.6kx}{\sqrt{1 + x^2}}$$

Let $C'(x) = 0$,

$$-k + \frac{1.6kx}{\sqrt{1 + k^2}} = 0$$
$$\frac{1.6x}{\sqrt{1 + k^2}} = 1$$
$$1.6x = \sqrt{1 + x^2}$$
$$2.56x^2 = 1 + x^2$$
$$1.56x^2 = 1$$
$$x^2 \doteq 0.64$$
$$x = 0.8, x > 0$$

Using max min Algorithm,

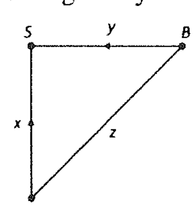
$A(0) = 9.6k$

$A(0.8) = k(8 - 0.8) + 1.6k\sqrt{1 + (0.8)^2} = 9.25k$

$A(8) = 12.9k$

The best way to cross the river is to run the pipe $8 - 0.8$ or 7.2 km along the river shore and then cross diagonally to the refinery.

19.



Let y represent the distance the westbound train is from the station and x the distance of the northbound train from the station S. Let t represent time after 10:00.

Then $x = 100t$, $y = (120 - 120t)$

Let the distance AB be z.

$$z = \sqrt{(100t)^2 + (120 - 120t)^2},\ 0 \le t \le 1$$

$$\frac{dz}{dt} = \frac{1}{2}[(100t)^2 + (120 - 120t)^2]^{-\frac{1}{2}}$$

$$\times [2 \times 100 \times 100t - 2 \times 120 \times (120(1 - t))]$$

Let $\frac{dz}{dt} = 0$, that is

$$\frac{2 \times 100 \times 100t - 2 \times 120 \times 120(1 - t)}{2\sqrt{(100t)^2 + (120 - 120t)^2}} = 0$$

or $20\,000t = 28\,800(1 - t)$

$48\,800t = 288\,000$

$$t = \frac{288}{488} \doteq 0.59 \text{ h or } 35.4 \text{ min.}$$

When $t = 0$, $z = 120$.

$t = 0.59$

$$z = \sqrt{(100 \times 0.59)^2 + (120 - 120 \times 0.59)^2}$$

$= 76.8$ km

$t = 1$, $z = 100$

The closest distance between trains is 76.8 km and occurs at 10:35.

20. Let the number of price increases be n.

New selling price $= 100 + 2n$.

Number sold $= 120 - n$.

Profit $=$ Revenue $-$ Cost

$P(n) = (100 + 2n)(120 - n) - 70(120 - n)$, $0 \le n \le 120$

$= 3600 + 210n - 2n^2$

$P'(n) = 210 - 4n$

Let $P'(n) = 0$

$210 - 4n = 0$

$n = 52.5$.

Therefore, $n = 52$ or 53.

Using max min Algorithm,

$P(0) = 3600$

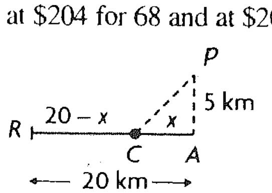
$P(52) = 9112$

$P(53) = 9112$

$P(120) = 0$

The maximum profit occurs when the portable MP3 are sold at \$204 for 68 and at \$206 for 67 portable MP3.

21.



Let x represent the distance AC.

Then, $RC = 20 - x$ and 4.

$PC = \sqrt{25 + x^2}$

The cost:

$C(x) = 100\,000\sqrt{25 + x^2} + 75\,000(20 - x)$, $0 \le x \le 20$

$$C'(x) = 100\,000 \times \frac{1}{2}(25 + x^2)^{-\frac{1}{2}}(2x) - 75\,000.$$

Let $C'(x) = 0$,

$$\frac{100\,000x}{\sqrt{25 + x^2}} - 75\,000 = 0$$

$$4x = 3\sqrt{25 + x^2}$$

$$16x^2 = 9(25 + x^2)$$

$$7x^2 = 225$$

$$x^2 \doteq 32$$

$$x \doteq 5.7.$$

Using max min Algorithm,

$A(0) = 100\,000\sqrt{25} + 75\,000(20) = 2\,000\,000$

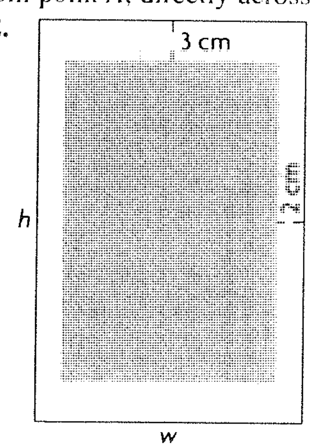
$A(5.7) = 100\,000\sqrt{25 + 5.7^2} + 75\,000(20 - 5.7)$

$= 1\,830\,721.60$

$A(20) = 2\,061\,552.81$.

The minimum cost is \$1 830 722 and occurs when the pipeline meets the shore at a point C, 5.7 km from point A, directly across from P.

22.



$$A = hw$$

$$81 = (h - 6)(w - 4)$$

$$\frac{81}{h - 6} = w - 4$$

$$\frac{81}{h - 6} + 4 = w$$

$$\frac{81 + 4(h - 6)}{h - 6} = w$$

$$\frac{4h + 57}{h - 6} = w$$

Substitute for w in the area equation and differentiate:

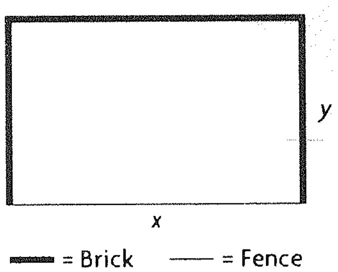
$$A = (h)\frac{4h + 57}{h - 6}$$
$$A = \frac{4h^2 + 57h}{h - 6}$$
$$A' = \frac{(8h + 57)(h - 6) - (4h^2 + 57h)}{(h - 6)^2}$$
$$A' = \frac{8h^2 + 9h - 342 - 4h^2 - 57h}{(h - 6)^2}$$
$$A' = \frac{4h^2 - 48h - 342}{(h - 6)^2}$$
Let $A' = 0$:
$$\frac{4h^2 - 48h - 342}{(h - 6)^2} = 0$$
Therefore, $4h^2 - 48h - 342 = 0$

Using the quadratic formula, $h = 17.02$ cm

h	t < 17.02	17.02	t > 17.02
A′(h)	−	0	+
Graph	Dec.	Local Min	Inc.

There is a local minimum at $h = 17.02$ cm, so that is the minimizing height.
$$81 = (h - 6)(w - 4)$$
$$81 = 11.02(w - 4)$$
$$7.35 = w - 4$$
$$w = 11.35 \text{ cm}$$
The dimensions of the page should be 11.35 cm × 17.02 cm.

23.

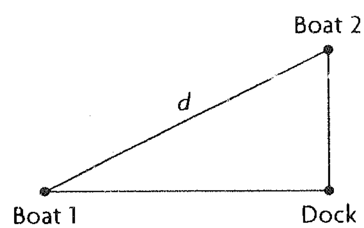


$$C = (192 + 48)x + 192(2y)$$
$$C = 240x + 284y$$
$$1000 = xy$$
$$\frac{1000}{y} = x$$
Substitute $\frac{1000}{y}$ for y in the cost equation and differentiate to find the minimizing value for x:
$$C = 240\frac{1000}{y} + 284y$$
$$C = \frac{240\,000}{y} + 284y$$
$$C' = \frac{-240\,000}{y^2} + 284$$
$$C' = \frac{284y^2 - 240\,000}{y^2}$$
Let $C' = 0$:
$$\frac{284y^2 - 240\,000}{y^2} = 0$$
Therefore $284y^2 - 240\,000 = 0$
$$284y^2 = 240\,000$$
$$y = 29.1 \text{ m}$$

y	y < 29.1	29.1	y > 29.1
C′(y)	−	0	+
Graph	Dec.	Local Min	Inc.

There is a local minimum at $y = 29.1$ m, so that is the minimizing value. To find x, use the equation
$$\frac{1000}{y} = x$$
$$\frac{1000}{29.1} = x$$
$$x = 34.4 \text{ m}$$
The fence and the side opposite it should be 34.4 m, and the other two sides should be 29.1 m.

24.



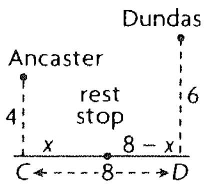
The distance between the boats is the hypotenuse of a right triangle. One side of the triangle is the distance from the first boat to the dock and the other side is the distance from the second boat to the dock. The distance is given by the equation

$d(t) = \sqrt{(15t)^2 + (12 - 12t)^2}$ where t is hours after 2:00
$$d(t) = \sqrt{369t^2 - 288t + 144}$$
To find the time that minimizes the distance, calculate the derivative and find the critical numbers:
$$d'(t) = \frac{738t - 288}{2\sqrt{81t^2 - 48t + 144}}$$
Let $d'(t) = 0$:
$$\frac{738t - 288}{2\sqrt{81t^2 - 48t + 144}} = 0$$
Therefore, $738t - 288 = 0$
$$738t = 288$$
$$t = .39 \text{ hours}$$

t	$t < .39$	.39	$t > .39$
$d'(t)$	$-$	0	$+$
Graph	Dec.	Local Min	Inc.

There is a local minimum at $t = .39$ hours, so the ships were closest together at 2:23.

25.



Let the distance from C to the rest stop be x and so the distance from the rest stop to D is $8 - x$, as shown. The distance from Ancaster to the rest stop is therefore

$\sqrt{4^2 + x^2} = \sqrt{16 + x^2}$, and the distance from the rest stop to Dundas is

$$\sqrt{6^2 + (8 - x)^2} = \sqrt{36 + 64 - 16x + x^2}$$
$$= \sqrt{100 - 16x + x^2}$$

So the total length of the trails is

$$L = \sqrt{16 + x^2} + \sqrt{100 - 16x + x^2}$$

The minimum cost can be found by expressing L as a function of x and examining its derivative to find critical points.

$L(x) = \sqrt{16 + x^2} + \sqrt{100 - 16x + x^2}$, which is defined for $0 \leq x \leq 8$

$$L'(x) = \frac{2x}{2\sqrt{16 + x^2}} + \frac{2x - 16}{2\sqrt{100 - 16x + x^2}}$$
$$= \frac{x\sqrt{100 - 16x + x^2} + (x - 8)\sqrt{16 + x^2}}{\sqrt{(16 + x^2)(100 - 16x + x^2)}}$$

The critical points of $A(r)$ can be found by setting $L'(x) = 0$:

$$x\sqrt{100 - 16x + x^2} + (x - 8)\sqrt{16 + x^2} = 0$$
$$x^2(100 - 16x + x^2) = (x^2 - 16x + 64)(16 + x^2)$$
$$100x^2 - 16x^3 + x^4 = x^4 - 16x^3 + 64x^2 + 16x^2 - 256x + 1024$$
$$20x^2 + 256x - 1024 = 0$$
$$4(5x - 16)(x + 16) = 0$$

So $x = 3.2$ and $x = -16$ are the critical points of the function. Only the positive root is within the interval of interest, however. The minimum total length therefore occurs at this point or at one of the endpoints of the interval:

$$L(0) = \sqrt{16 + 0^2} + \sqrt{100 - 16(0) + 0^2} = 14$$
$$L(3.2) = \sqrt{16 + 3.2^2} + \sqrt{100 - 16(3.2) + 3.2^2} \doteq 12.8$$
$$L(8) = \sqrt{16 + 8^2} + \sqrt{100 - 16(8) + 8^2} \doteq 14.9$$

So the rest stop should be built 3.2 km from point C.

26. a. $f(x) = x^2 - 2x + 6, -1 \leq x \leq 7$

$f'(x) = 2x - 2$

Set $f'(x) = 0$

$0 = 2x - 2$

$x = 1$

$f(-1) = (-1)^2 - 2(-1) + 6$

$f(-1) = 1 + 2 + 6$

$f(-1) = 9$

$f(7) = (7)^2 - 2(7) + 6$

$f(7) = 49 - 14 + 6$

$f(7) = 41$

$f(1) = 1^2 - 2(1) + 6$

$f(1) = 1 - 2 + 6$

$f(1) = 5$

Absolute Maximum: $f(7) = 41$

Absolute Minimum: $f(1) = 5$

b. $f(x) = x^3 + x^2, -3 \leq x \leq 3$

$f'(x) = 3x^2 + 2x$

Set $f'(x) = 0$

$0 = 3x^2 + 2x$

$0 = x(3x + 2)$

$x = -\frac{2}{3}$ or $x = 0$

$f(-3) = (-3)^3 + (-3)^2$

$f(-3) = -27 + 9$

$f(-3) = -18$

$$f\left(-\frac{2}{3}\right) = \left(-\frac{2}{3}\right)^3 + \left(-\frac{2}{3}\right)^2$$
$$f\left(-\frac{2}{3}\right) = -\frac{8}{27} + \frac{4}{9}$$
$$f\left(-\frac{2}{3}\right) = \frac{4}{27}$$

$f(0) = (0)^3 + (0)^2$

$f(0) = 0$

$f(3) = (3)^3 + (3)^2$

$f(3) = 27 + 9$

$f(3) = 36$

Absolute Maximum: $f(3) = 36$

Absolute Minimum: $f(-3) = -18$

c. $f(x) = x^3 - 12x + 2, -5 \leq x \leq 5$

$f'(x) = 3x^2 - 12$

Set $f'(x) = 0$

$$0 = 3x^2 - 12$$
$$x^2 = 4$$
$$x = -2 \text{ or } x = 2$$
$$f(-5) = (-5)^3 - 12(-5) + 2$$
$$f(-5) = -125 + 60 + 2$$
$$f(-5) = -63$$
$$f(2) = (2)^3 - 12(2) + 2$$
$$f(2) = 8 - 24 + 2$$
$$f(2) = -14$$
$$f(-2) = (-2)^3 - 12(-2) + 2$$
$$f(-2) = -8 + 24 + 2$$
$$f(-2) = 18$$
$$f(5) = (5)^3 - 12(5) + 2$$
$$f(5) = 125 - 60 + 2$$
$$f(5) = 67$$

Absolute Maximum: $f(5) = 67$
Absolute Minimum: $f(-5) = -63$

d. $f(x) = 3x^5 - 5x^3, -2 \leq x \leq 4$

$$f'(x) = 15x^4 - 15x^2$$

Set $f'(x) = 0$

$$0 = 15x^4 - 15x^2$$
$$0 = 15x^2(x^2 - 1)$$
$$0 = 15x^2(x - 1)(x + 1)$$
$$x = -1 \text{ or } x = 0 \text{ or } x = 1$$
$$f(-2) = 3(-2)^5 - 5(-2)^3$$
$$f(-2) = -96 + 40$$
$$f(-2) = -56$$
$$f(0) = 3(0)^5 + 5(0)^3$$
$$f(0) = 0$$

Note: (0, 0) is not a maximum or a minimum

$$f(4) = 3(4)^5 - 5(4)^3$$
$$f(4) = 3072 - 320$$
$$f(4) = 2752$$
$$f(-1) = 3(-1)^5 - 5(-1)^3$$
$$f(-1) = -3 + 5$$
$$f(-1) = 2$$
$$f(1) = 3(1)^5 - 5(1)^3$$
$$f(1) = 3 - 5$$
$$f(1) = -2$$

Absolute Maximum: $f(4) = 2752$
Absolute Minimum: $f(-2) = -56$

27. a. $s(t) = 20t - 0.3t^3$

$$s'(t) = 20 - 0.9t^2$$

The car stops when $s'(t) = 0$.

$$20 - 0.9t^2 = 0$$
$$0.9t^2 = 20$$
$$t = \sqrt{\frac{20}{0.9}}$$
$$t \doteq 4.714$$

(-4.714 is inadmissible)

$$s(4.714) = 20(4.714) - 0.3(4.714)^3$$
$$\doteq 62.9 \text{ m}$$

b. From the solution to a., the stopping time is about 4.7 s.

c. $s''(t) = -1.8t$

$$s''(2) = -1.8(2)$$
$$= -3.6 \text{ m/s}^2$$

The deceleration is 3.6 m/s^2.

28. a. $f'(x) = \frac{d}{dx}(5x^3 - x)$

$$= 15x^2 - 1$$
$$f''(x) = \frac{d}{dx}(15x^2 - 1)$$
$$= 30x$$

So $f''(2) = 30(2) = 60$

b. $f'(x) = \frac{d}{dx}(-2x^{-3} + x^2)$

$$= 6x^{-4} + 2x$$
$$f''(x) = \frac{d}{dx}(6x^{-4} + 2x)$$
$$= -24x^{-5} + 2$$

So $f''(-1) = -24(-1)^{-5} + 2 = 26$

c. $f'(x) = \frac{d}{dx}(4x - 1)^4$

$$= 4(4x - 1)^3(4)$$
$$= 16(4x - 1)^3$$
$$f''(x) = \frac{d}{dx}(16(4x - 1)^3)$$
$$= 16(3)(4x - 1)^2(4)$$
$$= 192(4x - 1)^2$$

So $f''(0) = 192(4(0) - 1)^2 = 192$

d. $f'(x) = \frac{d}{dx}\left(\frac{2x}{x - 5}\right)$

$$= \frac{(x - 5)(2) - (2x)(1)}{(x - 5)^2}$$
$$= \frac{-10}{(x - 5)^2}$$
$$f''(x) = \frac{d}{dx}\left(\frac{-10}{(x - 5)^2}\right)$$
$$= \frac{(x - 5)^2(0) - (-10)(2(x - 5))}{(x - 5)^4}$$

$$= \frac{20}{(x-5)^3}$$

So $f''(1) = \frac{20}{(1-5)^3} = -\frac{5}{16}$

e. $f(x)$ can be rewritten as $f(x) = (x+5)^{\frac{1}{2}}$. Then

$$f'(x) = \frac{d}{dx}\left((x+5)^{\frac{1}{2}}\right)$$

$$= \frac{1}{2}(x+5)^{-\frac{1}{2}}$$

$$f''(x) = \frac{d}{dx}\left(\frac{1}{2}(x+5)^{-\frac{1}{2}}\right)$$

$$= \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)(x+5)^{-\frac{3}{2}}$$

$$= -\frac{1}{4}(x+5)^{-\frac{3}{2}}$$

So $f''(4) = -\frac{1}{4}(4+5)^{-\frac{3}{2}} = -\frac{1}{108}$

f. $f(x)$ can be rewritten as $f(x) = x^{\frac{2}{3}}$. Then

$$f'(x) = \frac{d}{dx}\left(x^{\frac{2}{3}}\right)$$

$$= \left(\frac{2}{3}\right)x^{-\frac{1}{3}}$$

$$f''(x) = \frac{d}{dx}\left(\left(\frac{2}{3}\right)x^{-\frac{1}{3}}\right)$$

$$= \left(\frac{2}{3}\right)\left(-\frac{1}{3}\right)x^{-\frac{4}{3}}$$

$$= -\left(\frac{2}{9}\right)x^{-\frac{4}{3}}$$

So $f''(8) = -\left(\frac{2}{9}\right)(8)^{-\frac{4}{3}} = -\frac{1}{72}$

29. a. $s(t) = \frac{2t}{t+3}$

$$s'(t) = \frac{(t+3)(2) - 2t(1)}{(t+3)^2}$$

$$= \frac{2t + 6 - 2t}{(t+3)^2}$$

$$= \frac{6}{(t+3)^2}$$

$$s''(t) = \frac{(t+3)^2(0) - 6(2(t+3)+1)}{(t+3)^4}$$

$$= \frac{-6(2t+6)}{(t+3)^4}$$

$$= \frac{-12(t+3)}{(t+3)^4}$$

$$= \frac{-12}{(t+3)^3}$$

$$s(3) = \frac{2(3)}{3+3}$$

$$= \frac{6}{6}$$

$$= 1$$

$$s'(3) = \frac{6}{(3+3)^2}$$

$$= \frac{6}{36}$$

$$= \frac{1}{6}$$

$$s''(3) = \frac{-12}{(3+3)^3}$$

$$= \frac{-12}{216}$$

$$= -\frac{1}{18}$$

At $t = 3$, position is 1, velocity is $\frac{1}{6}$, acceleration is $-\frac{1}{18}$, and speed is $\frac{1}{6}$.

b. $s(t) = t + \frac{5}{t+2}$

$$s'(t) = 1 + \frac{(t+2)(0) - 5(1)}{(t+2)^2}$$

$$= 1 - \frac{5}{(t+2)^2}$$

$$s''(t) = 0 - \frac{(t+2)^2(0) - 5[2(t+2)(1)]}{(t+2)^4}$$

$$= \frac{10(t+2)}{(t+2)^4}$$

$$= \frac{10}{(t+2)^3}$$

$$s(1) = 1 + \frac{5}{1+2}$$

$$= 1 + \frac{5}{3}$$

$$= \frac{8}{3}$$

$$s'(1) = 1 - \frac{5}{(1+2)^2}$$

$$= 1 - \frac{5}{9}$$

$$= \frac{4}{9}$$

$$s''(1) = \frac{10}{(1+2)^3}$$
$$= \frac{10}{27}$$

At $t = 3$, position is $\frac{8}{3}$, velocity is $\frac{4}{9}$, acceleration is $\frac{10}{27}$, and speed is $\frac{4}{9}$.

30. a. $s(t) = (t^2 + t)^{\frac{2}{3}}, t \geq 0$

$$v(t) = \frac{2}{3}(t^2 + t)^{-\frac{1}{3}}(2t + 1)$$

$a(t)$

$$= \frac{2}{3}\left[-\frac{1}{3}(t^2 + t)^{-\frac{4}{3}}(2t + 1)(2t + 1) + 2(t^2 + 1)^{-\frac{1}{3}}\right]$$
$$= \frac{2}{3}\left(-\frac{1}{3}\right)(t^2 + t)^{-\frac{4}{3}}[(2t + 1)^2 - 6(t^2 + 1)]$$
$$= -\frac{2}{9}(t^2 + t)^{-\frac{4}{3}}(4t^2 + 4t + 1 - 6t^2 - 6t)$$
$$= \frac{2}{9}(t^2 + t)^{-\frac{4}{3}}(2t^2 + 2t - 1)$$

b. $v_{avg} = \frac{s(5) - s(0)}{5 - 0}$
$$= \frac{(5^2 + 5)^{\frac{2}{3}} - (0^2 + 0)^{\frac{2}{3}}}{5}$$
$$= \frac{30^{\frac{2}{3}} - 0}{5}$$
$$\doteq 1.931$$

The average velocity is approximately 1.931 m/s.

c. $v(5) = \frac{2}{3}(5^2 + 5)^{-\frac{1}{3}}(2(5) + 1)$
$$= \frac{2}{3}(30)^{-\frac{1}{3}}(11)$$
$$\doteq 2.360$$

The velocity at 5 s is approximately 2.36 m/s.

d. Average acceleration $= \frac{v(5) - v(0)}{5 - 0}$ which is undefined because $v(0)$ is undefined.

e. $a(5) = \frac{2}{9}(5^2 + 5)^{-\frac{4}{3}}(2(5)^2 + 2(5) \pm 1)$
$$= \frac{2}{9}\left(30^{-\frac{4}{3}}\right)(59)$$
$$\doteq 0.141$$

The acceleration at 5 s is approximately 0.141 m/s^2.

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1. a. $y = 7x^2 - 9x + 22$
$$y' = 14x - 9$$
$$y'' = 14$$

b. $f(x) = -9x^5 - 4x^3 + 6x - 12$
$$f'(x) = -45x^4 - 12x^2 + 6$$
$$f''(x) = -180x^3 - 24x$$

c. $y = 5x^{-3} + 10x^3$
$$y' = -15x^{-4} + 30x^2$$
$$y'' = 60x^{-5} + 60x$$

d. $f(x) = (4x - 8)^3$
$$f'(x) = 3(4x - 8)^2(4)$$
$$= 12(4x - 8)^2$$
$$f''(x) = 24(4x - 8)(4)$$
$$= 96(4x - 8)$$

2. a. $s(t) = -3t^3 + 5t^2 - 6t$
$$v(t) = -9t^2 + 10t - 6$$
$$v(3) = -9(9) + 30 - 6$$
$$= -57$$
$$a(t) = -18t + 10$$
$$a(3) = -18(3) + 10$$
$$= -44$$

b. $s(t) = (2t - 5)^3$
$$v(t) = 3(2t - 5)^2(2)$$
$$= 6(2t - 5)^2$$
$$v(2) = 6(4 - 5)^2$$
$$= 6$$
$$a(t) = 12(2t - 5)(2)$$
$$= 24(2t - 5)$$
$$a(2) = 24(4 - 5)$$
$$= -24$$

3. a. $s(t) = t^2 - 3t + 2$
$$v(t) = 2t - 3$$
$$a(t) = 2$$

b. $2t - 3 = 0$
$$t = 1.5 \text{ s}$$
$$s(1.5) = 1.5^2 - 3$$
$$(1.5) + 2 = -0.25$$

c. $t^2 - 3t + 2 = 0$
$$(t - 1)(t - 2) = 0$$
$$t = 1 \text{ or } t = 2$$
$$|v(1)| = |-1|$$
$$= 1$$
$$|v(2)| = |1|$$
$$= 1$$

The speed is 1 m/s when the position is 0.

d. The object moves to the left when $v(t) < 0$.
$$2t - 3 < 0$$
$$t < 1.5$$

The object moves to the left between $t = 0$ s and $t = 1.5$ s.

e. $v(5) = 10 - 3 = 7$ m/s

$v(2) = 4 - 3 = 1$ m/s

average velocity $= \dfrac{7 - 1}{5 - 2}$

$= 2$ m/s^2

4. a. $f(x) = x^3 - 12x + 2$

$f'(x) = 3x^2 - 12x$

$3x^2 - 12x = 0$

$3x(x - 4) = 0$

$x = 0$ or $x = 4$

Test the endpoints and the values that make the derivative 0.

$f(-5) = -125 + 60 + 2 = -63$ min

$f(0) = 2$

$f(4) = 64 - 48 + 2 = 18$

$f(5) = 125 - 60 + 2 = 67$ max

b. $f(x) = x + \dfrac{9}{x}$

$= x + 9x^{-1}$

$f'(x) = 1 - 9x^{-2}$

$1 - 9x^{-2} = 0$

$1 - \dfrac{9}{x^2} = 0$

$\dfrac{x^2 - 9}{x^2} = 0$

$x^2 - 9 = 0$

$x = \pm 3$

$x = -3$ is not in the given interval.

$f(1) = 1 + 9 = 10$ max

$f(3) = 3 + 3 = 6$ min

$f(6) = 6 + 1.5 = 7.5$

5. a. $h(t) = -4.9t^2 + 21t + 0.45$

$h'(t) = -9.8t + 21$

Set $h'(t) = 0$ and solve for t.

$-9.8t + 21 = 0$

$9.8t = 21$

$t \doteq 2.1$ s

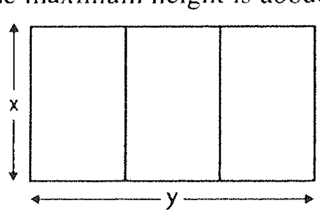
The graph has a max or min at $t = 2.1$ s. Since the equation represents a parabola, and the lead coefficient is negative, the value must be a maximum.

b. $h(2.1) = -4.9(2.1)^2 + 21(2.1) + 0.45$

$\doteq 22.9$

The maximum height is about 22.9 m.

6.



Let x represent the width of the field in m, $x > 0$.

Let y represent the length of the field in m.

$4x + 2y = 2000$ ①

$A = xy$ ②

From ①: $y = 1000 - 2x$. Restriction $0 < x < 500$

Substitute into ②:

$A(x) = x(1000 - 2x)$

$= 1000x - 2x^2$

$A'(x) = 1000 - 4x$.

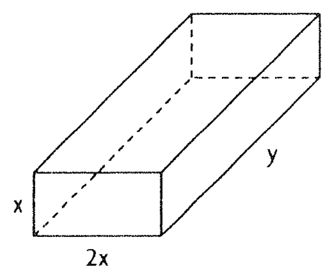
For a max min, $A'(x) = 0$, $x = 250$

x	$A(x) = x(1000 - 2x)$
0	$\lim_{x \to 0^+} A(x) = 0$
250	$A(250) = 125\,000$ max
1000	$\lim_{x \to 1000} A(x) = 0$

$x = 250$ and $y = 500$.

Therefore, each paddock is 250 m in width and $\frac{500}{3}$ m in length.

7.



Let x represent the height.

Let $2x$ represent the width.

Let y represent the length.

Volume $10\,000 = 2x^2y$

Cost:

$C = 0.02(2x)y + 2(0.05)(2x^2)$

$+ 2(0.05)(xy) + 0.1(2xy)$

$= 0.04xy + 0.2x^2 + 0.1xy + 0.2xy$

$= 0.34xy + 0.2x^2$

But $y = \dfrac{10\,000}{2x^2} = \dfrac{5000}{x^2}$.

Therefore, $C(x) = 0.34x\left(\dfrac{5000}{x^2}\right) + 0.2x^2$

$= \dfrac{1700}{x} + 0.2x^2, x \geq 0$

$C'(x) = \dfrac{-1700}{x^2} + 0.4x$.

Let $C'(x) = 0$:

$$\frac{-1700}{x^2} + 0.4x = 0$$

$$0.4x^3 = 1700$$

$$x^3 = 4250$$

$$x \doteq 16.2.$$

Using max min Algorithm,

$C(0) \to \infty$

$$C(16.2) = \frac{1700}{16.2} + 0.2(16.2)^2 = 157.4.$$

Minimum when $x = 16.2$, $2x = 32.4$ and $y = 19.0$.
The required dimensions are 162 mm by 324 mm by 190 mm.

8. Let x = the number of \$100 increases, $x \geq 0$.
The number of units rented will be $50 - 10x$.
The rent per unit will be $850 + 100x$.

$$R(x) = (850 + 100x)(50 - 10x)$$

$$R'(x) = (850 + 100x)(-10) + (50 - 10x)(100)$$

$$= -8500 - 1000x + 5000 - 1000x$$

$$= -2000x - 3500$$

Set $R'(x) = 0$

$0 = -3500 - 2000x$

$2000x = -3500$

$x = -1.75$ but $x \geq 0$

To maximize revenue the landlord should not increase rent. The residents should continue to pay \$850/month.

CHAPTER 4
Curve Sketching

Review of Prerequisite Skills, pp. 162–163

1. a. $2y^2 + y - 3 = 0$
$(2y + 3)(y - 1) = 0$
$y = -\frac{3}{2}$ or $y = 1$

b. $x^2 - 5x + 3 = 17$
$x^2 - 5x - 14 = 0$
$(x - 7)(x + 2) = 0$
$x = 7$ or $x = -2$

c. $4x^2 + 20x + 25 = 0$
$(2x + 5)(2x + 5) = 0$
$x = -\frac{5}{2}$

d. $y^3 + 4y^2 + y - 6 = 0$
$y = 1$ is a zero, so $y - 1$ is a factor. After synthetic division, the polynomial factors to $(y - 1)(y^2 + 5y + 6)$.
So $(y - 1)(y + 3)(y + 2) = 0$.
$y = 1$ or $y = -3$ or $y = -2$

2. a. $3x + 9 < 2$
$3x < -7$
$x < -\frac{7}{3}$

b. $5(3 - x) \geq 3x - 1$
$15 - 5x \geq 3x - 1$
$16 \geq 8x$
$8x \leq 16$
$x \leq 2$

c. $t^2 - 2t < 3$
$t^2 - 2t - 3 < 0$
$(t - 3)(t + 1) < 0$
Consider $t = 3$ and $t = -1$.

t values	$t < -1$	$-1 < t < 3$	$t > 3$
$(t + 1)$	−	+	+
$(t - 3)$	−	−	+
$(t - 3)(t + 1)$	+	−	+

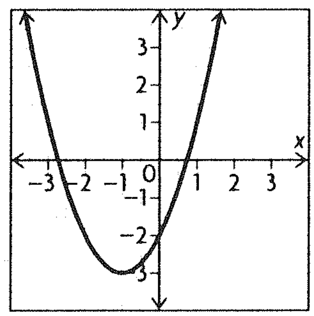
The solution is $-1 < t < 3$.

d. $x^2 + 3x - 4 > 0$
$(x + 4)(x - 1) > 0$
Consider $x = -4$ and $x = 1$.

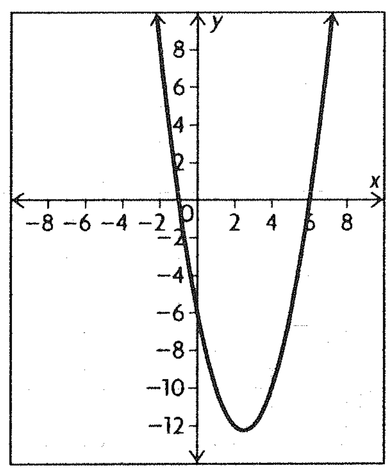
x values	$x < -4$	$-4 < x < 1$	$x > 1$
$(x + 4)$	−	+	+
$(x - 1)$	−	−	+
$(x + 4)(x - 1)$	+	−	+

The solution is $x < -4$ or $x > 1$.

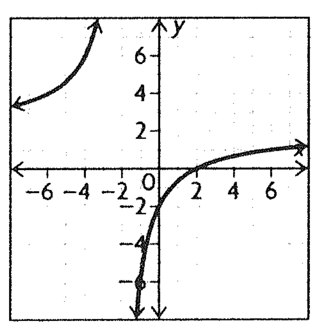
3. a.



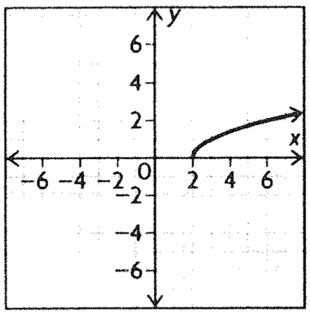
b.



c.



d.



4. a. $\lim\limits_{x\to 2^-}(x^2 - 4) = 2^2 - 4$

$= 0$

b. $\lim\limits_{x\to 2}\dfrac{x^2 + 3x - 10}{x - 2}$

$= \lim\limits_{x\to 2}\dfrac{(x + 5)(x - 2)}{x - 2}$

$= \lim\limits_{x\to 2}(x + 5)$

$= 7$

c. $\lim\limits_{x\to 3}\dfrac{x^3 - 27}{x - 3}$

$= \lim\limits_{x\to 3}\dfrac{(x - 3)(x^2 + 3x + 9)}{x - 3}$

$= \lim\limits_{x\to 3}(x^2 + 3x + 9)$

$= 3^2 + 3 \times 3 + 9$

$= 27$

d. $\lim\limits_{x\to 4^-}\sqrt{2x + 1}$

$= \sqrt{2 \times 4 + 1}$

$= 3$

5. a. $f(x) = \dfrac{1}{4}x^4 + 2x^3 - \dfrac{1}{x}$

$= \dfrac{1}{4}x^4 + 2x^3 - x^{-1}$

$f'(x) = x^3 + 6x^2 + x^{-2}$

b. $f(x) = \dfrac{x + 1}{x^2 - 3}$

$f'(x) = \dfrac{(x^2 - 3)(1) - (x + 1)(2x)}{(x^2 - 3)^2}$

$= \dfrac{x^2 - 3 - 2x^2 - 2x}{(x^2 - 3)^2}$

$= \dfrac{-x^2 - 2x - 3}{(x^2 - 3)^2}$

$= -\dfrac{x^2 + 2x + 3}{(x^2 - 3)^2}$

c. $f(x) = (3x^2 - 6x)^2$

$f'(x) = 2(3x^2 - 6x)(6x - 6)$

d. $f(t) = \dfrac{2t}{\sqrt{t - 4}}$

$f'(t) = \dfrac{2\sqrt{t - 4} - \dfrac{2t}{2\sqrt{t - 4}}}{t - 4}$

$f'(t) = \dfrac{\dfrac{4(t - 4)}{2\sqrt{t - 4}} - \dfrac{2t}{2\sqrt{t - 4}}}{t - 4}$

$f'(t) = \dfrac{4(t - 4) - 2t}{2(t - 4)^{\frac{3}{2}}}$

$= \dfrac{2t - 16}{2(t - 4)^{\frac{3}{2}}}$

$= \dfrac{t - 8}{(t - 4)^{\frac{3}{2}}}$

6. a.

$$\begin{array}{r} x - 8 \\ x + 3\overline{)x^2 - 5x + 4} \\ \underline{x^2 + 3x} \\ -8x + 4 \\ \underline{-8x - 24} \\ 28 \end{array}$$

$(x^2 - 5x - 4) \div (x + 3) = x - 8 + \dfrac{28}{x + 3}$

b.

$$\begin{array}{r} x + 7 \\ x - 1\overline{)x^2 + 6x - 9} \\ \underline{x^2 - x} \\ 7x - 9 \\ \underline{7x - 7} \\ -2 \end{array}$$

$(x^2 - 6x - 9) \div (x - 1) = x + 7 - \dfrac{2}{x - 1}$

7. $f(x) = x^3 + 0.5x^2 - 2x + 3$

$f'(x) = 3x^2 + x - 2$

Let $f'(x) = 0$:

$3x^2 + x - 2 = 0$

$(3x - 2)(x + 1) = 0$

$x = \dfrac{2}{3}$ or $x = -1$

The points are $\left(\frac{2}{3}, 2.19\right)$ and $(-1, 4.5)$.

8. a. If $f(x) = x^n$, where n is a real number, then $f'(x) = nx^{n-1}$.

b. If $f(x) = k$, where k is a constant, then $f'(x) = 0$.

c. If $k(x) = f(x)g(x)$, then $k'(x) = f'(x)g(x) - f(x)g'(x)$

d. If $h(x) = \frac{f(x)}{g(x)}$, then $h'(x)$

$= \dfrac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}, g(x) \neq 0.$

e. If f and g are functions that have derivatives, then the composite function $h(x) = f(g(x))$ has a derivative given by $h'(x) = f'(g(x))g'(x)$.
f. If u is a function of x, and n is a positive integer, then $\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$

9. a. $\lim_{x\to\infty} 2x^2 - 3x + 4 = \infty$

$\lim_{x\to-\infty} 2x^2 - 3x + 4 = \infty$

b. $\lim_{x\to\infty} 2x^3 + 4x - 1 = \infty$

$\lim_{x\to-\infty} 2x^3 + 4x - 1 = -\infty$

c. $\lim_{x\to\infty} -5x^4 + 2x^3 - 6x^2 + 7x - 1 = -\infty$

$\lim_{x\to-\infty} -5x^4 + 2x^3 - 6x^2 + 7x - 1 = -\infty$

10. a. $\frac{1}{f(x)} = \frac{1}{2x}$
Let $2x = 0$
$x = 0$, so the graph has a vertical asymptote at $x = 0$.

b. $\frac{1}{f(x)} = \frac{1}{-x + 3}$
Let $-x + 3 = 0$
$x = 3$, so the graph has a vertical asymptote at $x = 3$.

c. $\frac{1}{f(x)} = \frac{1}{(x + 4)^2 + 1}$
Let $(x + 4)^2 + 1 = 0$
There is no solution, so the graph has no vertical asymptotes.

d. $\frac{1}{f(x)} = \frac{1}{(x + 3)^2}$
Let $(x + 3)^2 = 0$
$x = -3$, so the graph has a vertical asymptote at $x = -3$.

11. a. $\lim_{x\to\infty}\frac{5}{x + 1} = 0$, so the horizontal asymptote is $y = 0$.

b. $\lim_{x\to\infty}\frac{4x}{x - 2} = 4$, so the horizontal asymptote is $y = 4$.

c. $\lim_{x\to\infty}\frac{3x - 5}{6x - 3} = \frac{1}{2}$, so the horizontal asymptote is $y = \frac{1}{2}$.

d. $\lim_{x\to\infty}\frac{10x - 4}{5x} = 2$, so the horizontal asymptote is $y = 2$.

12. a. i. $y = \frac{5}{x + 1}$
To find the x-intercept, let $y = 0$.
$\frac{5}{x + 1} = 0$
There is no solution, so there is no x-intercept.
The y-intercept is $y = \frac{5}{0 + 1} = 5$.

ii. $y = \frac{4x}{x - 2}$
To find the x-intercept, let $y = 0$.
$\frac{4x}{x - 2} = 0$
$x = 0$
The y-intercept is $y = \frac{0}{0 - 2} = 0$.

iii. $y = \frac{3x - 5}{6x - 3}$
To find the x-intercept, let $y = 0$:
$\frac{3x - 5}{6x - 3} = 0$
Therefore, $3x - 5 = 0$
$x = \frac{5}{3}$
The y-intercept is $y = \frac{0 - 5}{0 - 3} = \frac{5}{3}$.

iv. $y = \frac{10x - 4}{5x}$
To find the x-intercept, let $y = 0$.
$\frac{10x - 4}{5x} = 0$
Therefore, $10x - 4 = 0$
$x = \frac{2}{5}$
The y-intercept is $y = \frac{0 - 4}{0}$, which is undefined, so there is no y-intercept.

b. i. $y = \frac{5}{x + 1}$
Domain: $\{x \in \mathbf{R} | x \neq -1\}$
Range: $\{y \in \mathbf{R} | y \neq 0\}$

ii. $y = \frac{4x}{x - 2}$
Domain: $\{x \in \mathbf{R} | x \neq 2\}$
Range: $\{y \in \mathbf{R} | y \neq 4\}$

iii. $y = \frac{3x - 5}{6x - 3}$
Domain: $\left\{x \in \mathbf{R} | x \neq \frac{1}{2}\right\}$
Range: $\left\{y \in \mathbf{R} | y \neq \frac{1}{2}\right\}$

iv. $y = \frac{10x - 4}{5x}$
Domain: $\{x \in \mathbf{R} | x \neq 0\}$
Range: $\{y \in \mathbf{R} | y \neq 2\}$

4.1 Increasing and Decreasing Functions, pp. 169–171

1. a. $f(x) = x^3 + 6x^2 + 1$

$f'(x) = 3x^2 + 12x$

Let $f'(x) = 0$: $3x(x + 4) = 0$

$x = 0$ or $x = -4$

The points are $(0, 1)$ and $(-4, 33)$.

b. $f(x) = \sqrt{x^2 + 4}$

$= (x^2 + 4)^{\frac{1}{2}}$

$$f'(x) = \frac{1}{2}(x^2 + 4)^{-\frac{1}{2}}(2x)$$

$$= \frac{x}{\sqrt{x^2 + 4}}$$

Let $f'(x) = 0$:

$$\frac{x}{\sqrt{x^2 + 4}} = 0$$

So $x = 0$.

The point is $(0, 2)$.

c. $f(x) = (2x - 1)^2(x^2 - 9)$

$f'(x) = 2(2x - 1)(2)(x^2 - 9) + 2x(2x - 1)^2$

Let $f'(x) = 0$:

$2(2x - 1)(2(x^2 - 9) + x(2x - 1)) = 0$

$2(2x - 1)(4x^2 - x - 18) = 0$

$2(2x - 1)(4x - 9)(x + 2) = 0$

$x = \frac{1}{2}$ or $x = \frac{9}{4}$ or $x = -2$.

This points are $\left(\frac{1}{2}, 0\right)$, $(2.25, -48.2)$ and $(-2, -125)$.

d. $f(x) = \dfrac{5x}{x^2 + 1}$

$$f'(x) = \frac{5(x^2 + 1) - 5x(2x)}{(x^2 + 1)^2} = \frac{5(1 - x^2)}{(x^2 + 1)^2}$$

Let $f'(x) = 0$:

$$\frac{5(1 - x^2)}{(x^2 + 1)^2} = 0$$

Therefore, $5(1 - x^2) = 0$

$(1 - x)(1 + x) = 0$

$x = \pm 1$

The points are $\left(1, \frac{5}{2}\right)$ and $\left(-1, -\frac{5}{2}\right)$.

2. A function is increasing when $f'(x) > 0$ and is decreasing when $f'(x) < 0$.

3. a. i. $x < -1, x > 2$

ii. $-1 < x < 2$

iii. $(-1, 4), (2, -1)$

b. i. $-1 < x < 1$

ii. $x < -1, x > 1$

iii. $(-1, 2), (2, 4)$

c. i. $x < -2$

ii. $-2 < x < 2, 2 < x$

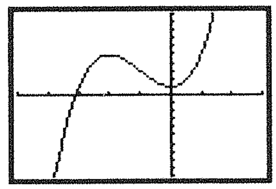
iii. none

d. i. $-1 < x < 2, 3 < x$

ii. $x < -1, 2 < x < 3$

iii. $(2, 3)$

4.



a. $f(x) = x^3 + 3x^2 + 1$

$f'(x) = 3x^2 + 6x$

Let $f'(x) = 0$

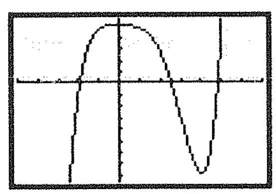
$3x^2 + 6x = 0$

$3x(x + 2) = 0$

$x = 0$ or $x = -2$

x	$x < -2$	-2	$-2 < x < 0$	0	$x > 0$
$f'(x)$	+	0	−	0	+
Graph	Increasing		Decreasing		Increasing

b.



$f(x) = x^5 - 5x^4 + 100$

$f'(x) = 5x^4 - 20x^3$

Let $f'(x) = 0$:

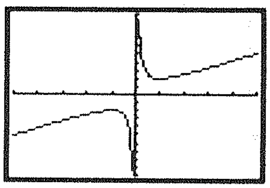
$5x^4 - 20x^3 = 0$

$5x^3(x - 4) = 0$

$x = 0$ or $x = 4$.

x	$x < 0$	0	$0 < x < 4$	4	$x > 4$
$f'(x)$	+	0	−	0	+
Graph	Increasing		Decreasing		Increasing

c.



$$f(x) = x + \frac{1}{x}$$

$$f'(x) = 1 - \frac{1}{x^2}$$

Let $f'(x) = 0$

$$1 - \frac{1}{x^2} = 0$$

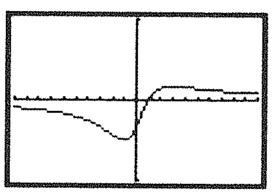
$$x^2 - 1 = 0$$

$$x = -1 \text{ or } x = 1$$

Also note that $f(x)$ is undefined for $x = 0$.

x	$x < -1$	-1	$-1 < x < 0$	0	$0 < x < 1$	1	$x > 1$
$f'(x)$	+	0	−	undefined	−	0	+
Graph	Increasing		Decreasing		Decreasing		Increasing

d.



$$f(x) = \frac{x - 1}{x^2 + 3}$$

$$f'(x) = \frac{x^2 + 3 - 2x(x - 1)}{(x^2 + 3)^2}$$

Let $f'(x) = 0$, therefore, $-x^2 + 2x + 3 = 0$.

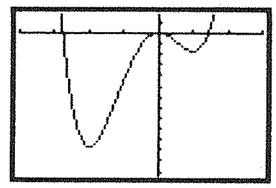
Or $x^2 - 2x - 3 = 0$

$(x - 3)(x + 1) = 0$

$x = 3$ or $x = -1$

x	$x < -1$	-1	$-1 < x < 3$	3	$x > 3$
$f'(x)$	−	0	+	0	−
Graph	Decreasing		Increasing		Decreasing

e.



$y = 3x^4 + 4x^3 - 12x^2$

$y' = 12x^3 + 12x^2 - 24x$

Intervals of increasing:

$12x^3 + 12x^2 - 24x > 0$

$x(x^2 + x - 2) > 0$

$x(x - 1)(x + 2) > 0$

Intervals of decreasing:

$12x^3 + 12x^2 - 24x < 0$

$x(x^2 + x - 2) < 0$

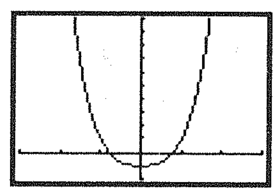
$x(x - 1)(x + 2) < 0$

	$x < -2$	$-2 < x < 0$	$0 < x < 1$	$x < 1$
x	−	−	+	+
$x - 1$	+	−	−	+
$x + 2$	−	+	+	+
y'	+	+	−	+

Intervals of increasing: $-2 < x < 0, x > 1$

Intervals of decreasing: $x < -2, 0 < x < 1$

f.



$y = x^4 + x^2 - 1$

$y' = 4x^3 + 2x$

Interval of increasing:

$4x^3 + 2x > 0$

$x(2x^2 + 1) > 0$

Interval of decreasing:

$4x^3 + 2x < 0$

$x(2x^2 + 1) < 0$

But $2x^2 + 1$ is always positive.

Interval of increasing: $x > 0$

Interval of decreasing: $x < 0$

5. $f'(x) = (x - 1)(x + 2)(x + 3)$

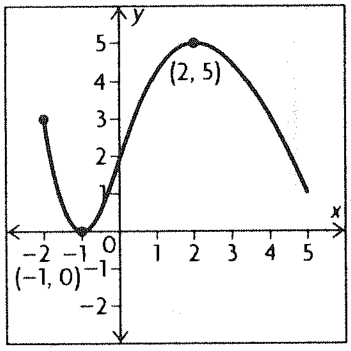
Let $f'(x) = 0$:

Then $(x - 1)(x + 2)(x + 3) = 0$

$x = 1$ or $x = -2$ or $x = -3$.

x	$x < -3$	-3	$-3 < x < -3$	-2	$-2 < x < 1$	1	$x > 1$
$f'(x)$	−	0	+	0	−	0	+
Graph	Decreasing		Increasing		Decreasing		Increasing

6.



7. $f(x) = x^3 + ax^2 + bx + c$

$f'(x) = 3x^2 + 2ax + b$

Since $f(x)$ increases to $(-3, 18)$ and then decreases, $f'(-3) = 0$.

Therefore, $27 - 6a + b = 0$ or $6a - b = 27$. (1)

Since $f(x)$ decreases to the point $(1, -14)$ and then increases $f'(1) = 0$.

Therefore, $3 + 2a + b = 0$ or $2a + b = -3$. (2)

Add (1) to (2) $8a = 24$ and $a = 3$.

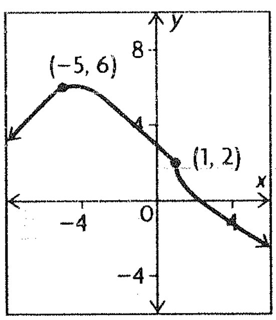
When $a = 3$, $b = 6 + b = -3$ or $b = -9$.

Since $(1, -14)$ is on the curve and $a = 3$, $b = -9$, then $-14 = 1 + 3 - 9 + c$

$c = -9$.

The function is $f(x) = x^3 + 3x^2 - 9x - 9$.

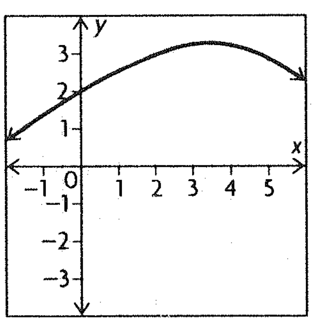
8.



9. **a. i.** $x < 4$

ii. $x > 4$

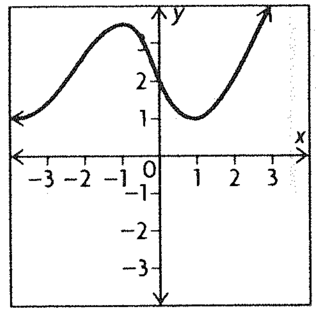
iii. $x = 4$



b. i. $x < -1, x > 1$

ii. $-1 < x < 1$

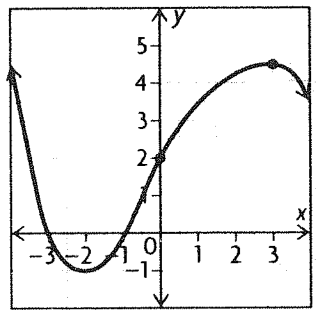
iii. $x = -1, x = 1$



c. i. $-2 < x < 3$

ii. $x < -2, x > 3$

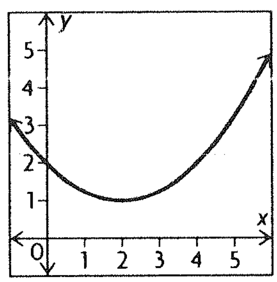
iii. $x = -2, x = 3$



d. i. $x > 2$

ii. $x < 2$

iii. $x = 2$



10. $f(x) = ax^2 + bx + c$

$f'(x) = 2ax + b$

Let $f'(x) = 0$, then $x = \frac{-b}{2a}$.

If $x < \frac{-b}{2a}$, $f'(x) < 0$, therefore the function is decreasing.

If $x > \frac{-b}{2a}$, $f'(x) > 0$, therefore the function is increasing.

11. $f(x) = x^4 - 32x + 4$

$f'(x) = 4x^3 - 32$

Let $f'(x) = 0$:

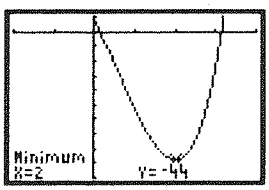
$4x^3 - 32 = 0$

$4x^3 = 32$

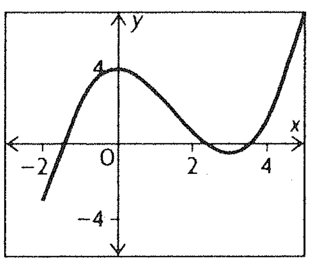
$x^3 = 8$
$x = 2$

x	$x < 2$	2	$x > 2$
$f(x)$	$-$	0	$+$
Graph	Dec.	Local Min	Inc

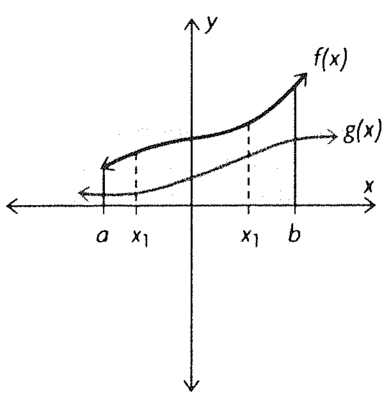
Therefore the function is decreasing for $x < 2$ and increasing for $x > 2$. The function has a local minimum at the point $(2, -44)$.



12.



13. Let $y = f(x)$ and $u = g(x)$.
Let x_1 and x_2 be any two values in the interval $a \leq x \leq b$ so that $x_1 < x_2$.
Since $x_1 < x_2$, both functions are increasing:
$f(x_2) > f(x_1)$ (1)
$g(x_2) > g(x_1)$ (2)
$yu = f(x) \cdot g(x)$.
(1) $\times$ (2) results in $f(x_2) \cdot g(x_2) > f(x_1)g(x_1)$.
The function yu or $f(x) \cdot g(x)$ is strictly increasing.



14. Let x_1, x_2 be in the interval $a \leq x \leq b$, such that $x_1 < x_2$. Therefore, $f(x_2) > f(x_1)$, and $g(x_2) > g(x_1)$. In this case, $f(x_1), f(x_2), g(x_1)$, and $g(x_2) < 0$. Multiplying an inequality by a negative will reverse its sign.

Therefore, $f(x_2) \cdot g(x_2) < f(x_1) \cdot g(x_1)$.
But LS > 0 and RS > 0.
Therefore, the function fg is strictly decreasing.

4.2 Critical Points, Relative Maxima, and Relative Minima, pp. 178–180

1. Finding the critical points means determining the points on the graph of the function for which the derivative of the function at the x-coordinate is 0.

2. a. Take the derivative of the function. Set the derivative equal to 0. Solve for x. Evaluate the original function for the values of x. The (x, y) pairs are the critical points.

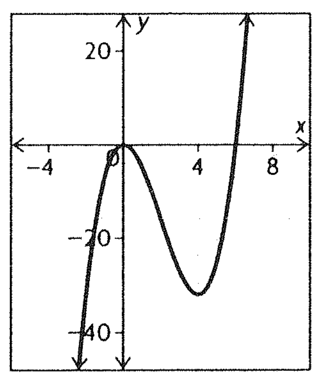
b. $y = x^3 - 6x^2$

$$\frac{dy}{dx} = 3x^2 - 12x$$
$$= 3x(x - 4)$$

Let $\frac{dy}{dx} = 0$.

$3x(x - 4) = 0$
$x = 0, 4$

The critical points are $(0, 0)$ and $(4, -32)$.



3. a. $y = x^4 - 8x^2$

$$\frac{dy}{dx} = 4x^3 - 16x = 4x(x^2 - 4)$$
$$= 4x(x + 2)(x - 2)$$

Let $\frac{dy}{dx} = 0$

$4x(x + 2)(x - 2) = 0$
$x = 0, \pm 2.$

The critical points are $(0, 0)$, $(-2, 16)$, and $(2, -16)$.

x	$x < -2$	-2	$-2 < x < 0$	0	$0 < x < 2$	2	$x < 2$
$\frac{dy}{dx}$	$-$	0	$+$	0	$-$	0	$+$
Graph	Dec.	Local Min	Inc.	Local Max	Dec.	Local Min	Inc.

Local minima at $(-2, -16)$ and $(2, -16)$
Local maximum at $(0, 0)$

b. $f(x) = \dfrac{2x}{x^2 + 9}$

$$f'(x) = \frac{2(x^2 + 9) - 2x(2x)}{(x^2 + 9)^2}$$
$$= \frac{18 - 2x^2}{(x^2 + 9)^2}$$

Let $f'(x) = 0$
Therefore, $18 - 2x^2 = 0$
$x^2 = 9$
$x = \pm 3.$

x	$x < -3$	-3	$-3 < x < 3$	3	$x > 3$
f'(x)	−	0	+	0	−
Graph	Decreasing	Local Min	Increasing	Local Max	Decreasing

Local minimum at $(-3, -0.3)$ and local maximum at $(3, 0.3)$.

c. $y = x^3 + 3x^2 + 1$

$$\frac{dy}{dx} = 3x^2 + 6x = 3x(x + 2)$$

Let $\dfrac{dy}{dx} = 0$

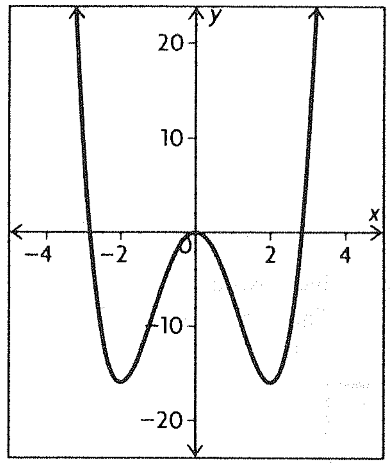
$3x(x + 2) = 0$
$x = 0, -2$

The critical points are $(0, 1)$ and $(-2, 5)$.

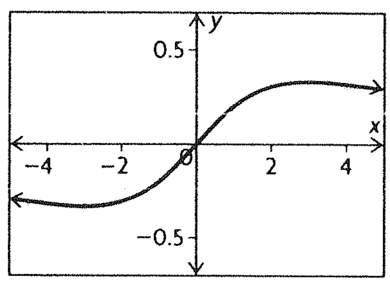
x	$x < -2$	-2	$-2 < x < 0$	0	$x < 0$
$\frac{dy}{dx}$	+	0	−	0	+
Graph	Inc.	Local Min		Local Max	Inc.

Local maximum at $(-2, 5)$
Local minimum at $(0, 1)$

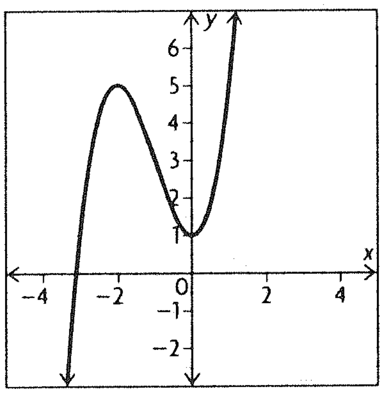
4. a. $y = x^4 - 8x^2$
To find the x-intercepts, let $y = 0$.
$x^4 - 8x^2 = 0$
$x^2(x^2 - 8) = 0$
$x = 0, \pm\sqrt{8}$
To find the y-intercepts, let $x = 0$.
$y = 0$



b. $f(x) = \dfrac{2x}{x^2 + 9}$
To find the x-intercepts, let $y = 0$.
$$\frac{2x}{x^2 + 9} = 0$$
Therefore, $2x = 0$
$x = 0$
To find the y-intercepts, let $x = 0$.
$$y = \frac{0}{9} = 0$$



c. $y = x^3 + 3x^2 + 1$
To find the x-intercepts, let $y = 0$.
$0 = x^3 + 3x^2 + 1$
The x-intercept cannot be easily obtained algebraically. Since the function has a local maximum when $x = -2$, it must have an x-intercept prior to this x-value. Since $f(-3) = 1$ and $f(-4) = -15$, an estimate for the x-intercept is about -3.1.
To find the y-intercepts, let $x = 0$.
$y = 1$



5. a. $h(x) = -6x^3 + 18x^2 + 3$
$h'(x) = -18x^2 + 36x$
Let $h'(x) = 0$:
$-18x^2 + 36x = 0$
$18x(2 - x) = 0$
$x = 0$ or $x = 2$
The critical points are (0, 3) and (2, 27).
Local minimum at (0, 3)
Local maximum at (2, 27)
Since the derivative is 0 at both points, the tangent is parallel to the horizontal axis for both.

b. $g(t) = t^5 + t^3$
$g'(t) = 5t^4 + 3t^2$
Let $g'(t) = 0$:
$5t^4 + 3t^2 = 0$
$t^2(5t^2 + 3) = 0$
$t = 0$

x	$x < 0$	0	$0 < x < 2$	0	$x > 2$
$\frac{dy}{dx}$	+	0	−	0	+
Graph	Inc.	Local Min	Dec.	Local Max	Inc.

The critical point is (0, 0).

t	$t < 0$	0	$t > 0$
$g'(x)$	+	0	+
Graph	Inc.	Local Min	Inc.

(0, 0) is neither a maximum nor a minimum
Since the derivative at (0, 0) is 0, the tangent is parallel to the horizontal axis there.

c. $y = (x - 5)^{\frac{1}{3}}$

$$\frac{dy}{dx} = \frac{1}{3}(x - 5)^{\frac{-2}{3}}$$

$$= \frac{1}{3(x - 5)^{\frac{2}{3}}}$$

$$\frac{dy}{dx} \neq 0$$

The critical point is at (5, 0), but is neither a maximum or minimum. The tangent is not parallel to the x-axis.

d. $f(x) = (x^2 - 1)^{\frac{1}{3}}$

$$f'(x) = \frac{1}{3}(x^2 - 1)^{-\frac{2}{3}}(2x)$$

Let $f'(x) = 0$:

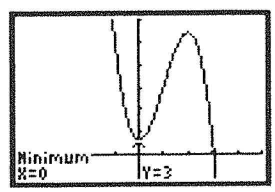
$$\frac{1}{3}(x^2 - 1)^{-\frac{2}{3}}(2x) = 0$$

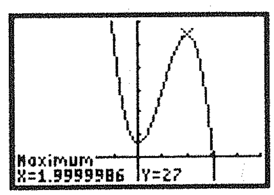
$$x = 0$$

There is a critical point at (0, −1). Since the derivative is undefined for $x = \pm 1$, (1, 0) and (−1, 0) are also critical points.

x	$x < -1$	−1	$-1 < x < 0$	0	$0 < x < 1$	0	$x < 1$
$\frac{dy}{dx}$	−	DNE	−	0	+	DNE	+
Graph	Dec.		Dec.	Local Min	Inc.		Inc.

Local minimum at (0, −1)
The tangent is parallel to the horizontal axis at (0, −1) because the derivative is 0 there. Since the derivative is undefined at (−1, 0) and (1, 0), the tangent is not parallel to the horizontal axis at either point.

6. a.


b.


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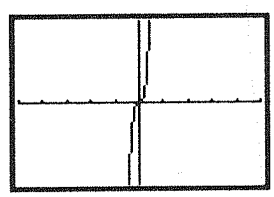
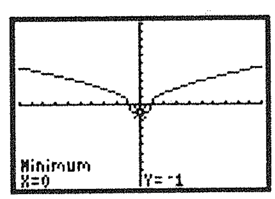
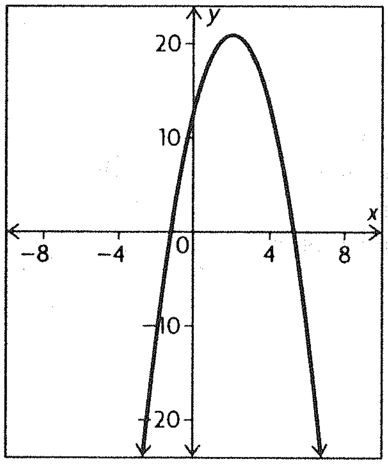


d.


7. a. $f(x) = -2x^2 + 8x + 13$
$f'(x) = -4x + 8$
Let $f'(x) = 0$:
$-4x + 8 = 0$
$x = 2$

The critical point is (2, 21).
Local maximum at (2, 21)

x	$x < 2$	2	$x > 2$
$f'(x)$	+	0	−
Graph	Inc.	Local Max.	Dec.



b. $f(x) = \frac{1}{3}x^3 - 9x + 2$

$f'(x) = x^2 - 9$

Let $f'(x) = 0$:

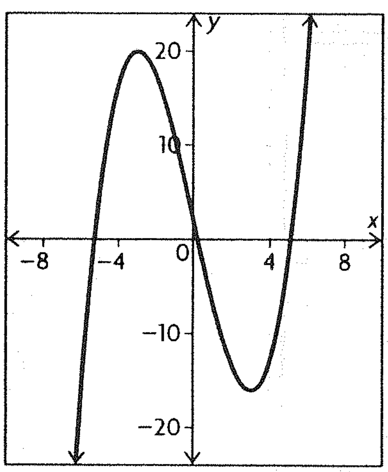
$x^2 - 9 = 0$

$x^2 = 9$

$x = \pm 3$

The critical points are $(-3, 20)$ and $(3, -16)$
Local maximum at $(-3, 20)$
Local minimum at $(3, -16)$

x	$x < -3$	−3	$-3 < x < 3$	3	$x > 3$
$\frac{dy}{dx}$	+	0	−	0	+
Graph	Inc.	Local Max	Dec.	Local Min	Inc.



c. $f(x) = 2x^3 + 9x^2 + 12x$

$f'(x) = 6x^2 + 18x + 12$

Let $f'(x) = 0$:

$6x^2 + 18x + 12 = 0$

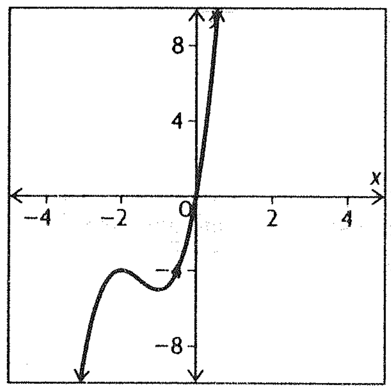
$6(x + 2)(x + 1) = 0$

$x = -2$ or $x = -1$

The critical points are $(-2, -4)$ and $(-1, -5)$.

x	$x < -2$	−2	$-2 < x < -1$	−1	$x > -1$
$\frac{dy}{dx}$	+	0	−	0	+
Graph	Inc.	Local Max	Dec.	Local Min	Inc.

Local maximum at $(-2, -4)$
Local minimum at $(-1, -5)$



d. $f(x) = -3x^3 - 5x$

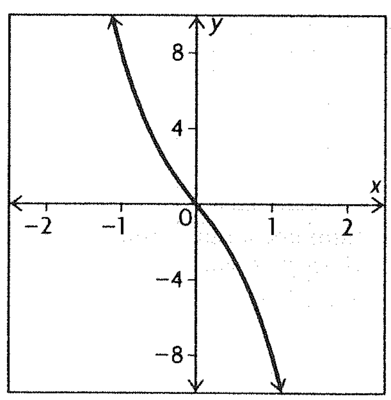
$f'(x) = -9x^2 - 5$

Let $f'(x) = 0$:

$-9x^2 - 5 = 0$

$x^2 = -\frac{5}{9}$

This equation has no solution, so there are no critical points.



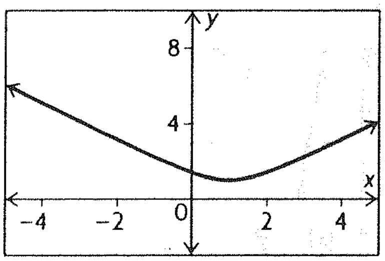
e. $f(x) = \sqrt{x^2 - 2x + 2}$

$f'(x) = \frac{2x - 2}{2\sqrt{x^2 - 2x + 2}} = \frac{x - 1}{\sqrt{x^2 - 2x + 2}}$

Let $f'(x) = 0$:
Therefore, $x - 1 = 0$
$x = 1$
The critical point is $(1, 1)$.
$\sqrt{x^2 - 2x + 2}$ is never undefined or equal to zero, so $(1, 1)$ is the only critical point.

x	$x < 1$	1	$x > 1$
$f'(x)$	−	0	+
Graph	Dec.	Local Min	Inc.

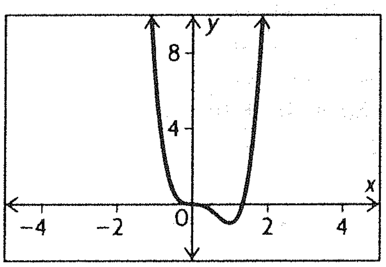
Local minimum at $(1, 1)$



f. $f(x) = 3x^4 - 4x^3$
$f'(x) = 12x^3 - 12x^2$
Let $f'(x) = 0$:
$12x^3 - 12x^2 = 0$
$12x^2(x - 1) = 0$
$x = 0$ or $x = 1$

x	$x < 0$	0	$0 < x < 1$	1	$x > 1$
$\frac{dy}{dx}$	−	0	−	0	+
Graph	Dec.		Dec.	Local Min	Inc.

There are critical points at $(0, 0)$ and $(1, -1)$.
Neither local minimum nor local maximum at $(0, 0)$
Local minimum at $(1, -1)$

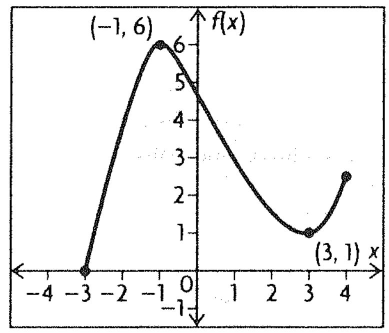


8. $f'(x) = (x + 1)(x - 2)(x + 6)$
Let $f'(x) = 0$:
$(x + 1)(x - 2)(x + 6) = 0$
$x = -6$ or $x = -1$ or $x = 2$
The critical numbers are -6, -1, and 2.

x	$x < -6$	−6	$-6 < x < -1$	−1	$-1 < x < 2$	2	$x < 2$
$\frac{dy}{dx}$	−	0	+	0	−	0	+
Graph	Dec.	Local Min	Inc.	Local Max	Dec.	Local Min	Inc.

Local minima at $x = -6$ and $x = 2$
Local maximum at $x = -1$

9.



10. $y = ax^2 + bx + c$
$\frac{dy}{dx} = 2ax + b$
Since a relative maximum occurs at $x = 3$, then $2ax + b = 0$ at $x = 3$. Or, $6a + b = 0$. Also, at $(0, 1)$, $1 = 0 + 0 + c$ or $c = 1$. Therefore, $y = ax^2 + bx + 1$. Since $(3, 12)$ lies on the curve,
$12 = 9a + 3b + 1$
$9a + 3b = 11$
$6a + b = 0$.
Since $b = -6a$,
Then $9a - 18a = 11$
or $a = -\frac{11}{9}$
$b = \frac{22}{3}$.
The equation is $y = -\frac{11}{9}x^2 + \frac{22}{3}x + 1$.

11. $f(x) = x^2 + px + q$
$f'(x) = 2x + p$
In order for 1 to be an extremum, $f'(1)$ must equal 0.
$2(1) + p = 0$
$p = -2$
To find q, substitute the known values for p and x into the original equation and set it equal to 5.

x	$x < 1$	1	$x > 1$
$f'(x)$	−	0	+
Graph	Dec.	Local Min	Inc.

$(1)^2 + (1)(-2) + q = 5$

$q = 6$

This extremum is a minimum value.

12. a. $f(x) = x^3 - kx$

$f'(x) = 3x^2 - k$

In order for f to have no critical numbers, $f'(x) = 0$ must have no solutions. Therefore, $3x^2 = k$ must have no solutions, so $k < 0$.

b. $f(x) = x^3 - kx$

$f'(x) = 3x^2 - k$

In order for f to have one critical numbers, $f'(x) = 0$ must have exactly one solution. Therefore, $3x^2 = k$ must have one solution, which occurs when $k = 0$.

c. $f(x) = x^3 - kx$

$f'(x) = 3x^2 - k$

In order for f to have two critical numbers, $f'(x) = 0$ must have two solutions. Therefore, $3x^2 = k$ must have two solutions, which occurs when $k > 0$.

13. $g(x) = ax^3 + bx^2 + cx + d$

$g'(x) = 3ax^2 + 2bx + c$

Since there are local extrema at $x = 0$ and $x = 2$,

$0a + 0b + c = 0$ and $12a + 4b + c = 0$

Therefore, $c = 0$ and $12a + 4b = 0$

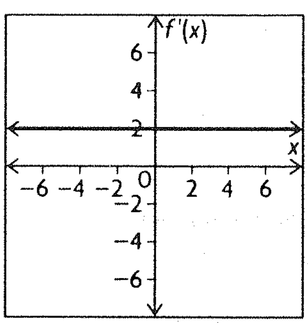
Going back to the original equation, we have the points (2, 4) and (0, 0). Substitute these values of x in the original function to get two more equations: $8a + 4b + 2c + d = 4$ and $d = 0$. We now know that $c = 0$ and $d = 0$. We are left with two equations to find a and b:

$12a + 4b = 0$

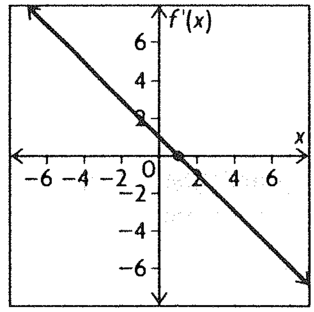
$8a + 4b = 4$

Subtract the second equation from the first to get $4a = -4$. Therefore $a = -1$, and $b = 3$.

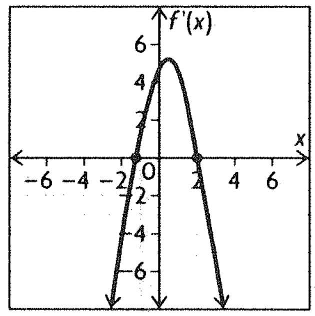
14. a.



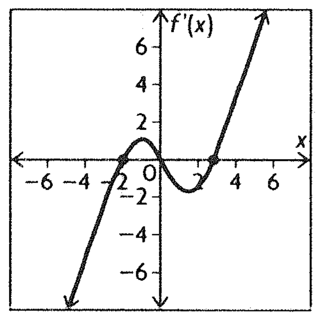
b.



c.



d.



15. $f(x) = 3x^4 + ax^3 + bx^2 + cx + d$

a. $f'(x) = 12x^3 + 3ax^2 + 2bx + c$

At $x = 0, f'(0) = 0$, then $f'(0) = 0 + 0 + 0 + c$ or $c = 0$.

At $x = -2, f'(-2) = 0$,

$-96 + 12a - 4b = 0.$ (1)

Since $(0, -9)$ lies on the curve,

$-9 = 0 + 0 + 0 + 0 + d$ or $d = -9$.

Since $(-2, -73)$ lies on the curve,

$-73 = 48 - 8a + 4b + 0 - 9$

$-8a + 4b = -112$

or $2a - b = 28$ (2)

Also, from (1): $3a - b = 24$

$2a - b = -28$

$a = -4$

$b = -36.$

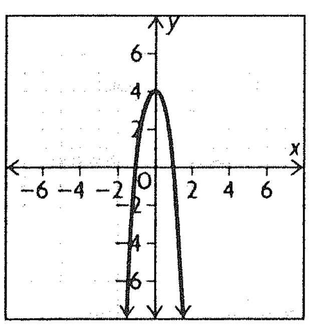
The function is $f(x) = 3x^4 - 4x^3 - 36x^2 - 9$.
b. $f'(x) = 12x^3 - 12x^2 - 72x$
Let $f'(x) = 0$:
$x^3 - x^2 - 6x = 0$
$x(x - 3)(x + 2) = 0$.
Third point occurs at $x = 3$,
$f(3) = -198$.
c.
Local minimum is at $(-2, -73)$ and $(3, -198)$.

x	$x < -2$	-2	$-2 < x < 0$	0	$0 < x < 3$	3	$x > 3$
$f'(x)$	$-$	0	$+$	0	$-$	0	$+$
Graph	Decreasing	Local Min	Increasing	Local Max	Decreasing	Local Min	Increasing

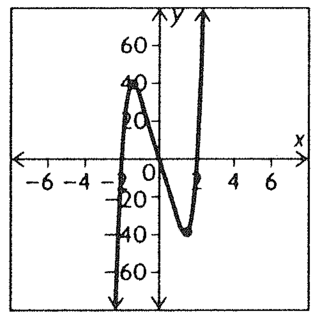
Local maximum is at $(0, -9)$.
16. a. $y = 4 - 3x^2 - x^4$
$$\frac{dy}{dx} = -6x - 4x^3$$
Let $\frac{dy}{dx} = 0$:
$-6x - 4x^3 = 0$
$-2x(2x^2 + 3) = 0$
$x = 0$ or $x^2 = -\frac{3}{2}$; inadmissible

x	$x < 0$	0	$x > 0$
$\frac{dy}{dx}$	$+$	0	$-$
Graph	Increasing	Local Max	Decreasing

Local maximum is at $(0, 4)$.



b. $y = 3x^5 - 5x^3 - 30x$
$$\frac{dy}{dx} = 15x^4 - 15x^2 - 30$$
Let $\frac{dy}{dx} = 0$:
$15x^4 - 15x^2 - 30 = 0$
$x^4 - x^2 - 2 = 0$
$(x^2 - 2)(x^2 + 1) = 0$
$x^2 = 2$ or $x^2 = -1$
$x = \pm\sqrt{2}$; inadmissible
At $x = 100$, $\frac{dy}{dx} > 0$.
Therefore, function is increasing into quadrant one, local minimum is at $(1.41, -39.6)$ and local maximum is at $(-1.41, 39.6)$.



17. $h(x) = \frac{f(x)}{g(x)}$
Since $f(x)$ has local maximum at $x = c$, then $f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$.
Since $g(x)$ has a local minimum at $x = c$, then $g'(x) < 0$ for $x < c$ and $g'(x) > 0$ for $x > c$.
$$h(x) = \frac{f(x)}{g(x)}$$
$$h'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}$$
If $x < c$, $f'(x) > 0$ and $g'(x) < 0$, then $h'(x) > 0$.
If $x > c$, $f'(x) < 0$ and $g'(x) > 0$, then $h'(x) < 0$.
Since for $x < c$, $h'(x) > 0$ and for $x > c$, $h'(x) < 0$.
Therefore, $h(x)$ has a local maximum at $x = c$.

4.3 Vertical and Horizontal Asymptotes, pp. 193–195

1. a. vertical asymptotes at $x = -2$ and $x = 2$; horizontal asymptote at $y = 1$
b. vertical asymptote at $x = 0$; horizontal asymptote at $y = 0$
2. $f(x) = \frac{g(x)}{h(x)}$
Conditions for a vertical asymptote: $h(x) = 0$ must have at least one solution s, and $\lim_{x \to \infty} f(x) = \infty$.
Conditions for a horizontal asymptote: $\lim_{x \to \infty} f(x) = k$, where $k \in \mathbf{R}$,
or $\lim_{x \to -\infty} f(x) = k$ where $k \in \mathbf{R}$.

Condition for an oblique asymptote is that the highest power of $g(x)$ must be one more than the highest power of $h(x)$.

3. a. $\lim_{x\to\infty}\frac{2x+3}{x-1}=\lim_{x\to\infty}\frac{x\left(2+\frac{3}{x}\right)}{x\left(x-\frac{1}{x}\right)}$

$=\lim_{x\to\infty}\frac{1+\frac{3}{2x}}{1-\frac{1}{x}}$

$=\frac{\lim_{x\to\infty}\left(2+\frac{3}{x}\right)}{\lim_{x\to\infty}\left(1-\frac{1}{x}\right)}$

$=\frac{2+0}{1-0}$

$=2$

Similarly, $\lim_{x\to-\infty}\frac{2x+3}{x-1}=2.$

b. $\lim_{x\to\infty}\frac{5x^2-3}{x^2+2}=\lim_{x\to\infty}\frac{x^2\left(5-\frac{3}{x^2}\right)}{x^2\left(1+\frac{2}{x^2}\right)}$

$=\lim_{x\to\infty}\frac{5-\frac{3}{x^2}}{1+\frac{2}{x^2}}$

$=\frac{\lim_{x\to\infty}\left(5-\frac{3}{x^2}\right)}{\lim_{x\to\infty}\left(1+\frac{2}{x^2}\right)}$

$=\frac{5-0}{1+0}$

$=5$

Similarly, $\lim_{x\to-\infty}\frac{5x^2-3}{x^2+2}=5.$

c. $\lim_{x\to\infty}\frac{-5x^2+3x}{2x^2-5}=\lim_{x\to\infty}\frac{x^2\left(-5+\frac{3}{x}\right)}{x^2\left(2-\frac{5}{x^2}\right)}$

$=\lim_{x\to\infty}\frac{-5+\frac{3}{x}}{2-\frac{5}{x^2}}$

$=\frac{\lim_{x\to\infty}\left(-5+\frac{3}{x}\right)}{\lim_{x\to\infty}\left(2-\frac{5}{x^2}\right)}$

$=\frac{-5+0}{2-0}$

$=-\frac{5}{2}$

Similarly, $\lim_{x\to-\infty}\frac{-5x^2+3x}{2x^2-5}=-\frac{5}{2}.$

d. $\lim_{x\to\infty}\frac{2x^5-3x^2+5}{3x^4+5x-4}=\lim_{x\to\infty}\frac{x^5\left(2-\frac{3}{x^3}+\frac{5}{x^5}\right)}{x^4\left(3+\frac{5}{x^3}-\frac{4}{x^4}\right)}$

$=\lim_{x\to\infty}\frac{x\left(2-\frac{3}{x^3}+\frac{5}{x^5}\right)}{3+\frac{5}{x^3}-\frac{4}{x^4}}$

$=\frac{\lim_{x\to\infty}\left(x\left(2-\frac{3}{x^3}+\frac{5}{x^5}\right)\right)}{\lim_{x\to\infty}\left(3+\frac{5}{x^3}-\frac{4}{x^4}\right)}$

$=\frac{\lim_{x\to\infty}(x)\times\lim_{x\to\infty}\left(2-\frac{3}{x^3}+\frac{5}{x^5}\right)}{\lim_{x\to\infty}\left(3+\frac{5}{x^3}-\frac{4}{x^4}\right)}$

$=\lim_{x\to\infty}(x)\times\frac{2-0+0}{3+0-0}$

$=\infty$

Similarly, $\lim_{x\to-\infty}\frac{2x^5-3x^2+5}{3x^4+5x-4}=\lim_{x\to\infty}(x)=-\infty.$

4. a. This function is discontinuous at $x=-5$. The numerator is non-zero at this point, so the function has a vertical asymptote there. The behaviour of the function near the asymptote is:

x-values	x	$x+5$	y	$\lim_{x\to c} y$
$x\to-5^-$	<0	<0	>0	$+\infty$
$x\to-5^+$	<0	>0	<0	$-\infty$

b. This function is discontinuous at $x=2$. The numerator is non-zero at this point, so the function has a vertical asymptote there. The behaviour of the function near the asymptote is:

x-values	$x+2$	$x-2$	$f(x)$	$\lim_{x\to c} f(x)$
$x\to2^-$	<0	<0	<0	$-\infty$
$x\to2^+$	>0	>0	>0	$+\infty$

c. This function is discontinuous at $t=3$. The numerator is non-zero at this point, so the function has a vertical asymptote there. The behaviour of the function near the asymptote is:

t-values	1	$(t-3)^2$	s	$\lim_{t\to c} s$
$x\to3^-$	>0	>0	>0	$+\infty$
$x\to3^+$	>0	>0	>0	$+\infty$

d. This function is discontinuous at $x = 3$. However, the numerator also has value 0 there, since $3^2 - 3 - 6 = 0$, so this function has no vertical asymptotes.

e. The denominator of the function has value 0 when

$(x + 3)(x - 1) = 0$

$x = -3$ or $x = 1$. The numerator is non-zero at these points, so the function has vertical asymptotes there. The behaviour of the function near the asymptotes is:

x-values	6	x + 3	x − 1	f(x)	$\lim_{x\to c} f(x)$
$x \to -3^-$	> 0	< 0	< 0	> 0	$+\infty$
$x \to -3^+$	> 0	> 0	< 0	< 0	$-\infty$
$x \to 1^-$	> 0	> 0	< 0	< 0	$-\infty$
$x \to 1^+$	> 0	> 0	> 0	> 0	$+\infty$

f. This function is discontinuous when

$x^2 - 1 = 0$

$(x + 1)(x - 1) = 0$

$x = -1$ or $x = 1$. The numerator is non-zero at these points, so the function has vertical asymptotes there. The behaviour of the function near the asymptotes is:

x-values	x^2	x + 1	x − 1	y	$\lim_{x\to c} y$
$x \to -1^-$	> 0	< 0	< 0	> 0	$+\infty$
$x \to -1^+$	> 0	> 0	< 0	< 0	$-\infty$
$x \to 1^-$	> 0	> 0	< 0	< 0	$-\infty$
$x \to 1^+$	> 0	> 0	> 0	> 0	$+\infty$

5. a. $\lim_{x\to\infty} \frac{x}{x + 4} = \lim_{x\to\infty} \frac{x}{x\left(1 + \frac{4}{x}\right)}$

$$= \lim_{x\to\infty} \frac{1}{1 + \frac{4}{x}}$$

$$= \frac{\lim_{x\to\infty}(1)}{\lim_{x\to\infty}\left(1 + \frac{4}{x}\right)}$$

$$= \frac{1}{1 + 0}$$

$$= 1$$

Similarly, $\lim_{x\to-\infty} \frac{x}{x + 4} = 1$, so $y = 1$ is a horizontal asymptote of the function.

At a point x, the difference between the function $y = \frac{x}{x + 4}$ and its asymptote $y = 1$ is

$$\frac{x}{x + 4} - 1 = \frac{x - (x + 4)}{x + 4}$$

$$= -\frac{4}{x + 4}.$$

When x is large and positive, this difference is negative, which means that the curve approaches the asymptote from below. When x is large and negative, this difference is positive, which means that the curve approaches the asymptote from above.

b. $\lim_{x\to\infty} \frac{2x}{x^2 - 1} = \lim_{x\to\infty} \frac{2x}{x^2\left(1 - \frac{1}{x^2}\right)}$

$$= \lim_{x\to\infty} \frac{2}{x\left(1 - \frac{1}{x^2}\right)}$$

$$= \frac{\lim_{x\to\infty}(2)}{\lim_{x\to\infty} x\left(1 - \frac{1}{x^2}\right)}$$

$$= \frac{\lim_{x\to\infty}(2)}{\lim_{x\to\infty}(x) \times \lim_{x\to\infty}\left(1 - \frac{1}{x^2}\right)}$$

$$= \lim_{x\to\infty} \frac{1}{x} \times \frac{1}{1 - 0}$$

$$= 0$$

Similarly, $\lim_{x\to-\infty} \frac{2x}{x^2 - 1} = 0$, so $y = 0$ is a horizontal asymptote of the function.

At a point x, the difference between the function $f(x) = \frac{2x}{x^2 - 1}$ and its asymptote $y = 0$ is $\frac{2x}{x^2 - 1}$.

When x is large and positive, this difference is positive, which means that the curve approaches the asymptote from above. When x is large and negative, this difference is negative, which means that the curve approaches the asymptote from below.

c. $\lim_{t\to\infty} \frac{3t^2 + 4}{t^2 - 1} = \lim_{t\to\infty} \frac{t^2\left(3 + \frac{4}{t^2}\right)}{t^2\left(1 - \frac{1}{t^2}\right)}$

$$= \lim_{t\to\infty} \frac{3 + \frac{4}{t^2}}{1 - \frac{1}{t^2}}$$

$$= \frac{\lim_{t\to\infty}\left(3 + \frac{4}{t^2}\right)}{\lim_{t\to\infty}\left(1 - \frac{1}{t^2}\right)}$$

$$= \frac{3+0}{1-0}$$
$$= 3$$

Similarly, $\lim_{x\to-\infty} \frac{3t^2+4}{t^2-1} = 3$, so $y = 3$ is a horizontal asymptote of the function.

At a point x, the difference between the function $g(t) = \frac{3t^2+4}{t^2-1}$ and its asymptote $y = 3$ is

$$\frac{3t^2+4}{t^2-1} - 3 = \frac{3t^2+4-3(t^2-1)}{t^2-1}$$
$$= \frac{7}{t^2-1}.$$

When x is large and positive, this difference is positive, which means that the curve approaches the asymptote from above. When x is large and negative, this difference is positive, which means that the curve approaches the asymptote from above.

d. $$\lim_{x\to\infty} \frac{3x^2-8x-7}{x-4} = \lim_{x\to\infty} \frac{x^2\left(3-\frac{8}{x}-\frac{7}{x^2}\right)}{x\left(1-\frac{4}{x}\right)}$$
$$= \lim_{x\to\infty} \frac{x\left(3-\frac{8}{x}-\frac{7}{x^2}\right)}{1-\frac{4}{x}}$$
$$= \frac{\lim_{x\to\infty}\left(x\left(3-\frac{8}{x}-\frac{7}{x^2}\right)\right)}{\lim_{x\to\infty}\left(1-\frac{4}{x}\right)}$$
$$= \frac{\lim_{x\to\infty}(x) \times \lim_{x\to\infty}\left(3-\frac{8}{x}-\frac{7}{x^2}\right)}{\lim_{x\to\infty}\left(1-\frac{4}{x}\right)}$$
$$= \lim_{x\to\infty}(x) \times \frac{3-0-0}{1-0}$$
$$= \infty$$

Similarly, $\lim_{x\to\infty} \frac{3x^2-8x-7}{x-4} = \lim_{x\to\infty}(x) = -\infty$, so this function has no horizontal asymptotes.

6. a. This function is discontinuous at $x = -5$. Since the numerator is not equal to 0 there, the function has a vertical asymptote at this point. The behaviour of the function near the asymptote is:

x-values	$x - 3$	$x + 5$	y	$\lim_{x\to c} y$
$x \to -5^-$	< 0	< 0	> 0	$+\infty$
$x \to -5^+$	< 0	> 0	< 0	$-\infty$

To check for a horizontal asymptote:

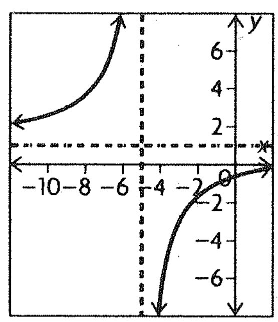
$$\lim_{x\to\infty} \frac{x-3}{x+5} = \lim_{x\to\infty} \frac{x\left(1-\frac{3}{x}\right)}{x\left(1+\frac{5}{x}\right)}$$
$$= \lim_{x\to\infty} \frac{1-\frac{3}{x}}{1+\frac{5}{x}}$$
$$= \frac{\lim_{x\to\infty}\left(1-\frac{3}{x}\right)}{\lim_{x\to\infty}\left(1+\frac{5}{x}\right)}$$
$$= \frac{1-0}{1+0}$$
$$= 1$$

Similarly, $\lim_{x\to-\infty} \frac{x-3}{x+5} = 1$, so $y = 1$ is a horizontal asymptote of the function.

At a point x, the difference between the function $y = \frac{x-3}{x+5}$ and its asymptote $y = 1$ is

$$\frac{x-3}{x+5} - 1 = \frac{x-3-(x+5)}{x+5} = -\frac{8}{x+5}.$$

When x is large and positive, this difference is negative, which means that the curve approaches the asymptote from below. When x is large and negative, this difference is positive, which means that the curve approaches the asymptote from above.



b. This function is discontinuous at $x = -2$. Since the numerator is non-zero there, the function has a vertical asymptote at this point. The behaviour of the function near the asymptote is:

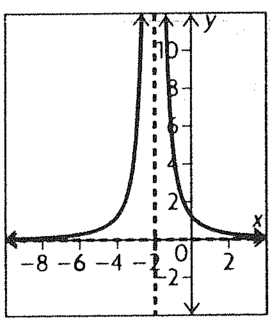
x-values	5	$(x+2)^2$	$f(x)$	$\lim_{x\to c} f(x)$
$x \to 2^-$	> 0	> 0	> 0	$+\infty$
$x \to 2^+$	> 0	> 0	> 0	$+\infty$

To check for a horizontal asymptote:

$$\lim_{x\to\infty}\frac{5}{(x+2)^2}=\lim_{x\to\infty}\frac{5}{x^2+4x+4}$$
$$=\lim_{x\to\infty}\frac{5}{x^2\left(1+\frac{4}{x}+\frac{4}{x^2}\right)}$$
$$=\frac{\lim_{x\to\infty}(5)}{\lim_{x\to\infty}\left(x^2\left(1+\frac{4}{x}+\frac{4}{x^2}\right)\right)}$$
$$=\frac{\lim_{x\to\infty}(5)}{\lim_{x\to\infty}(x^2)\times\lim_{x\to\infty}\left(1+\frac{4}{x}+\frac{4}{x^2}\right)}$$
$$=\lim_{x\to\infty}\frac{1}{x^2}\times\frac{5}{1+0+0}$$
$$=0$$

Similarly, $\lim_{x\to-\infty}\frac{5}{(x+2)^2}=0$, so $y=0$ is a horizontal asymptote of the function.

At a point x, the difference between the function $f(x)=\frac{5}{(x+2)^2}$ and its asymptote $y=0$ is $\frac{5}{(x+2)^2}$. When x is large and positive, this difference is positive, which means that the curve approaches the asymptote from above. When x is large and negative, this difference is positive, which means that the curve approaches the asymptote from above.

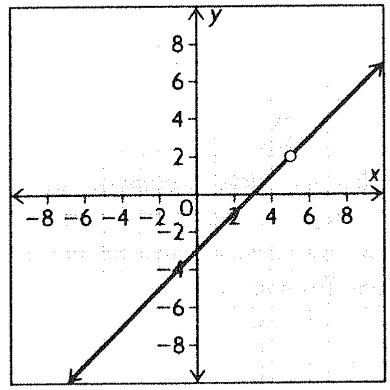


c. This function is discontinuous at $t=5$. However, the numerator is equal to zero there, since $5^2-2(5)-15=0$, so this function has no vertical asymptote.

To check for an oblique asymptote:

$$\begin{array}{r} t-3 \\ t-5\,\overline{)\,t^2-2t-15} \\ t^2-5t \\ 0+3t-15 \\ \underline{0+3t-15} \\ 0+0+0 \end{array}$$

So $g(t)$ can be written in the form $g(t)=t-3$



d. This function is discontinuous when

$x^2-3x=0$

$x(x-3)=0$

$x=0$ or $x=3$

The numerator is non-zero at these points, so the function has vertical asymptotes at both of them. The behaviour of the function near these asymptotes is:

x-values	**2 + x**	**3 − 2x**	**x**	**x − 3**	**y**	$\lim_{x\to c} y$
$x\to0^-$	>0	>0	<0	<0	>0	$+\infty$
$x\to0^+$	>0	>0	>0	<0	<0	$-\infty$
$x\to3^-$	>0	<0	>0	<0	>0	$+\infty$
$x\to3^+$	>0	<0	>0	>0	<0	$-\infty$

To check for horizontal asymptotes:

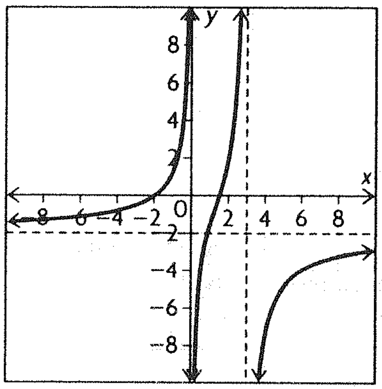
$$\lim_{x\to\infty}\frac{(2+x)(3-2x)}{x^2-3x}=\lim_{x\to\infty}\frac{-2x^2-x+6}{x^2-3x}$$
$$=\lim_{x\to\infty}\frac{x^2\left(-2-\frac{1}{x}+\frac{6}{x^2}\right)}{x^2\left(1-\frac{3}{x}\right)}$$
$$=\lim_{x\to\infty}\frac{-2-\frac{1}{x}+\frac{6}{x^2}}{1-\frac{3}{x}}$$
$$=\frac{\lim_{x\to\infty}\left(-2-\frac{1}{x}+\frac{6}{x^2}\right)}{\lim_{x\to\infty}\left(1-\frac{3}{x}\right)}$$
$$=\frac{-2-0+0}{1-0}$$
$$=-2$$

Similarly, $\lim_{x\to-\infty}\frac{(2+x)(3-2x)}{x^2-3x}=-2$, so $y=-2$ is a horizontal asymptote of the function.

At a point x, the difference between the function $y = \frac{-2x^2 - x + 6}{x^2 - 3x}$ and its asymptote $y = -2$ is

$$\frac{-2x^2 - x + 6}{x^2 - 3x} + 2 = \frac{-2x^2 - x + 6 + 2(x^2 - 3x)}{x^2 - 3x}$$
$$= \frac{-7x + 6}{x^2 - 3x}.$$

When x is large and positive, this difference is negative, which means that the curve approaches the asymptote from below. When x is large and negative, this difference is positive, which means that the curve approaches the asymptote from above.



7. a.

$$\begin{array}{r} 3x - 7 \\ x - 3 \overline{)3x^2 - 2x - 17} \\ \underline{3x^2 - 9x} \\ 7x - 17 \\ \underline{7x - 21} \\ 4 \end{array}$$

So $f(x)$ can be written in the form $f(x) = 3x - 7 + \frac{4}{x - 3}$. Since $\lim\limits_{x \to \infty} \frac{4}{x - 3} = 0$ and $\lim\limits_{x \to -\infty} \frac{4}{x - 3} = 0$, the line $y = 3x - 7$ is an asymptote to the function $f(x)$.

b.

$$\begin{array}{r} x + 3 \\ 2x + 3 \overline{)2x^2 + 9x + 2} \\ \underline{2x^2 + 3x} \\ 6x + 2 \\ \underline{6x + 9} \\ -7 \end{array}$$

So $f(x)$ can be written in the form $f(x) = x + 3 - \frac{7}{2x + 3}$. Since $\lim\limits_{x \to \infty} \frac{7}{2x + 3} = 0$ and $\lim\limits_{x \to -\infty} \frac{7}{2x + 3} = 0$, the line $y = x + 3$ is an asymptote to the function $f(x)$.

c.

$$\begin{array}{r} x - 2 \\ x^2 + 2x \overline{)x^3 + 0x^2 + 0x - 1} \\ \underline{x^3 + 2x^2} \\ -2x^2 + 0x - 1 \\ \underline{-2x^2 - 4x} \\ 4x - 1 \end{array}$$

So $f(x)$ can be written in the form $f(x) = x - 2 + \frac{4x - 1}{x^2 + 2x}$. Since

$$\lim_{x \to \infty} \frac{4x - 1}{x^2 + 2x} = \lim_{x \to \infty} \frac{x\left(4 - \frac{1}{x}\right)}{x^2\left(1 + \frac{2}{x}\right)}$$
$$= \lim_{x \to \infty} \frac{4 - \frac{1}{x}}{x\left(1 + \frac{2}{x}\right)}$$
$$= \frac{\lim\limits_{x \to \infty}\left(4 - \frac{1}{x}\right)}{\lim\limits_{x \to \infty}\left(x\left(1 + \frac{2}{x}\right)\right)}$$
$$= \frac{\lim\limits_{x \to \infty}\left(4 - \frac{1}{x}\right)}{\lim\limits_{x \to \infty}(x) \times \lim\limits_{x \to \infty}\left(1 + \frac{2}{x}\right)}$$
$$= \lim_{x \to \infty}\left(\frac{1}{x}\right) \times \frac{4 - 0}{1 + 0}$$
$$= 0,$$

and similarly $\lim\limits_{x \to -\infty} \frac{4x - 1}{x^2 + 2x} = 0$, the line $y = x - 2$ is an asymptote to the function $f(x)$.

d.

$$\begin{array}{r} x + 3 \\ x^2 - 4x + 3 \overline{)x^3 - x^2 - 9x + 15} \\ \underline{x^3 - 4x^2 + 3x} \\ 3x^2 - 12x + 15 \\ \underline{3x^2 - 12x + 9} \\ 6 \end{array}$$

So $f(x)$ can be written in the form $f(x) = x + 3 + \frac{6}{x^2 - 4x + 3}$. Since $\lim\limits_{x \to \infty} \frac{6}{x^2 - 4x + 3}$ and $\lim\limits_{x \to -\infty} \frac{6}{x^2 - 4x + 3} = 0$, the line $y = x + 3$ is an asymptote to the function $f(x)$.

8. a. At a point x, the difference between the function $f(x) = f(x) = 3x - 7 + \frac{4}{x - 3}$ and its oblique asymptote $y = 3x - 7$ is $3x - 7 + \frac{4}{x - 3} - (3x - 7) = \frac{4}{x - 3}$. When x is large and positive, this difference is positive, which means that the curve approaches the asymptote from above. When x is large and negative, this difference is negative, which means that the curve approaches the asymptote from below.

b. At a point x, the difference between the function $f(x) = x + 3 - \frac{7}{2x + 3}$ and its oblique asymptote $y = x + 3$ is $x + 3 - \frac{7}{2x + 3} - (x + 3) = -\frac{7}{2x + 3}$. When x is large and positive, this difference is negative, which means that the curve approaches the asymptote from below. When x is large and negative, this difference is positive, which means that the curve approaches the asymptote from above.

9. a. This function is discontinuous at $x = -5$. The numerator is non-zero at this point, so the function has a vertical asymptote there. The behaviour of the function near the asymptote is:

x-values	$3x - 1$	$x + 5$	$f(x)$	$\lim_{x \to c} f(x)$
$x \to -5^-$	< 0	< 0	> 0	$+\infty$
$x \to -5^+$	< 0	> 0	< 0	$-\infty$

To check for a horizontal asymptote:

$$\lim_{x \to \infty} \frac{3x - 1}{x + 5} = \lim_{x \to \infty} \frac{x\left(3 - \frac{1}{x}\right)}{x\left(1 + \frac{5}{x}\right)}$$

$$= \lim_{x \to \infty} \frac{3 - \frac{1}{x}}{1 + \frac{5}{x}}$$

$$= \frac{\lim_{x \to \infty}\left(3 - \frac{1}{x}\right)}{\lim_{x \to \infty}\left(1 + \frac{5}{x}\right)}$$

$$= \frac{3 - 0}{1 + 0}$$

$$= 1$$

Similarly, $\lim_{x \to \infty} \frac{3x - 1}{x + 5} = 3$, so $y = 3$ is a horizontal asymptote of the function.

b. This function is discontinuous at $x = 1$. The numerator is non-zero at this point, so the function has a vertical asymptote there. The behaviour of the function near the asymptote is:

To check for a horizontal asymptote:

x-values	$x^2 + 3x - 2$	$(x - 1)^2$	$g(x)$	$\lim_{x \to c} g(x)$
$x \to 1^-$	> 0	> 0	> 0	$+\infty$
$x \to 1^+$	> 0	> 0	> 0	$+\infty$

$$\lim_{x \to \infty} \frac{x^2 + 3x - 2}{(x - 1)^2} = \lim_{x \to \infty} \frac{x^2 + 3x - 2}{x^2 - 2x + 1}$$

$$= \lim_{x \to \infty} \frac{x^2\left(1 + \frac{3}{x} - \frac{2}{x^2}\right)}{x^2\left(1 - \frac{2}{x} + \frac{1}{x^2}\right)}$$

$$= \lim_{x \to \infty} \frac{1 + \frac{3}{x} - \frac{2}{x^2}}{1 - \frac{2}{x} + \frac{1}{x^2}}$$

$$= \frac{\lim_{x \to \infty}\left(1 + \frac{3}{x} - \frac{2}{x^2}\right)}{\lim_{x \to \infty}\left(1 - \frac{2}{x} + \frac{1}{x^2}\right)}$$

$$= \frac{1 + 0 - 0}{1 - 0 + 0}$$

$$= 1$$

Similarly, $\lim_{x \to \infty} \frac{x^2 + 3x - 2}{(x - 1)^2} = 1$, so $y = 1$ is a horizontal asymptote of the function.

c. This function is discontinuous when

$x^2 - 4 = 0$

$x^2 = 4$

$x = \pm 2.$

At $x = 2$ the numerator is 0, since $2^2 + 2 - 6 = 0$, so the function has no vertical asymptote there. At $x = -2$, however, the numerator is non-zero, so the function has a vertical asymptote there. The behaviour of the function near the asymptote is:

x-values	$x^2 + x - 6$	$x^2 - 4$	$h(x)$	$\lim_{x \to c} h(x)$
$x \to -2^-$	< 0	> 0	< 0	$-\infty$
$x \to -2^+$	< 0	< 0	> 0	$+\infty$

To check for a horizontal asymptote:

$$\lim_{x \to \infty} \frac{x^2 + x - 6}{x^2 - 4} = \lim_{x \to \infty} \frac{x^2\left(1 + \frac{1}{x} - \frac{6}{x^2}\right)}{x^2\left(1 - \frac{4}{x^2}\right)}$$

$$= \lim_{x \to \infty} \frac{1 + \frac{1}{x} - \frac{6}{x^2}}{1 - \frac{4}{x^2}}$$

$$= \frac{\lim_{x \to \infty}\left(1 + \frac{1}{x} - \frac{6}{x^2}\right)}{\lim_{x \to \infty}\left(1 - \frac{4}{x^2}\right)}$$

$$= \frac{1 + 0 - 0}{1 - 0}$$
$$= 1$$

Similarly, $\lim\limits_{x \to -\infty} \frac{x^2 + x - 6}{x^2 - 4} = 1$, so $y = 1$ is a horizontal asymptote of the function.

d. This function is discontinuous at $x = 2$. The numerator is non-zero at this point, so the function has a vertical asymptote there. The behaviour of the function near the asymptote is:

x-values	$5x^2 - 3x + 2$	$x - 2$	$m(x)$	$\lim\limits_{x \to c} m(x)$
$x \to 2^-$	> 0	< 0	< 0	$-\infty$
$x \to 2^+$	> 0	> 0	> 0	$+\infty$

To check for a horizontal asymptote:

$$\lim_{x \to \infty} \frac{x^2 + 3x - 2}{(x - 1)^2} = \lim_{x \to \infty} \frac{x^2 + 3x - 2}{x^2 - 2x + 1}$$
$$= \lim_{x \to \infty} \frac{x^2\left(1 + \frac{3}{x} - \frac{2}{x^2}\right)}{x^2\left(1 - \frac{2}{x} + \frac{1}{x^2}\right)}$$
$$= \lim_{x \to \infty} \frac{1 + \frac{3}{x} - \frac{2}{x^2}}{1 - \frac{2}{x} + \frac{1}{x^2}}$$
$$= \frac{\lim\limits_{x \to \infty}\left(1 + \frac{3}{x} - \frac{2}{x^2}\right)}{\lim\limits_{x \to \infty}\left(1 - \frac{2}{x} + \frac{1}{x^2}\right)}$$
$$= \frac{1 + 0 - 0}{1 - 0 + 0}$$
$$= 1$$

Similarly, $\lim\limits_{x \to \infty} \frac{x^2 + 3x - 2}{(x - 1)^2} = 1$, so $y = 1$ is a horizontal asymptote of the function.

10. a. $f(x) = \frac{3 - x}{2x + 5}$

Discontinuity is at $x = -2.5$.

$$\lim_{x \to -2.5^-} \frac{3 - x}{2x + 5} = -\infty$$
$$\lim_{x \to -2.5^+} \frac{3 - x}{2x + 5} = +\infty$$

Vertical asymptote is at $x = -2.5$.

Horizontal asymptote:

$$\lim_{x \to \infty} \frac{3 - x}{2x + 5} = -\frac{1}{2},$$
$$\lim_{x \to -\infty} \frac{3 - x}{2x + 5} = -\frac{1}{2}.$$

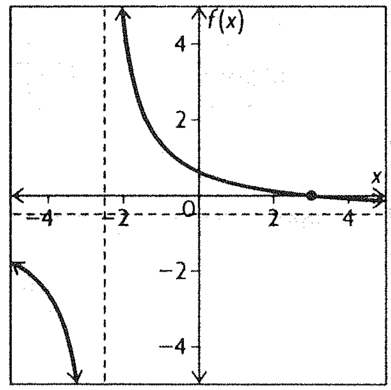
Horizontal asymptote is at $y = -\frac{1}{2}$.

$$f'(x) = \frac{-(2x + 5) - 2(3 - x)}{(2x + 5)^2} = \frac{-11}{(2x + 5)^2}$$

Since $f'(x) \neq 0$, there are no maximum or minimum points.

y-intercept, let $x = 0$, $y = \frac{3}{5} = 0.6$

x-intercept, let $y = 0$, $\frac{3 - x}{2x + 5} = 0$, $x = 3$



b. This function is a polynomial, so it is continuous for every real number. It has no horizontal, vertical, or oblique asymptotes.

The y-intercept can be found by letting $t = 0$, which gives $y = -10$.

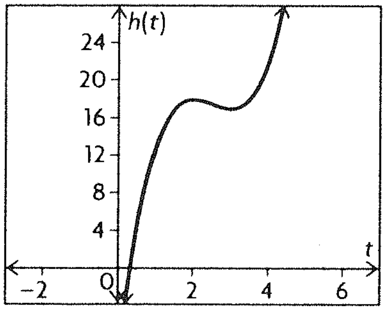
$h'(t) = 6t^2 - 30t + 36$

Set $h'(t) = 0$ and solve for t to determine the critical points.

$$6t^2 - 30t + 36 = 0$$
$$t^2 - 5t + 6 = 0$$
$$(t - 2)(t - 3) = 0$$
$$t = 2 \text{ or } t = 3$$

t	$t < 2$	$t = 2$	$2 < t < 3$	$t = 3$	$t > 3$
$h'(t)$	+	0	−	0	+
Graph	Inc.	Local Max	Dec.	Local Min	Inc.

The x-intercept cannot be easily obtained algebraically. Since the polynomial function has a local maximum when $x = 2$, it must have an x-intercept prior to this x-value. Since $f(0) = -10$ and $f(1) = 13$, an estimate for the x-intercept is about 0.3.



c. This function is discontinuous when

$x^2 + 4 = 0$

$x^2 = -4$

This equation has no real solutions, however, so the function is continuous everywhere.

To check for a horizontal asymptote:

$$\lim_{x\to\infty}\frac{20}{x^2+4} = \lim_{x\to\infty}\frac{20}{x^2\left(1+\frac{4}{x^2}\right)}$$

$$= \frac{\lim_{x\to\infty}(20)}{\lim_{x\to\infty}\left(x^2\left(1+\frac{4}{x^2}\right)\right)}$$

$$= \frac{\lim_{x\to\infty}(20)}{\lim_{x\to\infty}(x^2)\times\lim_{x\to\infty}\left(1+\frac{4}{x^2}\right)}$$

$$= \lim_{x\to\infty}\frac{1}{x^2}\times\frac{20}{1+0}$$

$$= 0$$

Similarly, $\lim_{x\to-\infty}\frac{20}{x^2+4} = 0$, so $y = 0$ is a horizontal asymptote of the function.

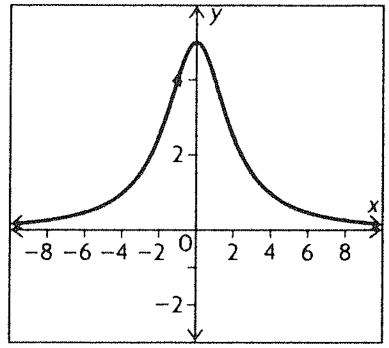
The y-intercept of this function can be found by letting $x = 0$, which gives $y = \frac{20}{0^2+4} = 5$. Since the numerator of this function is never 0, it has no x-intercept. The derivative can be found by rewriting the function as $y = 20(x^2+4)^{-1}$, then

$$y' = -20(x^2+4)^{-1}(2x)$$

$$= -\frac{40x}{(x^2+4)^2}$$

Letting $y' = 0$ shows that $x = 0$ is a critical point of the function.

x	$x < 0$	$x = 0$	$x > 0$
y′	+	0	−
Graph	Inc.	Local Max	Dec.



d. $s(t) = t + \frac{1}{t}$

Discontinuity is at $t = 0$.

$$\lim_{t\to0^+}\left(t+\frac{1}{t}\right) = +\infty$$

$$\lim_{t\to0^-}\left(t+\frac{1}{t}\right) = -\infty$$

Oblique asymptote is at $s(t) = t$.

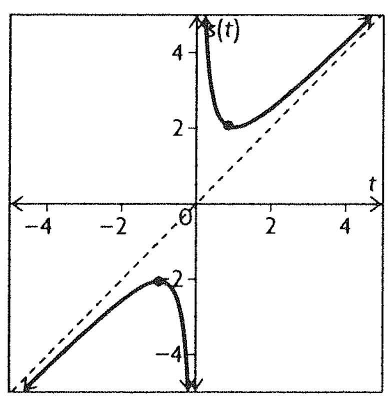
$$s'(t) = 1 - \frac{1}{t^2}$$

Let $s'(t) = 0$, $t^2 = 1$

$t = \pm 1$.

Local maximum is at $(-1, -2)$ and local minimum is at $(1, 2)$.

t	$t < -1$	$t = -1$	$-1 < t < 0$	$0 < t < 1$	$t = 1$	$t > 1$
s′(t)	+	0	−	−	0	+
Graph	Increasing	Local Max	Decreasing	Decreasing	Local Min	Increasing



e. $g(x) = \dfrac{2x^2 + 5x + 2}{x + 3}$

Discontinuity is at $x = -3$.

$\dfrac{2x^2 + 5x + 2}{x + 3} = 2x - 1 + \dfrac{5}{x + 3}$

Oblique asymptote is at $y = 2x - 1$.

$\lim\limits_{x \to -3^-} g(x) = +\infty, \ \lim\limits_{x \to -3^+} g(x) = -\infty$

$$g'(x) = \frac{(4x + 5)(x + 3) - (2x^2 + 5x + 2)}{(x + 3)^2}$$

$$= \frac{2x^2 + 12x + 13}{(x + 3)^2}$$

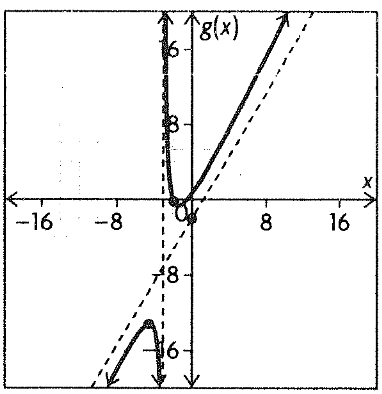
Let $g'(x) = 0$, therefore, $2x^2 + 12x + 13 = 0$:

$$x = \frac{-12 \pm \sqrt{144 - 104}}{4}$$

$x = -1.4$ or $x = -4.6$.

t	$x < -4.6$	-4.6	$-4.6 < x < -3$	-3	$-3 < x < -1.4$	$x = 1.4$	$x > -1.4$
s′(t)	+	0	−	Undefined	−	0	+
Graph	Increasing	Local Max	Decreasing	Vertical Asymptote	Decreasing	Local Min	Increasing

Local maximum is at $(-4.6, -10.9)$ and local minimum is at $(-1.4, -0.7)$.



f. $s(t) = \dfrac{t^2 + 4t - 21}{t - 3}, t \geq -7$

$= \dfrac{(t + 7)(t - 3)}{(t - 3)}$

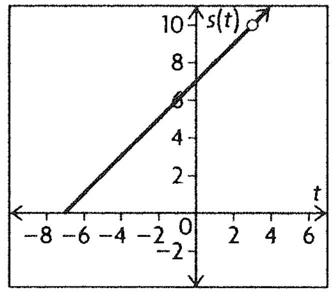
Discontinuity is at $t = 3$.

$\lim\limits_{x \to 3^-} \dfrac{(t + 7)(t - 3)}{(t - 3)} = \lim\limits_{x \to 3^-} (t + 7)$

$= 10$

$\lim\limits_{x \to 3^+} (t + 7) = 10$

There is no vertical asymptote. The function is the straight line $s = t + 7, t \geq -7$.



11. a. The horizontal asymptote occurs at $y = \frac{a}{c}$.

b. The vertical asymptote occurs when $cx + d = 0$ or $x = -\frac{d}{c}$.

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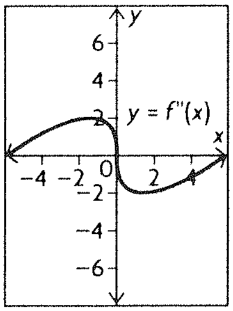
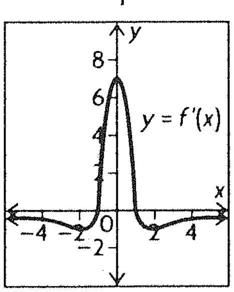
12. a. Since f is defined for all values of x, f' and f'' are also defined for all values of x. f has a horizontal asymptote at $y = 0$. So f' and f'' will have horizontal asymptotes there. f has a local maximum at $(0, 1)$ so f' will be 0 when $x = 0$. f has a point of inflection at $(-0.7, 0.6)$ and $(0.7, 0.6)$, so f'' will be 0 at $x = \pm 0.7$. At $x = 0.7$, f changes from concave up to concave down, so the sign of f'' changes from positive to negative. At $x = 0.7$, f changes from concave down to concave up, so the sign of f'' changes from negative to positive there. f is increasing for $x < 0$, so f' will be positive. f is decreasing for $x > 0$, so f' will be negative. The graph of f is concave up for $x < -0.7$ and $x > 0.7$, so f'' is positive for $x < -0.7$ and $x > 0.7$. The graph of f is concave down for $-0.7 < x < 0.7$, so f'' is negative for $-0.7 < x < 0.7$.

Also, since f'' is 0 at $x = \pm 0.7$, the graph of f' will have a local minimum or local maximum at these points. Since the sign of f'' changes from negative to positive at $x = 0.7$, it must be a local minimum point. Since the sign of f'' changes from positive to negative at $x = -0.7$, it must be a local maximum point.

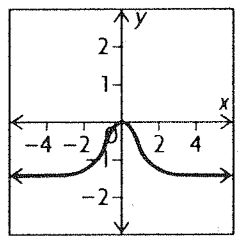


b. Since f is defined for all values of x, f' and f'' are also defined for all values of x. f has a horizontal asymptote at $y = 0$ so f' and f'' will have a horizontal asymptote there. f has a local maximum at $(1, 3.5)$ so f' will be 0 when $x = 1$. f has a local minimum at $(-1, -3.5)$ so f' will be 0 when $x = -1$. f has a point of inflection at $(-1.7, -3)$, $(1.7, 3)$ and $(90, 0)$ so f'' will be 0 at $x = \pm 1.7$ and $x = 0$. At $x = 0$, f changes from concave up to concave down, so the sign of f'' changes from positive to negative. At $x = -1.7$, f changes from concave down to concave up, so the sign of f'' changes from negative to positive. At $x = 1.7$, f changes from concave down to concave up, so the sign of f'' changes from negative to positive. f is decreasing for $x < -1$ and $x > 1$, so f' will be negative. The graph of f is concave up for $-1.7 < x < 0$ and $x > 1.7$, so f'' is positive for $-1.7 < x < 0$ and $x > 1.7$. The graph of f is concave down for $x < -1.7$ and $0 < x < 1.7$, so f'' is negative for $x < -1.7$ and $0 < x < 1.7$. Also, since f'' is 0 when $x = 0$ and $x = \pm 1.7$, the graph of f' will have a local maximum or minimum at these points. Since the sign of f'' changes from negative to positive at $x = -1.7$, f' has a local minimum at $x = -1.7$. Since the sign of f'' changes from positive to negative at $x = 0$, it must be a local maximum point. Since the sign of f'' changes from negative to positive at $x = 1.7$, it must be a local minimum point.

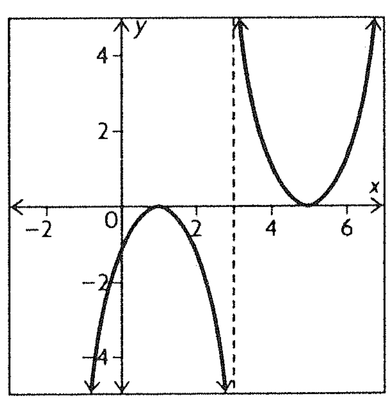


13. a. If $f(x)$ is increasing, then $f'(x) > 0$. From the graph of f', $f'(x) > 0$ for $x < 0$. If $f(x)$ is decreasing, then $f'(x) < 0$. From the graph of f', $f'(x) < 0$ for $x > 0$. At a stationary point, $f'(x) = 0$. From the graph, the zero of $f'(x)$ occurs at $(0, 0)$. At $x = 0$, The graph changes from positive to negative, so f has a local maximum there. If the graph of f is concave up, $f''(x)$ is positive. From the slope of f', the graph of f is concave up for $x < -0.6$ and $x > 0.6$. If the graph of f is concave down, $f''(x)$ is negative and concave down for $-0.6 < x < 0.6$. Graphs will vary slightly.

An example showing the shape of the curve is illustrated.



b. If $f(x)$ is increasing, then $f'(x) > 0$. From the graph of f', $f'(x) > 0$ for $x < 1$ and $x > 5$. If $f(x)$ is decreasing, then $f'(x) < 0$. From the graph of f', $f'(x) < 0$ for $1 < x < 3$ and $3 < x < 5$. At a stationary point, $f'(x) = 0$. From the graph, the zeros of $f'(x)$ occur at $x = 1$ and $x = 5$. At $x = 1$, the graph changes from positive to negative, so f has a local maximum there. At $x = 5$, the graph changes from negative to positive, so f has a local minimum there. If the graph of f is concave up, $f''(x)$ is positive. From the slope of f', the graph is concave up for $x > 3$. If the graph of f is concave down, $f''(x)$ is negative. From the slope of f', the graph of f is concave down for $x < 3$. There is a vertical asymptote at $x = 3$ since f' is not defined there. Graphs will vary slightly. An example showing the shape of the curve is illustrated.



14. a. $f(x)$ and $r(x)$: $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} r(x)$ exist.

b. $h(x)$: the highest degree of x in the numerator is exactly one degree higher than the highest degree of x in the denominator.

c. $h(x)$: the denominator is defined for all $x \in \mathbf{R}$.

$f(x) = \dfrac{-x-3}{(x-7)(x+2)}$ has vertical asymptotes at $x = 7$ and $x = -2$. $f(-2.001) = -110.99$ so as $x \to -2^-, f(x) \to -\infty$

$f(-1.999) = 111.23$ so as $x \to -2^+, f(x) \to \infty$

$f(6.999) = 111.12$ so as $x \to 7^-, f(x) \to \infty$

$f(7.001) = -111.10$ so as $x \to 7^+, f(x) \to -\infty$

$f(x)$ has a horizontal asymptote at $y = 0$.

$g(x)$ has a vertical asymptote at $x = 3$.

$g(2.999) = 23\,974.009$ so as $x \to 3^-, g(x) \to \infty$

$g(3.001) = -24\,026.009$ so as $x \to 3^+, g(x) \to -\infty$

By long division, $h(x) = x + \left(\dfrac{-4x-1}{x^2+1}\right)$ so $y = x$ is an oblique asymptote.

$r(x) = \dfrac{(x+3)(x-2)}{(x-4)(x+4)}$ has vertical asymptotes at $x = -4$ and $x = 4$.

$r(-4.001) = 750.78$ so as $x \to -4^-, r(x) \to \infty$

$r(-3.999) = -749.22$ so as $x \to -4^+, r(x) \to -\infty$

$r(3.999) = -1749.09$ so as $x \to 4^-, r(x) \to -\infty$

$r(4.001) = 1750.91$ so as $x \to 4^+, r(x) \to \infty$

$r(x)$ has a horizontal asymptote at $y = 1$.

15. $f(x) = \dfrac{ax+5}{3-bx}$

Vertical asymptote is at $x = -4$.

Therefore, $3 - bx = 0$ at $x = -5$.

That is, $3 - b(-5) = 0$

$$b = \frac{3}{5}.$$

Horizontal asymptote is at $y = -3$.

$$\lim_{x\to\infty}\left(\frac{ax+5}{3-bx}\right) = -3$$

$$\lim_{x\to\infty}\left(\frac{ax+5}{3-bx}\right) = \lim_{x\to\infty}\left(\frac{a+\frac{5}{x}}{\frac{3}{x}-b}\right) = \frac{-a}{b}$$

But $-\frac{a}{b} = -3$ or $a = 3b$.

But $b = \frac{3}{5}$, then $a = \frac{9}{5}$.

16. a. $\displaystyle\lim_{x\to\infty}\frac{x^2+1}{x+1} = \lim_{x\to\infty}\frac{x+\frac{1}{x}}{1+\frac{1}{x}}$

$= \infty$

$$\lim_{x\to\infty}\frac{x^2+2x+1}{x+1} = \lim_{x\to\infty}\frac{(x+1)(x+1)}{(x+1)} = \lim_{x\to\infty}(x+1) = \infty$$

b. $\displaystyle\lim_{x\to\infty}\left[\frac{x^2+1}{x+1} - \frac{x^2+2x+1}{x+1}\right]$

$$= \lim_{x\to\infty}\frac{x^2+1-x^2-2x-1}{x+1}$$

$$= \lim_{x\to\infty}\frac{-2x}{x+1}$$

$$= \lim_{x\to\infty}\frac{-2}{1+\frac{1}{x}} = -2$$

17. $f(x) = \dfrac{2x^2-2x}{x^2-9}$

Discontinuity is at $x^2 - 9 = 0$ or $x = \pm 3$.

$\lim_{x\to 3^+} f(x) = +\infty$

$\lim_{x\to 3^-} f(x) = -\infty$

$\lim_{x\to -3^+} f(x) = -\infty$

$\lim_{x\to -3^-} f(x) = +\infty$

Vertical asymptotes are at $x = 3$ and $x = -3$.

Horizontal asymptote:

$\lim_{x\to\infty} f(x) = 2$ (from below)

$\lim_{x\to -\infty} f(x) = 2$ (from above)

Horizontal asymptote is at $y = 2$.

$$f'(x) = \frac{(4x-2)(x^2-9) - 2x(2x^2-2x)}{(x^2-9)^2}$$

$$= \frac{4x^3 - 2x^2 - 36x + 18 - 4x^3 + 4x^2}{(x^2-9)^2}$$

$$= \frac{2x^2 - 36x + 18}{(x^2-9)^2}$$

Let $f'(x) = 0$,

$2x^2 - 36x + 18 = 0$ or $x^2 - 18x + 9 = 0$.

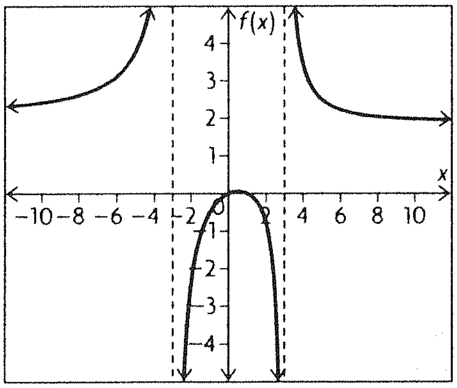
$$x = \frac{18 \pm \sqrt{18^2 - 36}}{2}$$

$x = 0.51$ or $x = 17.5$

$y = 0.057$ or $y = 1.83$.

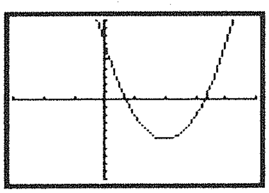
Local maximum is at $(0.51, 0.057)$ and local minimum is at $(17.5, 1.83)$.

t	$-3 < x < 0.51$	0.51	$0.51 < x < 3$	$3 < x < 17.5$	17.5	$x > 17.5$
s′(t)	+	0	–	–	0	+
Graph	Increasing	Local Max	Decreasing	Decreasing	Local Min	Increasing



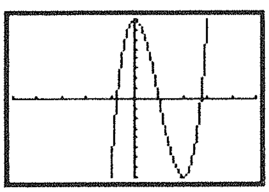
Mid-Chapter Review, pp. 196–197

1. a.



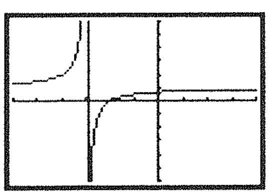
The function appears to be decreasing on $(-\infty, 2)$ and increasing on $(2, \infty)$.

b.



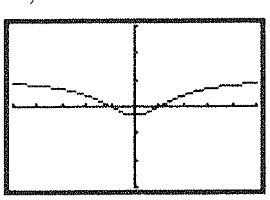
The function appears to be increasing on $(-\infty, 0)$ and $(2, \infty)$ and decreasing on $(0, 2)$.

c.



The function is increasing on $(-\infty, -3)$ and $(-3, \infty)$.

d.



The function appears to be decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$.

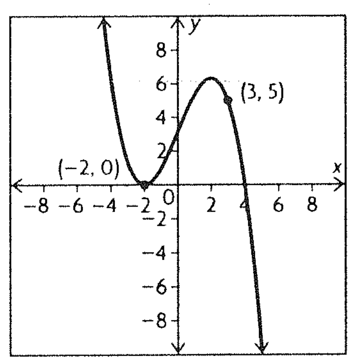
2. The slope of a general tangent to the graph $g(x) = 2x^3 - 3x^2 - 12x + 15$ is given by $\frac{dg}{dx} = 6x^2 - 6x - 12$. We first determine values of x for which $\frac{dg}{dx} = 0$.

$$\begin{aligned} \text{So } 6x^2 - 6x - 12 &= 0 \\ 6(x^2 - x - 2) &= 0 \\ 6(x + 1)(x - 2) &= 0 \\ x = -1, x &= 2 \end{aligned}$$

Since $\frac{dg}{dx}$ is defined for all values of x, and since $\frac{dg}{dx} = 0$ only at $x = -1$ and $x = 2$, it must be either positive or negative for all other values of x. Consider the intervals between $x < -1$, $-1 < x < 2$, and $x > 2$.

Value of x	$x < -1$	$-1 < x < 2$	$x > 2$
Value of $\frac{dg}{dx} = 6x^2 - 6x - 12$	$\frac{dg}{dx} > 0$	$\frac{dg}{dx} < 0$	$\frac{dg}{dx} > 0$
Slope of Tangents	positive	negative	positive
y-values Increasing or Decreasing	increasing	decreasing	increasing

3.



4. The critical numbers can be found when $\frac{dy}{dx} = 0$.

a. $\frac{dy}{dx} = -4x + 16$. When $\frac{dy}{dx} = 0$,

$$\frac{dy}{dx} = -4(x + 4) = 0$$

$$x = -4$$

	$x < 1$	$1 < x < 2$	$x > 2$
$x - 1$	$-$	$+$	$+$
$x - 2$	$-$	$-$	$+$
$(x - 1)(x - 2)$	$(-)(-) = +$	$(+)(-) = -$	$(+)(+) = +$
$\frac{dy}{dx}$	> 0	< 0	> 0
$g(x) = 2x^3 - 9x^2 + 12x$	increasing	decreasing	increasing

b. $\frac{dy}{dx} = x^3 - 27x$. When $\frac{dy}{dx} = 0$,

$\frac{dy}{dx} = x(x^2 - 27) = 0$

$x = 0, x = \pm 3\sqrt{3}$

c. $\frac{dy}{dx} = 4x^3 - 8x$. When $\frac{dy}{dx} = 0$,

$\frac{dy}{dx} = 4x(x^2 - 2) = 0$

$x = 0, x = \pm\sqrt{2}$

d. $\frac{dy}{dx} = 15x^4 - 75x^2 + 60$. When $\frac{dy}{dx} = 0$,

$\frac{dy}{dx} = 15(x^4 - 5x^2 + 4) = 0$

$\frac{dy}{dx} = 15(x^2 - 1)(x^2 - 4) = 0$

$x = \pm 1, x = \pm 2$

e. $\frac{dy}{dx} = \frac{2x(x^2 + 1) - (x^2 - 1)(2x)}{(x^2 + 1)^2}$. When $\frac{dy}{dx} = 0$, the numerator equals 0. So $\frac{dy}{dx} = 2x(x^2 + 1) - (x^2 - 1)(2x) = 0$. After simplifying, $\frac{dy}{dx} = 4x = 0$.
$x = 0$

f. $\frac{dy}{dx} = \frac{(x^2 + 2) - x(2x)}{(x^2 + 2)^2}$. When $\frac{dy}{dx} = 0$, the numerator equals 0. So after simplifying,

$\frac{dy}{dx} = -x^2 + 2 = 0$.

$x = \pm\sqrt{2}$

5. a. $\frac{dg}{dx} = 6x^2 - 18x + 12$

To find the critical numbers, set $\frac{dg}{dx} = 0$. So

$6x^2 - 18x + 12 = 0$

$6(x - 1)(x - 2) = 0$

$x = 1, x = 2$

From the table above, $x = 1$ is the local maximum and $x = 2$ is the local minimum.

b. $\frac{dg}{dx} = 3x^2 - 4x - 4$

To find the critical numbers, set $\frac{dg}{dx} = 0$.

$3x^2 - 4x - 4 = 0$

$(3x + 2)(x - 2) = 0$

$x = -\frac{2}{3}$ or $x = 2$

	$x < -\frac{2}{3}$	$-\frac{2}{3} < x < 2$	$x > 2$
$3x + 2$	$-$	$+$	$+$
$x - 2$	$-$	$-$	$+$
$\frac{dg}{dx}$	$+$	$-$	$+$
$g(x)$	increasing	decreasing	increasing

The function has a local maximum at $x = -\frac{2}{3}$ and a local minimum at $x = 2$

6. $\frac{df}{dx} = 2x + k$

To have a local minimum value, $\frac{df}{dx} = 0$. This occurs when $x = -\frac{k}{2}$. So $f\left(-\frac{k}{2}\right) = 1$.

$\frac{k^2}{4} - \frac{k^2}{2} + 2 = 1$

$-\frac{k^2}{4} + 2 = 1$

$-\frac{k^2}{4} = -1$

$k^2 = 4$

$k = \pm 2$

7. $f'(x) = 4x^3 - 32$

To find the critical numbers, set $f'(x) = 0$.

$4x^3 - 32 = 0$

$4(x^3 - 8) = 0$

$x = 2$

	$x < 2$	$x > 2$
$f'(x) = 4x^3 - 32$	−	+
$f(x)$	decreasing	increasing

The function has a local minimum at $x = 2$.

8. a. Since $x + 2 = 0$ for $x = -2$, $x = -2$ is a vertical asymptote. Large and positive to left of asymptote, large and negative to right of asymptote.

b. Since $9 - x^2 = 0$ for $x = \pm 3$, $x = -3$ and $x = 3$ are vertical asymptotes. For $x = -3$: large and negative to left of asymptote, large and positive to right of asymptote.

c. Since $3x + 9 = 0$ for $x = -3$, $x = -3$ is a vertical asymptote. Large and negative to left of asymptote, large and positive to right of asymptote.

d. Since $3x^2 - 13x - 10 = 0$ when $x = -\frac{2}{3}$ and $x = 5$, $x = -\frac{2}{3}$ and $x = 5$ are vertical asymptotes. For $x = -\frac{2}{3}$ large and positive to left of asymptote, large and negative to right of asymptote. For $x = 3$: large and positive to left of asymptote, large and negative to right of asymptote.

9. a. $$f(x) = \frac{3x - 1}{x + 5} = \frac{3x\left(1 - \frac{1}{3x}\right)}{x\left(1 + \frac{5}{x}\right)}$$

$$= \frac{3\left(1 - \frac{1}{3x}\right)}{1 + \frac{5}{x}}$$

$$\lim_{x \to +\infty} f(x) = \frac{3\left[\lim_{x \to -\infty}\left(1 - \frac{1}{3x}\right)\right]}{\lim_{x \to -\infty}\left(1 + \frac{5}{x}\right)}$$

$$= \frac{3(1 - 0)}{(1 + 0)}$$

$$= 3$$

So the horizontal asymptote is $y = 3$. Similarly, we can prove $\lim_{x \to -\infty} f(x) = 3$. If x is large and positive, for example, if $x = 1000$, $f(x) = \frac{2999}{1005}$, which is smaller than 3. If x is large and negative, for example, if $x = -1000$, $f(x) = \frac{-3001}{-995}$, which is larger than 3. So $f(x)$ approaches $y = 3$ from below when x is large and positive and approached $y = 3$ from above when x is large and negative.

b. $$f(x) = \frac{x^2 + 3x - 2}{(x - 1)^2} = \frac{x^2\left(1 + \frac{3}{x} - \frac{2}{x^2}\right)}{x^2\left(1 - \frac{2}{x} + \frac{1}{x^2}\right)}$$

$$= \frac{\left(1 + \frac{3}{x} - \frac{2}{x^2}\right)}{\left(1 - \frac{2}{x} + \frac{1}{x^2}\right)}$$

$$\lim_{x \to +\infty} \frac{\left(1 + \frac{3}{x} - \frac{2}{x^2}\right)}{\left(1 - \frac{2}{x} + \frac{1}{x^2}\right)} = \frac{(1 + 0 - 0)}{(1 - 0 + 0)}$$

$$= 1$$

So the horizontal asymptote is 1. Similarly, we can prove $\lim_{x \to -\infty} f(x) = 1$. If x is large and positive, for example, $x = 1000$, $f(x) = \frac{1000^2 + 3(1000) - 2}{(1000 - 1)^2} = \frac{996\,998}{1\,002\,001}$, which is greater than 1. If x is large and negative, for example, $x = -1000$, $f(x) = \frac{(-1000)^2 + 3(-1000) - 2}{(-1000 - 1)^2} = \frac{996\,998}{1\,002\,001}$, which is less than 1. So $f(x)$ approaches $y = 1$ from above when x is large and positive and approaches $y = 1$ from below when x is large and negative.

10. a. Since $(x - 5)^2 = 0$ when $x = 5$, $x = 5$ is a vertical asymptote.

$$f(x) = \frac{x}{(x - 5)^2} = \frac{x}{x^2\left(1 - \frac{10}{x} + \frac{25}{x^2}\right)}$$

$$= \frac{1}{x\left(1 - \frac{10}{x} + \frac{25}{x^2}\right)}$$

$$\lim_{x \to +5} f(x) = \frac{\lim_{x \to +5}(1)}{\lim_{x \to +5}\left(x\left(1 - \frac{10}{x} + \frac{25}{x^2}\right)\right)} = +\infty$$

This limit gets larger as it approaches 5 from the right. Similarly, we can prove that the limit goes to $+\infty$ as it approaches 5 from the left. For example, if $x = 1000$ $f(x) = \frac{1}{1000\left(1 - \frac{10}{1000} + \frac{25}{1000^2}\right)}$, which gets larger as x gets larger. Thus, $f(x)$ approaches $+\infty$ on both sides of $x = 5$.

b. There are no discontinuities because $x^2 + 9$ never equals zero.

c. Using the quadratic formula, we find that $x^2 - 12x + 12 = 0$ when $x = 6 \pm 2\sqrt{6}$. So $x = 6 \pm 2\sqrt{6}$ are vertical asymptotes.

$$f(x) = \frac{x-2}{x^2 - 12x + 12} = \frac{x\left(1 - \frac{2}{x}\right)}{x^2\left(1 - \frac{12}{x} + \frac{12}{x^2}\right)}$$

$$\lim_{x \to +6+2\sqrt{6}} f(x) = \frac{\lim_{x \to +6+2\sqrt{6}} x\left(1 - \frac{2}{x}\right)}{\lim_{x \to +6+2\sqrt{6}} x^2\left(1 - \frac{12}{x} + \frac{12}{x^2}\right)}.$$

This limit gets smaller as it approaches $6 + 2\sqrt{6}$ from the right and get larger as it approaches $6 + 2\sqrt{6}$ from the left. Similarly, we can show that the limit gets smaller as it approaches $6 - 2\sqrt{6}$ from the left and gets larger as it approaches from the right.

11. a. $f'(x) > 0$ implies that $f(x)$ is increasing.

b. $f'(x) < 0$ implies that $f(x)$ is decreasing.

12. a. $h(t) = -4.9t^2 + 9.5t + 2.2$

Note that $h(0) = 2.2 < 3$ because when the diver dives, the board is curved down.

$h'(t) = -9.8t + 9.5$

Set $h'(t) = 0$

$0 = -9.8t + 9.5$

$t \doteq 0.97$

	$0 < t < 0.97$	$t > 0.97$
$-9.8t + 9.5$	+	−
Sign of $h'(t)$	+	−
Behaviour of $h(t)$	increasing	decreasing

b. $h'(t) = v(t)$

$v(t) = -9.8t + 9.5$

$v'(t) = -9.8 < 0$

The velocity is decreasing all the time.

13. $C(t) = \frac{t}{4} + 2t^{-2}$

$C'(t) = \frac{1}{4} - 4t^{-3}$

Set $C'(t) = 0$

$0 = \frac{1}{4} - 4t^{-3}$

$\frac{1}{4} = 4t^{-3}$

$t^3 = 16$

$t \doteq 2.5198$

	$t < 2.5198$	$t > 2.5198$
$\frac{1}{4} - 4t^{-3}$	−	+
Sign of $C'(t)$	−	+
Behaviour of $C(t)$	decreasing	increasing

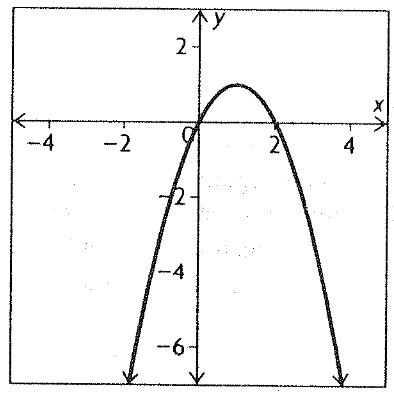
14. For $f(x)$ the derivative function $f'(0) = 0$ and $f'(2) = 0$.

Therefore, $f'(x)$ passes through $(0, 0)$ and $(2, 0)$.

When $x < 0$, $f(x)$ is decreasing, therefore, $f'(x) < 0$.

When $0 < x < 2$, $f(x)$ is increasing, therefore, $f'(x) > 0$.

When $x > 2$, $f(x)$ is decreasing, therefore, $f'(x) < 0$.



15. a. $f(x) = x^2 - 7x - 18$

i. $f'(x) = 2x - 7$

Set $f'(x) = 0$

$0 = 2x - 7$

$x = \frac{7}{2}$

ii.

	$x < \frac{7}{2}$	$x > \frac{7}{2}$
$2x - 7$	−	+
Sign of $f'(x)$	−	+
Behaviour of $f(x)$	decreasing	increasing

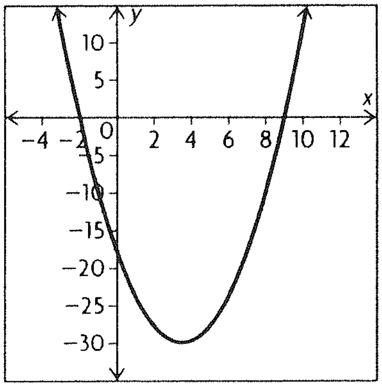
iii. From ii., there is a minimum at $x = \frac{7}{2}$.

$f\left(\frac{7}{2}\right) = \left(\frac{7}{2}\right)^2 - 7\left(\frac{7}{2}\right) - 18$

$f\left(\frac{7}{2}\right) = \frac{49}{4} - \frac{49}{2} - 18$

$f\left(\frac{7}{2}\right) = -\frac{121}{4}$

iv.



b. $f(x) = -2x^3 + 9x^2 + 3$

i. $f'(x) = -6x^2 + 18x$

Set $f'(x) = 0$

$0 = -6x^2 + 18x$

$0 = -6x(x - 3)$

$x = 0$ or $x = 3$

ii.

	$x < 0$	$0 < x < 3$	$x > 3$
$-6x$	+	−	−
$x - 3$	−	−	+
Sign of $f'(x)$	$(+)(-) = -$	$(-)(-) = +$	$(-)(+) = -$
Behaviour of $f(x)$	decreasing	increasing	decreasing

iii. From ii., there is a minimum at $x = 0$ and a maximum at $x = 3$.

$f(0) = -2(0)^3 + 9(0)^2 + 3$

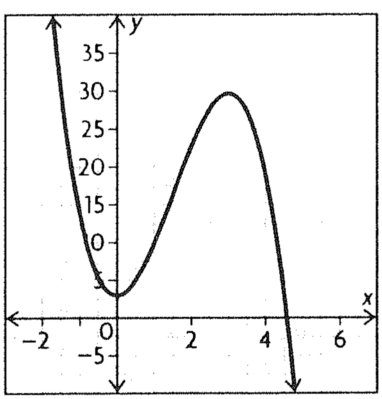
$f(0) = 3$

$f(3) = -2(3)^3 + 9(3)^2 + 3$

$f(3) = -54 + 81 + 3$

$f(3) = 30$

iv.



c. $f(x) = 2x^4 - 4x^2 + 2$

i. $f'(x) = 8x^3 - 8x$

$f'(x) = 0$

$0 = 8x^3 - 8x$

$0 = 8x(x^2 - 1)$

$0 = 8x(x - 1)(x + 1)$

$x = -1$ or $x = 0$ or $x = 1$

ii.

	$x < -1$	$-1 < x < 0$	$0 < x < 1$	$x > 1$
$8x$	−	−	+	+
$x - 1$	−	−	−	+
$x + 1$	−	+	+	+
Sign of $f'(x)$	$(-)(-)(-) = -$	$(-)(-)(+) = +$	$(+)(-)(+) = -$	$(+)(+)(+) = +$
Behaviour of $f(x)$	decreasing	increasing	decreasing	increasing

iii. From ii., there is a maximum at $x = 0$ and minima at $x = -1$ and $x = 1$

$f(-1) = 2(-1)^4 - 4(-1)^2 + 2$

$f(-1) = 2 - 4 + 2$

$f(-1) = 0$

$f(0) = 2(0)^4 - 4(0)^2 + 2$

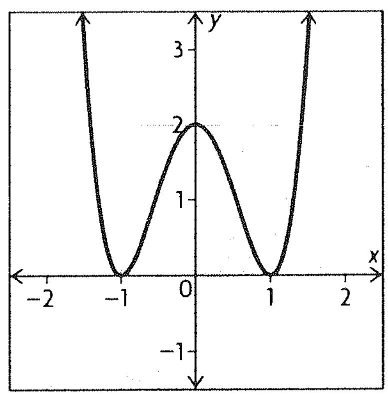
$f(0) = 2$

$f(1) = 2(1)^4 - 4(1)^2 + 2$

$f(1) = 2 - 4 + 2$

$f(1) = 0$

iv.



d. $f(x) = x^5 - 5x$

i. $f'(x) = 5x^4 - 5$

Set $f'(x) = 0$

$0 = 5x^4 - 5$

$0 = 5(x^4 - 1)$

$0 = 5(x^2 - 1)(x^2 + 1)$

$0 = 5(x - 1)(x + 1)(x^2 + 1)$

$x = -1$ or $x = 1$

ii.

	$x < -1$	$-1 < x < 1$	$x > 1$
5	+	+	+
$x - 1$	−	−	+
$x + 1$	−	+	+
$x^2 + 1$	+	+	+
Sign of $f'(x)$	(+)(−)(−)(+) = +	(+)(−)(+)(+) = −	(+)(+)(+)(+) = +
Behaviour of $f(x)$	increasing	decreasing	increasing

iii. From ii., there is a maximum at $x = -1$ and a minimum at $x = 1$

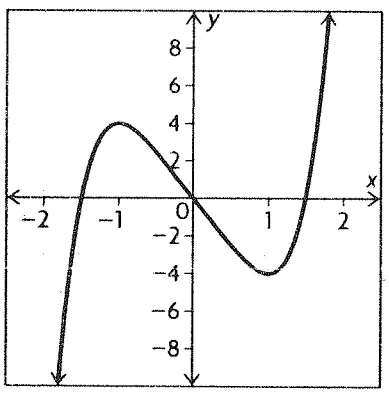
$f(-1) = (-1)^5 - 5(-1)$

$f(-1) = -1 + 5$

$f(-1) = 4$

$f(1) = (1)^5 - 5(1)$

$f(1) = -4$



16. a. vertical asymptote: $x = -\frac{1}{2}$, horizontal asymptote $y = \frac{1}{2}$; as x approaches $\frac{1}{2}$ from the left, graph approaches infinity; as x approaches $\frac{1}{2}$ from the right, graph approaches negative infinity.

b. vertical asymptote: $x = -2$, horizontal asymptote: $y = 1$; as x approaches -2 from the left, graph approaches infinity; as x approaches -2 from the right, graph decreases to $(-0.25, -1.28)$ and then approaches to infinity.

c. vertical asymptote: $x = -3$, horizontal asymptote: $y = -1$; as x approaches -3 from the left, graph approaches infinity; as x approaches -3 from the right, graph approaches infinity

d. vertical asymptote: $x = -4$, no horizontal asymptote; as x approaches -4 from the left, graph increases to $(-7.81, -30.23)$ and then decreases to -4; as x approaches -4 from the right, graph decreases to $(-0.19, 0.23)$ then approaches infinity.

17. a. $\lim_{x\to\infty} \frac{3 - 2x}{3x}$

$= \lim_{x\to\infty} \frac{\frac{3}{x} - \frac{2x}{x}}{\frac{3x}{x}}$

$= \frac{0 - 2}{3}$

$= -\frac{2}{3}$

b. $\lim_{x\to\infty} \frac{x^2 - 2x + 5}{6x^2 + 2x - 1}$

$= \lim_{x\to\infty} \frac{\frac{x^2}{x^2} - \frac{2x}{x^2} + \frac{5}{x^2}}{\frac{6x^2}{x^2} + \frac{2x}{x^2} - \frac{1}{x^2}}$

$= \frac{1 - 0 + 0}{6 + 0 - 0}$

$= \frac{1}{6}$

c. $\lim_{x\to\infty} \frac{7 + 2x^2 - 3x^3}{x^3 - 4x^2 + 3x}$

$= \lim_{x\to\infty} \frac{\frac{7}{x^3} + \frac{2x^2}{x^3} - \frac{3x^3}{x^3}}{\frac{x^3}{x^3} - \frac{4x^2}{x^3} + \frac{3x}{x^3}}$

$= \frac{0 + 0 - 3}{1 - 0 + 0}$

$= -3$

d. $\lim_{x\to\infty} \frac{5 + 2x^3}{x^4 - 4x}$

$= \lim_{x\to\infty} \frac{\frac{5}{x^4} - \frac{2x^3}{x^4}}{\frac{x^4}{x^4} - \frac{4x}{x^4}}$

$= \frac{0 - 0}{1 - 0}$

$= 0$

e. $\lim_{x\to\infty} \frac{2x^5 - 1}{3x^4 - x^2 - 2} = \lim_{x\to\infty} \left(\frac{2}{3}x + \frac{\frac{2}{3}x^3 + \frac{4}{3}x - 1}{3x^4 - x^2 - 2} \right)$

$= \lim_{x\to\infty} \frac{2}{3}x + \lim_{x\to\infty} \frac{\frac{\frac{2}{3}x^3}{x^4} + \frac{\frac{4}{3}x}{x^4} - \frac{1}{x^4}}{\frac{3x^4}{x^4} - \frac{x^2}{x^4} - \frac{2}{x^4}}$

$= \infty$

f. $\lim_{x \to \infty} \frac{x^2 + 3x - 18}{(x - 3)^2}$

$= \lim_{x \to \infty} \frac{x^2 + 3x - 18}{x^2 - 6x + 9}$

$= \lim_{x \to \infty} \frac{\frac{x^2}{x^2} + \frac{3x}{x^2} - \frac{18}{x^2}}{\frac{x^2}{x^2} - \frac{6x}{x^2} + \frac{9}{x^2}}$

$= \frac{1 + 0 - 0}{1 - 0 + 0}$

$= 1$

g. $\lim_{x \to \infty} \frac{x^2 - 4x - 5}{x^2 - 1}$

$= \lim_{x \to \infty} \frac{\frac{x^2}{x^2} - \frac{4x}{x^2} - \frac{5}{x^2}}{\frac{x^2}{x^2} - \frac{1}{x^2}}$

$= \lim_{x \to \infty} \frac{1 - 0 - 0}{1 - 0}$

$= 1$

h. $\lim_{x \to \infty} \left(5x + 4 - \frac{7}{x + 3}\right)$

$= \lim_{x \to \infty} 5x + \lim_{x \to \infty} 4 - \lim_{x \to \infty} \frac{\frac{7}{x}}{\frac{x}{x} + \frac{3}{x}}$

$= \infty$

4.4 Concavity and Points of Inflection, pp. 205–206

1. a. A: negative; B: negative; C: positive; D: positive
b. A: negative; B: negative; C: positive; D: negative
2. a. $y = x^3 - 6x^2 - 15x + 10$

$\frac{dy}{dx} = 3x^2 - 12x - 15$

For critical values, we solve $\frac{dy}{dx} = 0$:
$3x^2 - 12x - 15 = 0$
$x^2 - 4x - 5 = 0$
$(x - 5)(x + 1) = 0$
$x = 5$ or $x = -1$
The critical points are $(5, -105)$ and $(-1, 20)$.
Now, $\frac{d^2y}{dx^2} = 6x - 12$.

At $x = 5$, $\frac{d^2y}{dx^2} = 18 > 0$. There is a local minimum at this point.

At $x = -1$, $\frac{d^2y}{dx^2} = -18 < 0$. There is a local maximum at this point.

The local minimum is $(5, -105)$ and the local maximum is $(-1, 20)$

b. $y = \frac{25}{x^2 + 48}$

$\frac{dy}{dx} = -\frac{50x}{(x^2 + 48)^2}$

For critical values, solve $\frac{dy}{dx} = 0$ or $\frac{dy}{dx}$ does not exist.
Since $x^2 + 48 > 0$ for all x, the only critical point is $\left(0, \frac{25}{48}\right)$

$\frac{d^2y}{dx^2} = -50(x^2 + 48)^{-2} + 100x(x^2 + 48)^{-3}(2x)$

$= -\frac{50}{(x^2 + 48)^2} + \frac{200x^2}{(x^2 + 48)^3}$

At $x = 0$, $\frac{d^2y}{dx^2} = -\frac{50}{48^2} < 0$. The point $\left(0, \frac{25}{48}\right)$ is a local maximum.

c. $s = t + t^{-1}$

$\frac{ds}{dt} = 1 - \frac{1}{t^2}, t \neq 0$

For critical values, we solve $\frac{ds}{dt} = 0$:

$1 - \frac{1}{t^2} = 0$
$t^2 = 1$
$t = \pm 1.$
The critical points are $(-1, -2)$ and $(1, 2)$

$\frac{d^2s}{dt^2} = \frac{2}{t^3}$

At $t = -1$, $\frac{d^2s}{dt} = -2 < 0$. The point $(-1, -2)$ is a local maximum. At $t = 1$, $\frac{d^2s}{dt^2} = 2 > 0$. The point $(1, 2)$ is a local minimum.

d. $y = (x - 3)^3 + 8$

$\frac{dy}{dx} = 3(x - 3)^2$

$x = 3$ is a critical value.
The critical point is $(3, 8)$

$\frac{d^2y}{dx^2} = 6(x - 3)$

At $x = 3$, $\frac{d^2y}{dx^2} = 0$.
The point $(3, 8)$ is neither a relative (local) maximum or minimum.

3. a. For possible point(s) of inflection, solve $\frac{d^2y}{dx^2} = 0$:

$6x - 8 = 0$

$x = \frac{4}{3}.$

Interval	$x < \frac{4}{3}$	$x = \frac{4}{3}$	$x > \frac{4}{3}$
$f''(x)$	<0	$=0$	>0
Graph of $f(x)$	Concave Down	Point of Inflection	Concave Up

The point $\left(\frac{4}{3}, -14\frac{20}{27}\right)$ is point of inflection.

b. For possible point(s) of inflection, solve

$$\frac{d^2y}{dx^2} = 0:$$

$$200x^2 - 50x^2 - 2400 = 0$$

$$150x^2 = 2400.$$

Since $x^2 + 48 > 0$:

$x = \pm 4.$

Interval	$x < -4$	$x = -4$	$-4 < x < 4$	$x = 4$	$x > 4$
$f''(x)$	>0	$=0$	<0	$=0$	>0
Graph of $f(x)$	Concave Up	Point of Inflection	Concave Down	Point of Inflection	Concave Up

$\left(-4, \frac{25}{64}\right)$ and $\left(4, \frac{25}{64}\right)$ are points of inflection.

c. $\frac{d^2s}{dt^2} = \frac{3}{t^2}$

Interval	$t < 0$	$t = 0$	$t > 0$
$f''(t)$	<0	Undefined	>0
Graph of $f(t)$	Concave Down	Undefined	Concave Up

The graph does not have any points of inflection.

d. For possible points of inflection, solve

$$\frac{d^2y}{dx^2} = 0:$$

$$6(x - 3) = 0$$

$$x = 3.$$

Interval	$x < 3$	$x = 3$	$x > 3$
$f''(x)$	<0	$=0$	>0
Graph of $f(x)$	Concave Down	Point of Inflection	Concave Up

(3, 8) is a point of inflection.

4. a. $f(x) = 2x^3 - 10x + 3$ at $x = 2$

$$f'(x) = 6x^2 - 10$$

$$f''(x) = 12x$$

$$f''(2) = 24 > 0$$

The curve lies above the tangent at $(2, -1)$.

b. $g(x) = x^2 - \frac{1}{x}$ at $x = -1$

$$g'(x) = 2x + \frac{1}{x^2}$$

$$g''(x) = 2 - \frac{2}{x^3}$$

$$g''(-1) = 2 + 2 = 4 > 0$$

The curve lies above the tangent line at $(-1, 2)$.

c. $p(w) = \frac{w}{\sqrt{w^2 + 1}}$ at $w = 3$

$$p(w) = w(w^2 + 1)^{-\frac{1}{2}}$$

$$\frac{dp}{dw} = (w^2 + 1)^{-\frac{1}{2}} + w\left(-\frac{1}{2}\right)(w^2 + 1)^{-\frac{3}{2}}(2w)$$

$$= (w^2 + 1)^{-\frac{1}{2}} - w^2(w^2 + 1)^{-\frac{3}{2}}$$

$$\frac{d^2p}{dw^2} = -\frac{1}{2}(w^2 + 1)^{-\frac{3}{2}}(2w) - 2w(w^2 + 1)^{-\frac{3}{2}}$$

$$+ w^2\left(\frac{3}{2}\right)(w^2 + 1)^{-\frac{5}{2}}(2w)$$

At $w = 3$, $\frac{d^2p}{dw^2} = -\frac{3}{10\sqrt{10}} - \frac{6}{10\sqrt{10}} + \frac{81}{100\sqrt{10}}$

$$= -\frac{9}{100\sqrt{10}} < 0.$$

The curve is below the tangent line at $\left(3, \frac{3}{\sqrt{10}}\right)$.

d. The first derivative is

$$s'(t) = \frac{(t - 4)(2) - (2t)(1)}{(t - 4)^2}$$

$$= \frac{-8}{(t - 4)^2}$$

The second derivative is

$$s''(t) = \frac{(t - 4)^2(0) - (-8)2(t - 4)^1}{(t - 4)^4}$$

$$= \frac{16}{(t - 4)^3}$$

So $s''(-2) = \frac{16}{(-2 - 4)^3}$

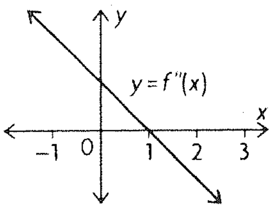
$$= -\frac{16}{216} = -\frac{2}{27}$$

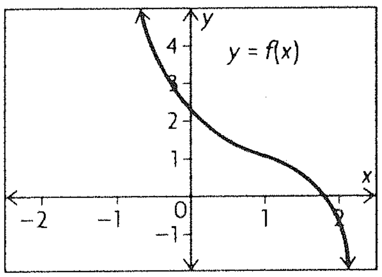
Since the second derivative is negative at this point, the function lies below the tangent there.

5. For the graph on the left: **i.** $f''(x) > 0$ for $x < 1$ Thus, the graph of $f(x)$ is concave up on $x < 1$. $f''(x) \leq 0$ for $x > 1$. The graph of $f(x)$ is concave down on $x > 1$.

ii. There is a point of inflection at $x = 1$.

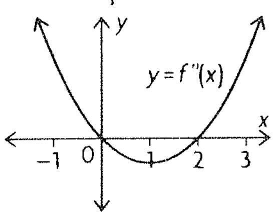
iii.

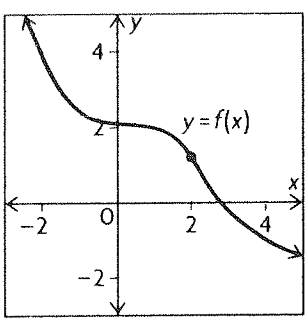




For the graph on the right: **i.** $f''(x) > 0$ for $x < 0$ or $x > 2$

The graph of $f(x)$ is concave up on $x < 0$ or $x > 2$.
The graph of $f(x)$ is concave down on $0 < x < 2$.
ii. There are points of inflection at $x = 0$ and $x = 2$.
iii.





6. For any function $y = f(x)$, find the critical points, i.e., the values of x such that $f'(x) = 0$ or $f'(x)$ does not exist. Evaluate $f''(x)$ for each critical value.
If the value of the second derivative at a critical point is positive, the point is a local minimum. If the value of the second derivative at a critical point is negative, the point is a local maximum.
7. Step 4: Use the first derivative test or the second derivative test to determine the type of critical points that may be present.
8. a. $f(x) = x^4 + 4x^3$
i. $f'(x) = 4x^3 + 12x^2$
$f''(x) = 12x^2 + 24x$

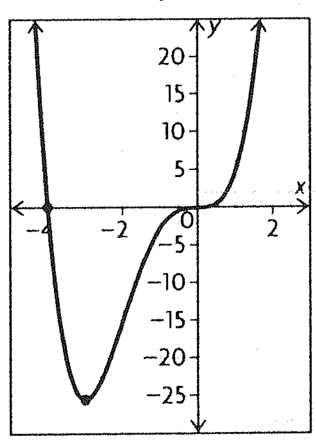
For possible points of inflection, solve $f''(x) = 0$:
$12x^2 + 24x = 0$
$12x(x + 2) = 0$
$x = 0$ or $x = -2$.

Interval	$x < -2$	$x = -2$	$-2 < x < 0$	$x = 0$	$x > 0$
$f''(x)$	>0	$=0$	<0	$=0$	>0
Graph of $f(x)$	Concave Up	Point of Inflection	Concave Down	Point of Inflection	Concave Up

The points of inflection are $(-2, -16)$ and $(0, 0)$.
ii. If $x = 0$, $y = 0$.
For critical points, we solve $f'(x) = 0$:
$4x^3 + 12x^2 = 0$
$4x^2(x + 3) = 0$
$x = 0$ and $x = -3$.

Interval	$x < -3$	$x = -3$	$-3 < x < 0$	$x = 0$	$x > 0$
$f'(x)$	<0	$=0$	>0	$=0$	>0
Graph of $f(x)$	Decreasing	Local Min	Increasing		Increasing

If $y = 0$, $x^4 + 4x^3 = 0$
$x^3(x + 4) = 0$
$x = 0$ or $x = -4$
The x-intercepts are 0 and -4.



b. d. $g(w) = \dfrac{4w^2 - 3}{w^3}$
$= \dfrac{4}{3} - \dfrac{3}{w^3}, w \neq 0$

i. $g'(w) = -\dfrac{4}{w^2} + \dfrac{9}{w^4}$
$= \dfrac{9 - 4w^2}{w^4}$

$g''(w) = \dfrac{8}{w^3} - \dfrac{36}{w^3}$
$= \dfrac{8w^2 - 36}{w^5}$

For possible points of inflection, we solve $g''(w) = 0$:

$8w^2 - 36 = 0$, since $w^5 \neq 0$

$$w^2 = \frac{9}{2}$$

$$w = \pm\frac{3}{\sqrt{2}}.$$

Interval	$w < -\frac{3}{\sqrt{2}}$	$w = -\frac{3}{\sqrt{2}}$	$-\frac{3}{\sqrt{2}} < w < 0$	$< w < 0 < \frac{3}{\sqrt{2}}$	$w = \frac{3}{\sqrt{2}}$	$w > \frac{3}{\sqrt{2}}$
$g'(w)$	<0	$=0$	>0	<0	0	>0
Graph of $g(w)$	Concave Down	Point of Inflection	Concave Up	Concave Down	Point of Inflection	Concave Up

The points of inflection are $\left(-\frac{3}{\sqrt{2}}, -\frac{8\sqrt{2}}{9}\right)$ and $\left(\frac{3}{\sqrt{2}}, -\frac{8\sqrt{2}}{9}\right)$.

ii. There is no y-intercept.

The x-intercept is $\pm\frac{3}{\sqrt{2}}$.

For critical values, we solve $g'(w) = 0$:

$9 - 4w^2 = 0$ since $w^4 \neq 0$

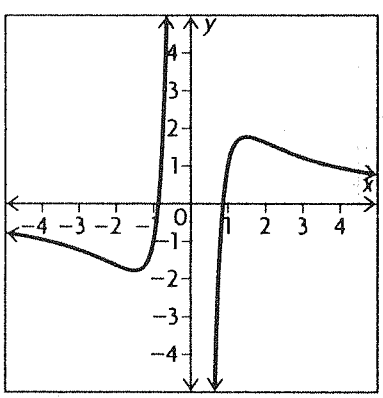
$$w = \pm\frac{3}{2}.$$

Interval	$w < -\frac{3}{2}$	$w = -\frac{3}{2}$	$-\frac{3}{2} < w < 0$	$< 0 < w < \frac{3}{2}$	$w = \frac{3}{2}$	$w > \frac{3}{2}$
$g'(w)$	<0	$=0$	>0	>0	0	<0
Graph of $g(w)$	Decreasing Down	Local Min	Increasing	Increasing	Local Max	Decreasing

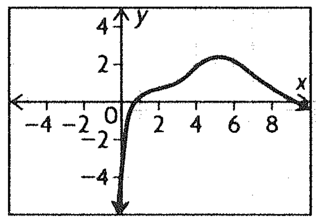
$$\lim_{w \to 0^+} \frac{4w^2 - 3}{w^3} = \infty, \lim_{w \to 0^-} \frac{4w^2 - 3}{w^3} = -\infty$$

$$\lim_{w \to -\infty} \left(\frac{4}{w} - \frac{3}{w^3}\right) = 0, \lim_{w \to \infty} \left(\frac{4}{w} - \frac{3}{w^3}\right) = 0$$

Thus, $y = 0$ is a horizontal asymptote and $x = 0$ is a vertical asymptote.



9. The graph is increasing when $x < 2$ and when $2 < x < 5$.
The graph is decreasing when $x > 5$.
The graph has a local maximum at $x = 5$.
The graph has a horizontal tangent line at $x = 2$.
The graph is concave down when $x < 2$ and when $4 < x < 7$.
The graph is concave up when $2 < x < 4$ and when $x > 7$.
The graph has points of inflection at $x = 2$, $x = 4$, and $x = 7$.
The y-intercept of the graph is -4.



10. $f(x) = ax^3 + bx^2 + c$

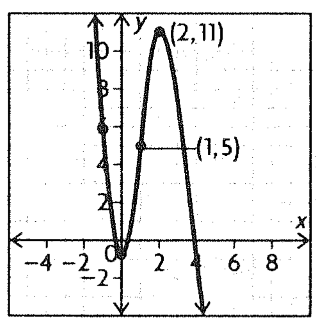
$f'(x) = 3ax^2 + 2bx$

$f''(x) = 6ax + 2b$

Since $(2, 11)$ is a relative extremum,
$f(2) = 12a + 4b = 0$.
Since $(1, 5)$ is an inflection point,
$f''(1) = 6a + 2b = 0$.
Since the points are on the graph, $a + b + c = 5$ and
$8a + 4b + c = 11$

$$7a + 3b = 6$$
$$9a + 3b = 0$$
$$2a = -6$$
$$a = -3$$
$$b = 9$$
$$c = -1.$$

Thus, $f(x) = -3x^3 + 9x^2 - 1$.



11. $f(x) = (x + 1)^{\frac{1}{2}} + bx^{-1}$

$$f'(x) = \frac{1}{2}(x + 1)^{\frac{1}{2}} - bx^{-2}$$

$$f''(x) = -\frac{1}{4}(x + 1)^{\frac{3}{2}} + 2bx^{-3}$$

Since the graph of $y = f(x)$ has a point of inflection at $x = 3$:

$$-\frac{1}{4}(4)^{-\frac{3}{2}} + \frac{2b}{27} = 0$$

$$-\frac{1}{32} + \frac{2b}{27} = 0$$

$$b = \frac{27}{64}.$$

12. $f(x) = ax^4 + bx^3$

$f'(x) = 4ax^3 + 3bx^2$

$f''(x) = 12ax^2 + 6bx$

For possible points of inflection, we solve $f''(x) = 0$:

$12ax^2 + 6bx = 0$

$6x(2ax + b) = 0$

$x = 0$ or $x = -\frac{b}{2a}$.

The graph of $y = f''(x)$ is a parabola with x-intercepts 0 and $-\frac{b}{2a}$.

We know the values of $f''(x)$ have opposite signs when passing through a root. Thus at $x = 0$ and at $x = -\frac{b}{2a}$, the concavity changes as the graph goes through these points. Thus, $f(x)$ has points of inflection at $x = 0$ and $x = -\frac{b}{2a}$. To find the x-intercepts, we solve $f(x) = 0$

$$x^3(ax + b) = 0$$

$$x = 0 \text{ or } x = -\frac{b}{a}.$$

The point midway between the x-intercepts has x-coordinate $-\frac{b}{2a}$.

The points of inflection are (0, 0) and $\left(-\frac{b}{2a}, -\frac{b}{16a^3}\right)$.

13. a. $y = \frac{x^3 - 2x^2 + 4x}{x^2 - 4} = x - 2 + \frac{8x - 8}{x^2 - 4}$ (by division of polynomials). The graph has discontinuities at $x = \pm 2$.

$$\lim_{x \to -2}\left(x - 2 + \frac{8x - 8}{x^2 - 4}\right) = -\infty$$

$$\lim_{x \to -2^-}\left(x - 2 + \frac{8x - 8}{x^2 - 4}\right) = -\infty$$

$$\lim_{x \to -2^+}\left(x - 2 + \frac{8x - 8}{x^2 - 4}\right) = -\infty$$

$$\lim_{x \to -2^-}\left(x - 2 + \frac{8x - 8}{x^2 - 4}\right) = -\infty$$

When $x = 0$, $y = 0$.

Also, $y = \frac{x(x^2 - 2x + 4)}{x^2 - 4} = \frac{x[(x - 1)^2 + 3]}{x^2 - 4}$.

Since $(x - 1)^2 + 3 > 0$, the only x-intercept is $x = 0$.

Since $\lim_{x \to \infty} \frac{8x - 8}{x^2 - 4} = 0$, the curve approaches the value $x - 2$ as $x \to \infty$. This suggests that the line $y = x - 2$ is an oblique asymptote. It is verified by the limit $\lim_{x \to \infty}[x - 2 - f(x)] = 0$. Similarly, the curve approaches $y = x - 2$ as $x \to -\infty$.

$$\frac{dy}{dx} = 1 + \frac{8(x^2 - 4) - 8(x - 1)(2x)}{(x^2 - 4)^2}$$

$$= 1 - \frac{8(x^2 - 2x + 4)}{(x^2 - 4)^2}$$

We solve $\frac{dy}{dx} = 0$ to find critical values:

$8x^2 - 16x + 32 = x^4 - 8x^2 + 16$

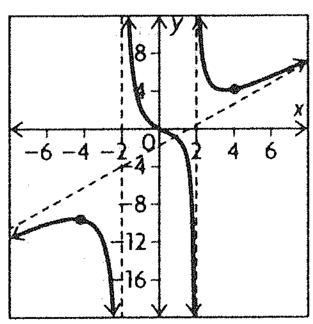
$x^4 - 16x^2 - 16 = 0$

$x^2 = 8 + 4\sqrt{5}$ $\left(8 - 4\sqrt{5} \text{ is inadmissible}\right)$

$x \doteq \pm 4.12.$

$\lim_{x \to \infty} y = \infty$ and $\lim_{x \to -\infty} y = -\infty$

Interval	$x < -4.12$	$x = -4.12$	$-4.12 < x < 2$	$-2 < x < 2$	$2 < x -4.12$	$x = 4.12$	$x > 4.12$
$\frac{dy}{dx}$	>0	$=0$	<0	<0	<0	0	>0
Graph of y	Increasing	Local Max	Decreasing	Decreasing	Decreasing	Local Min	Increasing



b. Answers may vary. For example, there is a section of the graph that lies between the two sections of the graph that approach the asymptote.

14. For the various values of n, $f(x) = (x - c)^n$ has the following properties:

n	$n = 1$	$n = 2$	$n = 3$	$n = 4$
$f'(x)$	1	$2(x - c)$	$3(x - c)^2$	$4(x - c)^3$
$f''(x)$	0	2	$6(x - c)$	$12(x - c)^2$
Infl. Pt.	None	None	$x = c$	$x = c$

It appears that the graph of f has an inflection point at $x = c$ when $n \geq 3$.

4.5 An Algorithm for Curve Sketching, pp. 212–213

1. A cubic polynomial that has a local minimum must also have a local maximum. If the local minimum is to the left of the local maximum, then $f(x) \to +\infty$ as $x \to -\infty$ and $f(x) \to -\infty$ as $x \to +\infty$. If the local minimum is to the right of the local maximum, then $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to +\infty$ as $x \to +\infty$.

2. Since each local maximum and minimum of a function corresponds to a zero of its derivative, the number of zeroes of the derivative is the maximum number of local extreme values that the function can have. For a polynomial of degree n, the derivative has degree $n - 1$, so it has at most $n - 1$ zeroes, and thus at most $n - 1$ local extremes. A polynomial of degree three has at most 2 local extremes. A polynomial of degree four has at most 3 local extremes.

3. a. This function is discontinuous when

$x^2 + 4x + 3 = 0$

$(x + 3)(x + 1) = 0$

$x = -3$ or $x = -1$. Since the numerator is non-zero at both of these points, they are both equations of vertical asymptotes.

b. This function is discontinuous when

$x^2 - 6x + 12$

$$x = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(12)}}{2(1)}$$

$$= \frac{6 \pm \sqrt{-12}}{2}$$

This equation has no real solutions, so the function has no vertical asymptotes.

c. This function is discontinuous when

$x^2 - 6x + 9 = 0$

$(x - 3)^2 = 0$

$x = 3$. Since the numerator is non-zero at this point, it is the equation of a vertical asymptote.

4. a. $y = x^3 - 9x^2 + 15x + 30$

We know the general shape of a cubic polynomial with leading coefficient positive. The local extrema will help refine the graph.

$$\frac{dy}{dx} = 3x^2 - 18x + 15$$

Set $\frac{dy}{dx} = 0$ to find the critical values:

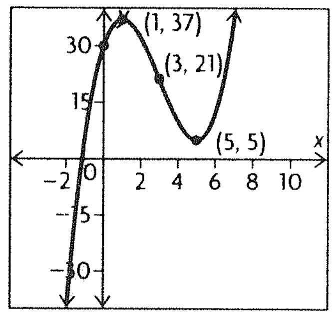
$3x^2 - 18x + 15 = 0$

$x^2 - 6x + 5 = 0$

$(x - 1)(x - 5) = 0$

$x = 1$ or $x = 5$.

The local extrema are $(1, 37)$ and $(5, 5)$.



b. $f(x) = 4x^3 + 18x^2 + 3$

The graph is that of a cubic polynomial with leading coefficient negative. The local extrema will help refine the graph.

$$\frac{dy}{dx} = 12x^2 + 36x$$

To find the critical values, we solve $\frac{dy}{dx} = 0$:

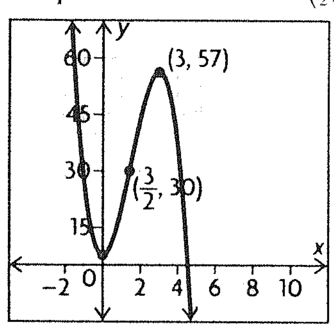
$-12x(x - 3) = 0$

$x = 0$ or $x = 3$.

The local extrema are $(0, 3)$ and $(3, 57)$.

$$\frac{d^2y}{dx^2} = -24x + 36$$

The point of inflection is $\left(\frac{3}{2}, 30\right)$.

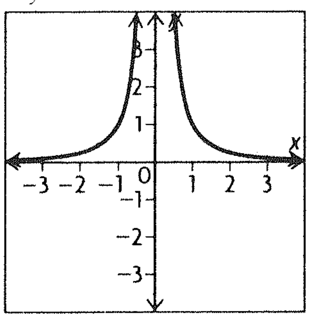


c. $y = 3 + \frac{1}{(x+2)^2}$

We observe that $y = 3 + \frac{1}{(x+2)^2}$ is just a translation of $y = \frac{1}{x^2}$.

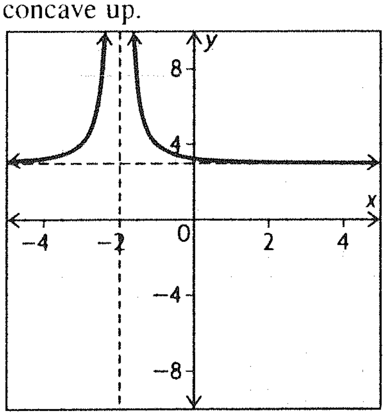
The graph of $y = \frac{1}{x^2}$ is

The reference point (0, 0) for $y = \frac{1}{x^2}$ becomes the point $(-2, 3)$ for $y = 3 + \frac{1}{(x+2)^2}$. The vertical asymptote is $x = -2$, and the horizontal asymptote is $y = 3$.



$\frac{dy}{dx} = -\frac{2}{(x+2)^3}$, hence there are no critical points.

$\frac{d^2y}{dx^2} = \frac{6}{(x+2)^4} > 0$, hence the graph is always concave up.



d. $f(x) = x^4 - 4x^3 - 8x^2 + 48x$

We know the general shape of a fourth degree polynomial with leading coefficient positive. The local extrema will help refine the graph.

$f'(x) = 4x^3 - 12x^2 - 16x + 48$

For critical values, we solve $f'(x) = 0$

$x^3 - 3x^2 - 4x + 12 = 0.$

Since $f'(2) = 0$, $x - 2$ is a factor of $f'(x)$.

The equation factors are

$(x-2)(x-3)(x+2) = 0.$

The critical values are $x = -2, 2, 3.$

$f''(x) = 12x^2 - 24x - 16$

Since $f''(-2) = 80 > 0$, $(-2, -80)$ is a local minimum.

Since $f''(2) = -16 < 0$, $(2, 48)$ is a local maximum.

Since $f''(3) = 20 > 0$, $(3, 45)$ is a local minimum.

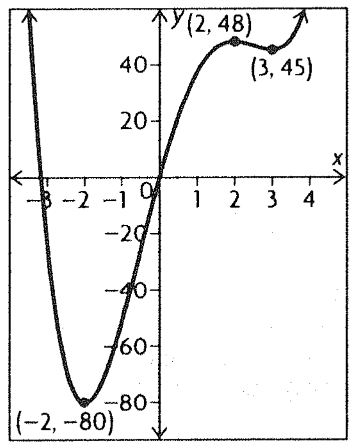
The graph has x-intercepts 0 and -3.2

The points of inflection can be found by solving $f''(x) = 0$:

$3x^2 - 6x - 4 = 0$

$x = \frac{6 \pm \sqrt{84}}{6}$

$x \doteq -\frac{1}{2}$ or $\frac{5}{2}$.



e. $y = \frac{2x}{x^2 - 25}$

There are discontinuities at $x = -5$ and $x = 5$.

$\lim_{x\to 5^-}\left(\frac{2x}{x^2-25}\right) = -\infty$ and $\lim_{\to -5^-}\left(\frac{2x}{x^2-25}\right) = \infty$

$\lim_{x\to 5^-}\left(\frac{2x}{x^2-25}\right) = -\infty$ and $\lim_{\to -5^-}\left(\frac{2x}{x^2-25}\right) = \infty$

$x = -5$ and $x = 5$ are vertical asymptotes.

$\frac{dy}{dx} = \frac{2(x^2-25) - 2x(2x)}{(x^2-25)^2} = -\frac{2x^2+50}{(x^2-25)^2} < 0$ for all x in the domain. The graph is decreasing throughout the domain.

$$\lim_{x\to\infty}\left(\frac{2x}{x^2-25}\right) = \lim_{x\to\infty}\left(\frac{\frac{2}{x}}{1 - \frac{25}{x^2}}\right) = 0$$

$$\lim_{x\to-\infty}\left(\frac{\frac{2}{x}}{1 - \frac{25}{x^2}}\right) = 0$$

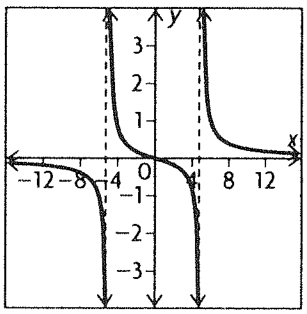
$y = 0$ is a horizontal asymptote.

$$\frac{d^2y}{dx^2} = -\frac{4x(x^2 - 25)^2 - (2x^2 + 50)(2)(x^2 - 25)(2x)}{(x^2 - 25)^4}$$

$$= \frac{4x^3 + 300x}{(x^2 - 25)^3} = \frac{4x(x^2 + 75)}{(x^2 - 25)^3}$$

There is a possible point of inflection at $x = 0$.

Interval	$x < -5$	$-5 < x < 0$	$x = 0$	$0 < x < 5$	$x > 5$
$\frac{d^2y}{dx^2}$	<0	>0	$=0$	<0	>0
Graph of y	Concave Down	Point of Up	Concave Inflection	Point of Down	Concave Up



f. This function is discontinuous when

$x^2 - 4x = 0$

$x(x - 4) = 0$

$x = 0$ or $x = 4$. The numerator is non-zero at these points, so the function has vertical asymptotes at both of them. The behaviour of the function near these asymptotes is:

x-values	1	x	$x - 4$	$f(x)$	$\lim_{x \to c} f(x)$
$x \to 0^-$	> 0	< 0	< 0	> 0	$+\infty$
$x \to 0^+$	> 0	> 0	< 0	< 0	$-\infty$
$x \to 4^-$	> 0	> 0	< 0	< 0	$-\infty$
$x \to 4^+$	> 0	> 0	> 0	> 0	$+\infty$

To check for a horizontal asymptote:

$$\lim_{x\to\infty} \frac{1}{x^2 - 4x} = \lim_{x\to\infty} \frac{1}{x^2\left(1 - \frac{4}{x}\right)}$$

$$= \frac{\lim_{x\to\infty} (1)}{\lim_{x\to\infty} \left(x^2\left(1 - \frac{4}{x}\right)\right)}$$

$$= \frac{\lim_{x\to\infty} (1)}{\lim_{x\to\infty} (x^2) \times \lim_{x\to\infty} \left(1 - \frac{4}{x}\right)}$$

$$= \lim_{x\to\infty} \frac{1}{x^2} \times \frac{1}{1 + 0}$$

$$= 0$$

Similarly, $\lim_{x\to-\infty} \frac{1}{x^2 - 4x} = 0$, so $y = 0$ is a horizontal asymptote of the function.

Since $y = 0$ and $x = 0$ are both asymptotes of the function, it has no x- or y- intercepts.

The derivative is

$$f'(x) = \frac{(x^2 - 4x) - (1)(2x - 4)}{(x^2 - 4x)^2}$$

$$= \frac{4 - 2x}{(x^2 - 4x)^2}$$, and the second derivative is

$$f''(x) = \frac{(x^2 - 4x)^2(-2) - (4 - 2x)(2(x^2 - 4x)(2x - 4))}{(x^2 - 4x)^4}$$

$$= \frac{-2x^2 + 8x + 8x^2 - 32x + 32}{(x^2 - 4x)^3}$$

$$= \frac{6x^2 - 24x + 32}{(x^2 - 4x)^3}$$

Letting $f'(x) = 0$ shows that $x = 2$ is a critical point of the function. The inflection points can be found by letting $f''(x) = 0$, so

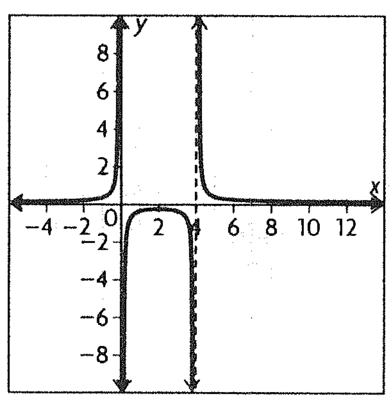
$2(3x^2 - 12x + 16) = 0$

$$x = \frac{12 \pm \sqrt{(-12)^2 - 4(3)(16)}}{2(3)}$$

$$= \frac{12 \pm \sqrt{-48}}{6}$$

This equation has no real solutions, so the graph of f has no inflection points.

x	$x < 0$	$0 < x < 2$	$x = 0$	$2 < x < 4$	$x > 4$
$f'(x)$	+	+	0	−	−
Graph	Inc.	Inc.	Local Max	Dec.	Dec.
$f''(x)$	+	−	−	−	+
Concavity	Up	Down	Down	Down	Up



g. $y = \dfrac{6x^2 - 2}{x^3}$

$= \dfrac{6}{x} - \dfrac{2}{x^3}$

There is a discontinuity at $x = 0$.

$\lim_{x \to 0} \dfrac{6x^2 - 2}{x^3} = \infty$ and $\lim_{x \to 0^-} \dfrac{6x^2 - 2}{x^3} = -\infty$

The y-axis is a vertical asymptote. There is no y-intercept. The x-intercept is a vertical asymptote. There is no y-intercept. The x-intercept is $\pm\frac{1}{\sqrt{3}}$.

$\dfrac{dy}{dx} = -\dfrac{6}{x^2} + \dfrac{6}{x^4} = \dfrac{-6x^2 + 6}{x^4}$

$\dfrac{dy}{dx} = 0$ when $6x^2 = 6$

$x = \pm 1$

Interval	$x < -1$	$x = -1$	$-1 < x < 0$	$0 < x < 1$	$x = 1$	$x > 1$
$\dfrac{dy}{dx}$	<0	$=0$	>0	>0	$=0$	<0
Graph of $y = f(x)$	Decreasing	Local Min	Increasing	Increasing	Local Max	Decreasing

There is a local minimum at $(-1, -4)$ and a local maximum at $(1, 4)$.

$\dfrac{d^2y}{dx^2} = \dfrac{12}{x^3} = \dfrac{24}{x^3} = \dfrac{12x^2 - 24}{x^3}$

For possible points of inflection, we solve $\dfrac{d^2y}{dx^2} = 0$ $(x^3 \neq 0)$:

$12x^2 = 24$

$x = \pm\sqrt{2}.$

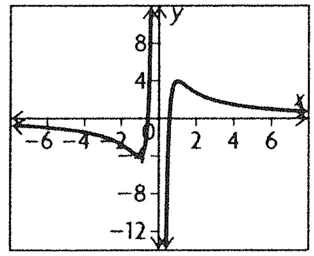
Interval	$x < -\sqrt{2}$	$x < -\sqrt{2}$	$-\sqrt{2} < x < 0$	$0 < x < \sqrt{2}$	$x = \sqrt{2}$	$x > \sqrt{2}$
$\dfrac{d^2y}{dx^2}$	<0	$=0$	>0	<0	$=0$	>0
Graph of $y = f(x)$	Concave Down	Point of Inflection	Concave Up	Concave Down	Point of Inflection	Convave Up

There are points of inflection at $\left(-\sqrt{2}, -\frac{5}{\sqrt{2}}\right)$ and $\left(\sqrt{2}, \frac{5}{\sqrt{2}}\right)$.

$\lim_{x \to \infty} \dfrac{6x^2 - 2}{x^3} = \lim_{x \to \infty} \dfrac{\frac{6}{x} - \frac{2}{x^3}}{1} = 0$

$\lim_{x = -\infty} \dfrac{\frac{6}{x} - \frac{2}{x^3}}{x} = 0$

The x-axis is a horizontal asymptote.



h. $y = \dfrac{x + 3}{x^2 - 4}$

There are discontinuities at $x = -2$ and at $x = 2$.

$\lim_{x \to -2} \left(\dfrac{x + 3}{x^2 - 4}\right) = \infty$ and $\lim_{x \to 2} \left(\dfrac{x + 3}{x^2 - 4}\right) = -\infty$

$\lim_{x \to 2} \left(\dfrac{x + 3}{x^2 - 4}\right) = -\infty$ and $\lim_{x \to 2^+} \left(\dfrac{x + 3}{x^2 - 4}\right) = \infty$

There are vertical asymptotes at $x = -2$ and $x = 2$.

When $x = 0$, $y = -\frac{3}{4}$. The x-intercept is -3.

$\dfrac{dy}{dx} = \dfrac{(1)(x^2 - 4) - (x + 3)(2x)}{(x^2 - 4)^2}$

$= \dfrac{-x^2 - 6x - 4}{(x^2 - 4)^2}$

For critical values, we solve $\dfrac{dy}{dx} = 0$:

$x^2 + 6x + 4 = 0$

$x = -\dfrac{6 \pm \sqrt{36 - 16}}{2}$

$= -3 \pm \sqrt{5}$

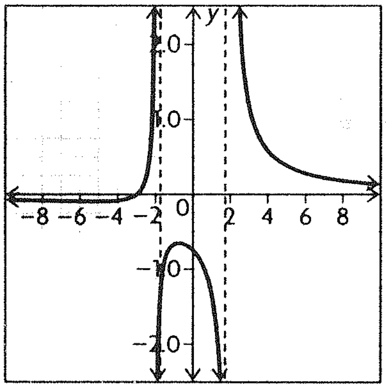
$\doteq -5.2$ or -0.8.

Interval	$x < -5.2$	$x = -5.2$	$-5.2 < x < -2$	$-2 < x < -0.8$	$x = -0.8$	$-0.8 < x < 2$	$x > 2$
$\dfrac{dy}{dx}$	< 0	$= 0$	> 0	> 0	$= 0$	< 0	< 0
Graph of y	Decreasing	Local Min	Increasing	Increasing	Local Max	Decreasing	Decreasing

$\lim_{x \to \infty} y = \lim_{x \to \infty} \left(\dfrac{\frac{1}{x} + \frac{3}{x^2}}{1 - \frac{4}{x^2}}\right) = 0$

$\lim_{x \to \infty} \left(\dfrac{\frac{1}{x} + \frac{3}{x^2}}{1 - \frac{4}{x^2}}\right) = 0$

The x-axis is a horizontal asymptote.



i. $y = \dfrac{x^2 - 3x + 6}{x - 1}$

$= x - 2 + \dfrac{4}{x - 1}$

$$\begin{array}{r|l} & x - 2 \\ x - 1 & x^2 - 3x + 6 \\ & \underline{x^2 - x} \\ & -2x + 6 \\ & \underline{-2x + 2} \\ & 4 \end{array}$$

There is a discontinuity at $x = 1$.

$\lim\limits_{x \to 1^-} \left(\dfrac{x^2 - 3x + 6}{x - 1}\right) = -\infty$

$\lim\limits_{x \to 1^+} \left(\dfrac{x^2 - 3x + 6}{x - 1}\right) = \infty$

Thus, $x = 1$ is a vertical asymptote.

The y-intercept is -6.

There are no x-intercepts ($x^2 - 3x + 6 > 0$ for all x in the domain).

$\dfrac{dy}{dx} = 1 - \dfrac{4}{(x - 1)^2}$

For critical values, we solve $\dfrac{dy}{dx} = 0$:

$1 - \dfrac{4}{(x - 1)^2} = 0$

$(x - 1)^2 = 4$

$x - 1 = \pm 2$

$x = -1$ or $x = 3$.

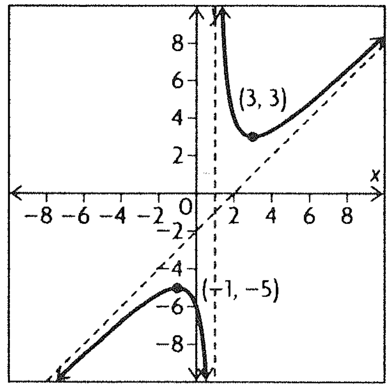
Interval	$x < -1$	$x = -1$	$-1 < x < 1$	$1 < x < 3$	$x = 3$	$x > 3$
$\frac{dy}{dx}$	>0	=0	<0	<0	=0	>0
Graph of y	Increasing	Local Max	Decreasing	Decreasing	Local Min	Increasing

$\dfrac{d^2y}{d^2x} = \dfrac{8}{(x - 1)^3}$

For $x < 1$, $\dfrac{d^2y}{dx^2} < 0$ and y is always concave down.

For, $x > 1$, $\dfrac{d^2y}{dx^2} > 0$ and y is always concave up.

The line $y = x - 2$ is an oblique asymptote.



j. This function is continuous everywhere, so it has no vertical asymptotes. It also has no horizontal asymptote, because $\lim\limits_{x \to \infty} (x - 4)^{\frac{2}{3}} = \infty$ and $\lim\limits_{x \to -\infty} (x - 4)^{\frac{2}{3}} = \infty$.

The x-intercept of the function is found by letting $f(x) = 0$, which gives

$(x - 4)^{\frac{2}{3}} = 0$

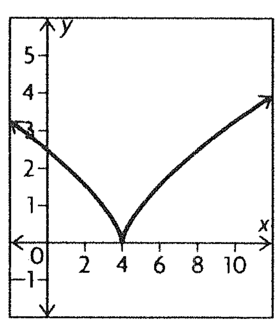
$x = 4$

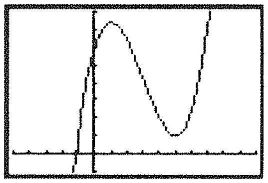
The y-intercept is found by letting $x = 0$, which gives $f(0) = (0 - 4)^{\frac{2}{3}} \doteq 2.5$.

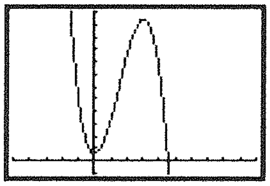
The derivative of the function is

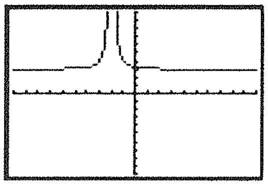
$f'(x) = \left(\frac{2}{3}\right)(x - 4)^{-\frac{1}{3}}$ and the second derivative is $f''(x) = \left(-\frac{2}{9}\right)(x - 4)^{-\frac{4}{3}}$. Neither of these derivatives has a zero, but each is undefined for $x = 4$, so it is a critical value and a possible point of inflection.

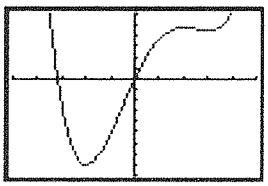
x	$x < 4$	$x = 4$	$x > 4$
f'(x)	−	Undefined	+
Graph	Dec.	Local Min	Inc.
f''(x)	−	Undefined	−
Concavity	Down	Undefined	Down

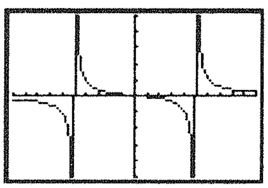


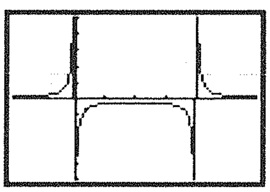
5. a. 

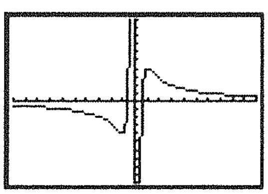
b. 

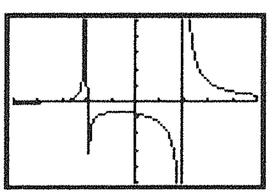
c. 

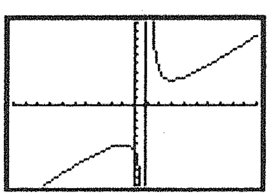
d. 

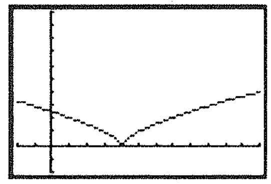
e. 

f. 

g. 

h. 

i. 

j. 

6. $y = ax^3 + bx^2 + cx + d$

Since $(0, 0)$ is on the curve $d = 0$:

$\frac{dy}{dx} = 3ax^2 + 2bx + c$

At $x = 2$, $\frac{dy}{dx} = 0$.

Thus, $12a + 4b + c = 0$.

Since $(2, 4)$ is on the curve, $8a + 4b + 2c = 4$ or $4a + 2b + c = 2$.

$$\frac{d^2y}{dx^2} = 6ax + 2b$$

Since $(0, 0)$ is a point of inflection, $\frac{d^2y}{dx^2} = 0$ when $x = 0$.

Thus, $2b = 0$

$b = 0$.

Solving for a and c:

$12a + c = 0$

$4a + c = 2$

$8a = -2$

$a = -\frac{1}{4}$

$c = 3$.

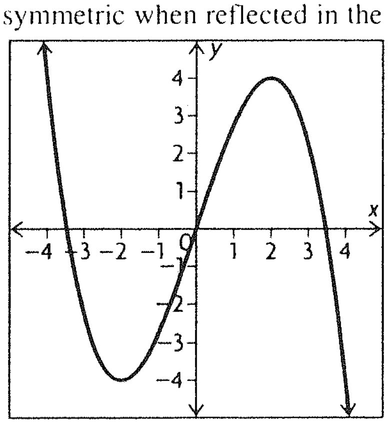
The cubic polynomial is $y = -\frac{1}{4}x^3 + 3x$.

The y-intercept is 0. The x-intercepts are found by setting $y = 0$:

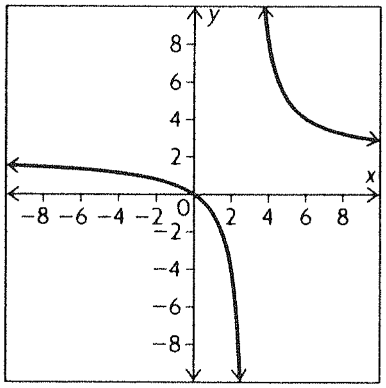
$-\frac{1}{4}x(x^2 - 12) = 0$

$x = 0$, or $x = \pm 2\sqrt{3}$.

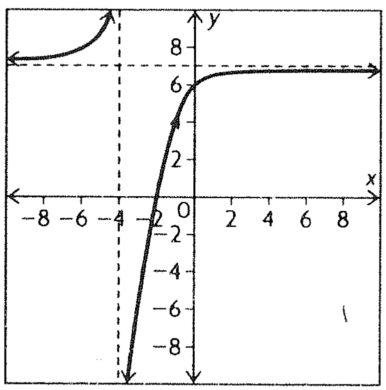
Let $y = f(x)$. Since $f(-x) = \frac{1}{4}x^3 - 3x = -f(x)$, $f(x)$ is an odd function. The graph of $y = f(x)$ is symmetric when reflected in the origin.



7. a. Answers may vary. For example:



b. Answers may vary. For example:



8. $f(x) = \dfrac{k - x}{k^2 + x^2}$

There are no discontinuities.

The y-intercept is $\frac{1}{k}$ and the x-intercept is k.

$$f'(x) = \frac{(-1)(k^2 + x^2) - (k - x)(2x)}{(k^2 + x^2)^2}$$

$$= \frac{x^2 - 2kx - k^2}{(k^2 + x^2)^2}$$

For critical points, we solve $f'(x) = 0$:

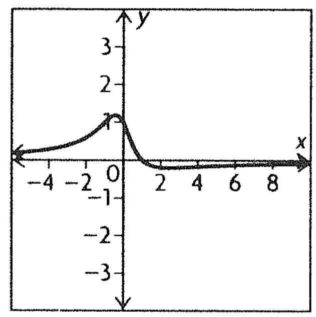
$$x^2 - 2kx - k^2 = 0$$
$$x^2 - 2kx + k^2 = 2k^2$$
$$(x - k)^2 = 2k^2$$
$$x - k = \pm\sqrt{2}k$$
$$x = \left(1 + \sqrt{2}\right)k \text{ or } x = \left(1 - \sqrt{2}\right)k.$$

Interval	$x < -0.41k$	$x = 0.41k$	$-0.41k < x < 2.41k$	$x = 2.41k$	$x > 2.41k$
f(x)	>0	<0	<0	=0	>0
Graph of f(x)	Increasing	Local Max	Decreasing	Local Min	Increasing

$$\lim_{x\to\infty}\left(\frac{k - x}{k^2 + x^2}\right) = \lim_{x\to\infty}\left(\frac{\frac{k}{x^2} - \frac{1}{x}}{\frac{k^2}{x^2} + 1}\right) = 0$$

$$\lim_{x\to\infty}\left(\frac{\frac{k}{x^2} - \frac{1}{x}}{\frac{k^2}{x^2} + 1}\right) = 0$$

Hence, the x-axis is a horizontal asymptote.



9. $g(x) = x^{\frac{1}{3}}(x + 3)^{\frac{2}{3}}$

There are no discontinuities.

$$g'(x) = \frac{1}{3}x^{-\frac{2}{3}}(x + 3)^{\frac{2}{3}} + x^{\frac{1}{3}}\left(\frac{2}{3}\right)(x + 3)^{-\frac{1}{3}}(1)$$

$$= \frac{x + 3 + 2x}{3x^{\frac{2}{3}}(x + 3)^{\frac{1}{3}}} = \frac{3(x + 1)}{3x^{\frac{2}{3}}(x + 3)^{\frac{1}{3}}}$$

$$= \frac{x + 1}{3x^{\frac{2}{3}}(x + 3)^{\frac{1}{3}}}$$

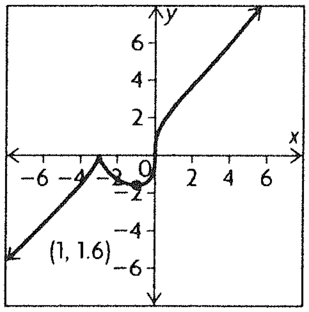
$g'(x) = 0$ when $x = -1$.

$g'(x)$ doesn't exist when $x = 0$ or $x = -3$.

Interval	$x < -3$	$x = -3$	$-3 < x < -1$	$x = -1$	$-1 < x < 0$	$x = 0$	$x < 0$
$g'(x)$	>0	Does not Exist	<0	=0	>0	Does not Exist	>0
Graph of g(x)	Increasing	Local Max	Decreasing	Local Min	Increasing		Increasing

There is a local maximum at $(-3, 0)$ and a local minimum at $(-1, -1.6)$. The second derivative is algebraically complicated to find.

Interval	$x < -3$	$x = -3$	$-3 < x < 0$	$x = 0$	$x > 0$
$g''(x)$	>0	Does Not Exist	>0	Does Not Exist	>0
Graph $g''(x)$	Concave Down	Cusp	Concave Up	Point of Inflection	Concave Down



10. a. $f(x) = \dfrac{x}{\sqrt{x^2 + 1}}$

$= \dfrac{x}{|x|\sqrt{1 + \dfrac{1}{x^2}}}$

$\lim\limits_{x \to \infty} f(x) = \lim\limits_{x \to \infty} \dfrac{x}{x\sqrt{1 + \dfrac{1}{x^2}}}$, since $x > 0$

$= \lim\limits_{x \to \infty} \dfrac{1}{\sqrt{1 + \dfrac{1}{x^2}}}$

$= 1$

$y = 1$ is a horizontal asymptote to the right-hand branch of the graph.

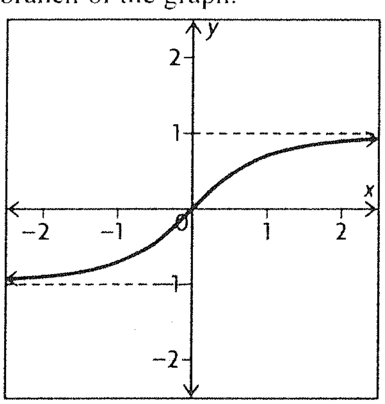
$\lim\limits_{x \to -\infty} f(x) = \lim\limits_{x \to -\infty} \dfrac{x}{-x\sqrt{1 + \dfrac{1}{x^2}}}$, since $|x| = -x$

for $x < 0$

$= \lim\limits_{x \to -\infty} \dfrac{x}{-\sqrt{1 + \dfrac{1}{x^2}}}$

$= -1$

$y = -1$ is a horizontal asymptote to the left-hand branch of the graph.



b. $g(t) = \sqrt{t^2 + 4t} - \sqrt{t^2 + t}$

$= \dfrac{(\sqrt{t^2 + 4t} - \sqrt{t^2 + t})(\sqrt{t^2 + 4t} + \sqrt{t^2 + t})}{\sqrt{t^2 + 4t} + \sqrt{t^2 + t}}$

$= \dfrac{3t}{\sqrt{t^2 + 4t} + \sqrt{t^2 + t}}$

$= \dfrac{3t}{|t|\sqrt{1 + \dfrac{4}{t}} + |t|\sqrt{1 + \dfrac{1}{t}}}$

$\lim\limits_{x \to \infty} g(t) = \frac{3}{2} = \frac{3}{2}$, since $|t| = t$ for $t > 0$

$\lim\limits_{x \to -\infty} g(t) = \frac{3}{-1-1} = -\frac{3}{2}$, since $|t| = -t$ for $t < 0$

$y = \frac{3}{2}$ and $y = -\frac{3}{2}$ are horizontal asymptotes.

11. $y = ax^3 + bx^2 + cx + d$

$\dfrac{dy}{dx} = 3ax^2 + 2bx + c$

$\dfrac{d^2y}{dx^2} = 6ax + 2b = 6a\left(x + \dfrac{b}{3a}\right)$

For possible points of inflection, we solve $\frac{d^2y}{dx^2} = 0$:

$x = -\dfrac{b}{3a}$.

The sign of $\frac{d^2y}{dx^2}$ changes as x goes from values less than $-\frac{b}{3a}$ to values greater than $-\frac{b}{3a}$. Thus, there is a point of inflection at $x = -\frac{b}{3a}$.

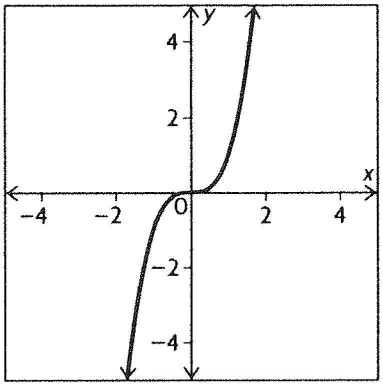
At $x = \dfrac{b}{3a}$, $\dfrac{dy}{dx} = 3a\left(-\dfrac{b}{3a}\right)^2 + 2b\left(-\dfrac{b}{3a}\right) + c$

$= c - \dfrac{b^2}{3a}$.

Review Exercise, pp. 216–219

1. a. i. $x < 1$
ii. $x > 1$
iii. $(1, 20)$
b. i. $x < -3$, $-3 < x < 1$, $x > 6.5$
ii. $1 < x < 3$, $3 < x < 6.5$
iii. $(1, -1)$, $(6.5, -1)$

2. No. A counter example is sufficient to justify the conclusion. The function $f(x) = x^3$ is always increasing yet the graph is concave down for $x < 0$ and concave up for $x > 0$.



3. a. $f(x) = -2x^3 + 9x^2 + 20$
$f'(x) = -6x^2 + 18x$
For critical values, we solve:
$f'(x) = 0$
$-6x(x - 3) = 0$
$x = 0$ or $x = 3$.
$f''(x) = -12x + 18$
Since $f''(0) = 18 > 0$, $(0, 20)$ is a local minimum point. The tangent to the graph of $f(x)$ is horizontal at $(0, 20)$. Since $f''(3) = -18 < 0$, $(3, 47)$ is a local maximum point. The tangent to the graph of $f(x)$ is horizontal at $(3, 47)$.
b. $f(x) = x^4 - 8x^3 + 18x^2 + 6$
$f(x) = 4x^3 - 24x^2 + 36x$
$f(x) = 4x(x^2 - 6x + 9)$
$f(x) = 4x(x - 3)^2$
Let $f(x) = 0$:
$4x(x - 3)^2 = 0$
$x = 0$ or $x = 3$
The critical points are $(0, 6)$ and $(3, 33)$.

x	$x < 0$	0	$0 < x < 3$	3	$x > 3$
$\frac{dy}{dx}$	−	0	+	0	+
Graph	Dec	Local Min	Inc		Inc

Local minimum at $(0, 6)$
$(3, 33)$ is neither a local maximum nor a local minimum.
The tangent is parallel to the x-axis at both points because the derivative is defined at both points.
c. $h(x) = \dfrac{x - 3}{x^2 + 7}$

$$h(x) = \frac{(1)(x^2 + 7) - (x - 3)(2x)}{(x^2 + 7)^2}$$
$$= \frac{7 + 6x - x^2}{(x^2 + 7)^2}$$
$$= \frac{(7 - x)(1 + x)}{(x^2 + 7)^2}$$

Since $x^2 + 7 > 0$ for all x, the only critical values occur when $h'(x) = 0$. The critical values are $x = 7$ and $x = -1$.

Interval	$x < -1$	$x = -1$	$-1 < x < 7$	$x = 7$	$x > 7$
$h'(x)$	< 0	$= 0$	> 0	$= 0$	< 0
Graph of $h(t)$	Decreasing	Local Min	Increasing	Local Max	Decreasing

There is a local minimum at $(-1, -\frac{1}{2})$ and a local maximum at $(7, \frac{1}{14})$. At both points, the tangents are parallel to the x-axis.
d) $g(x) = (x - 1)^{\frac{1}{3}}$

$g'(x) = \dfrac{1}{3}(x - 1)^{-\frac{2}{3}}$

Let $g'(x) = 0$:

$\dfrac{1}{3}(x - 1)^{-\frac{2}{3}} = 0$

There are no solutions, but $g'(x)$ is undefined for $x = 1$, so the point $(1, 0)$ is a critical point.

x	$x < 1$	1	$x > 1$
$f'(x)$	+	Undefined	+
Graph	Inc.		Inc.

$(1, 0)$ is neither a local maximum nor a local minimum.
The tangent is not parallel to the x-axis because it is not defined for $x = 1$.
4. a. $a < x < b$, $x > e$
b. $b < x < c$
c. $x < a$, $d < x < e$
d. $c < x < d$
5. a. $y = \dfrac{2x}{x - 3}$
There is a discontinuity at $x = 3$.

$$\lim_{x \to 3^-}\left(\frac{2x}{x - 3}\right) = -\infty \text{ and } \lim_{x \to 3^+}\left(\frac{2x}{x - 3}\right) = \infty$$

Therefore, $x = 3$ is a vertical asymptote.
b. $g(x) = \dfrac{x - 5}{x + 5}$
There is a discontinuity at $x = -5$.

$$\lim_{x \to -5^-}\left(\frac{x - 5}{x + 5}\right) = \infty \text{ and } \lim_{x \to -5^+}\left(\frac{x - 5}{x + 5}\right) = -\infty$$

Therefore, $x = -5$ is a vertical asymptote.
c. $f(x) = \dfrac{x^2 - 2x - 15}{x + 3}$

$$= \frac{(x+3)(x-5)}{x+3}$$
$$= x - 5, x \neq -3$$

There is a discontinuity at $x = -3$.

$\lim_{x \to -3^-} f(x) = -8$ and $\lim_{x \to -3^+} f(x) = -8$

There is a hole in the graph of $y = f(x)$ at $(-3, -8)$.

d. $g(x) = \frac{5}{x^2 - x - 20}$

$$g(x) = \frac{5}{(x-5)(x+4)}$$

To find vertical asymptotes, set the denominator equal to 0:

$(x-5)(x+4) = 0$

$x = -4$ or $x = 5$

Vertical asymptotes at $x = -4$ and $x = 5$

$$\lim_{x \to -4^-} \frac{5}{(x-5)(x+4)} = \infty$$
$$\lim_{x \to -4^+} \frac{5}{(x-5)(x+4)} = -\infty$$
$$\lim_{x \to 5^-} \frac{5}{(x-5)(x+4)} = -\infty$$
$$\lim_{x \to 5^+} \frac{5}{(x-5)(x+4)} = \infty$$

6. $y = x^3 + 5$

$y' = 3x^2$

$y'' = 6x$

Let $y'' = 0$

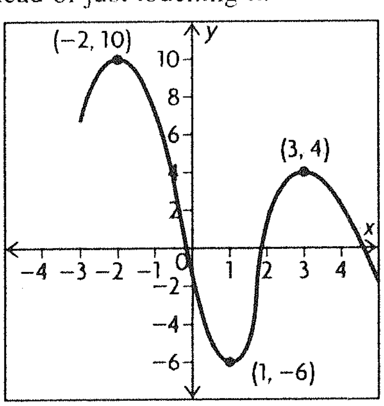
$6x = 0$

$x = 0$

The point of inflection is $(0, 5)$

Since the derivative is 0 at $x = 0$, the tangent line is parallel to the x-axis at that point. Because the derivative is always positive, the function is always increasing and therefore must cross the tangent line instead of just touching it.

7.

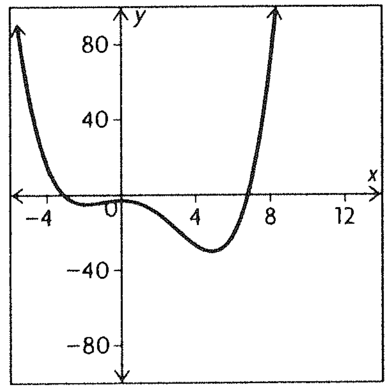


8. a. i. Concave up: $-1 < x < 3$

Concave down: $x < -1, 3 < x$

ii. Points of inflection at $x = -1$ and $x = 3$

iii.

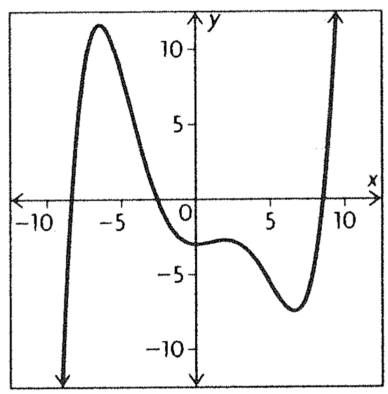


b. i. Concave up: $-4.5 < x < 1, 5 < x$

Concave down: $x < -4.5, 1 < x < 5$

ii. Points of inflection at $x = -4.5, x = 1$, and $x = 5$

iii.



9. a. $g(x) = \frac{ax + b}{(x-1)(x-4)}$

$$= \frac{ax + b}{x^2 - 5x + 4}$$

$$g'(x) = \frac{a(x^2 - 5x + 4) - (ax + b)(2x - 5)}{(x^2 - 5x + 4)^2}$$

Since the tangent at $(2, -1)$ has slope 0, $g'(2) = 0$.

Hence, $\frac{-2a + 2a + b}{4} = 0$ and $b = 0$.

Since $(2, -1)$ is on the graph of $g(x)$:

$$-1 = \frac{2a + b}{-2}$$

$2a + 0 = 2$

$a = 1$.

Therefore $g(x) = \frac{x}{(x-1)(x-4)}$.

b. There are discontinuities at $x = 1$ and $x = 4$.

$\lim_{x \to 1^-} g(x) = \infty$ and $\lim_{x \to 1^+} g(x) = -\infty$

$\lim_{x \to 4^-} g(x) = -\infty$ and $\lim_{x \to 4^+} g(x) = \infty$

$x = 1$ and $x = 4$ are vertical asymptotes.

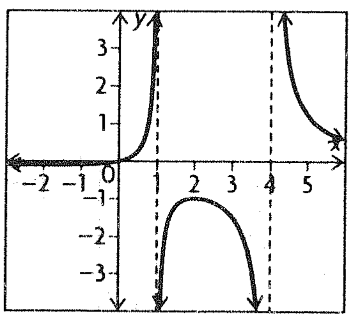
The y-intercept is 0.

$$g'(x) = \frac{4 - x^2}{(x^2 - 5x + 4)^2}$$

$g'(x) = 0$ when $x = \pm 2$.

Interval	$x < -2$	$x = -2$	$-2 < x < 1$	$1 < x < 2$	$x = 2$	$2 < x < 4$	$x > 4$
$g'(x)$	< 0	0	> 0	> 0	0	< 0	< 0
Graph of $g(x)$	Decreasing	Local Min	Increasing	Increasing	Local Max	Decreasing	Decreasing

There is a local minimum at $\left(-2, -\frac{1}{9}\right)$ and a local maximum at $(2, -1)$.



10. a. $y = x^4 - 8x^2 + 7$

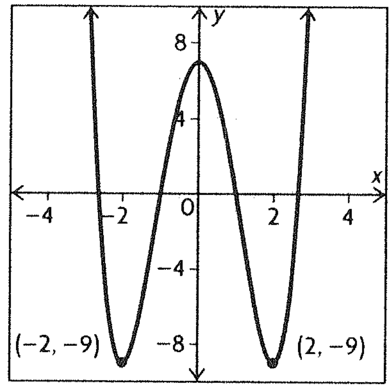
This is a fourth degree polynomial and is continuous for all x. The y-intercept is 7.

$$\frac{dy}{dx} = 4x^3 - 16x$$
$$= 4x(x - 2)(x + 2)$$

The critical values are $x = 0$, -2 and 2.

Interval	$x < -2$	$x = -2$	$-2 < x < 0$	$x = 0$	$0 < x < 2$	$x = 2$	$x > 2$
$\frac{dy}{dx}$	< 0	$= 0$	> 0	$= 0$	< 0	$= 0$	> 0
Graph of y	Decreasing	Local Min	Increasing	Local Max	Decreasing	Local Min	Increasing

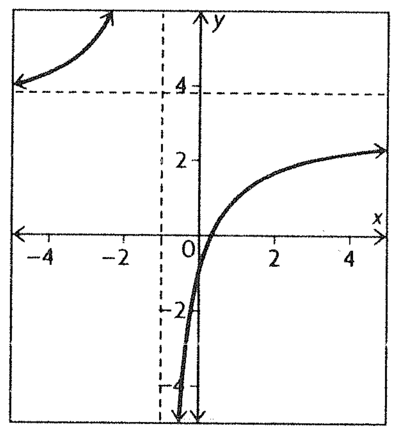
There are local minima at $(-2, -9)$ and at $(2, -9)$, and a local maximum at $(0, 7)$.



b. $f(x) = \dfrac{3x - 1}{x + 1}$

$$= 3 - \frac{4}{x + 1}$$

From experience, we know the graph of $y = -\frac{1}{x}$ is



The graph of the given function is just a transformation of the graph of $y = -\frac{1}{x}$. The vertical asymptote is $x = -1$ and the horizontal asymptote is $y = 3$. The y-intercept is -1 and there is an x-intercept at $\frac{1}{3}$.

c. $g(x) = \dfrac{x^2 + 1}{4x^2 - 9}$

$$= \frac{x^2 + 1}{(2x - 3)(2x + 3)}$$

The function is discontinuous at $x = -\frac{3}{2}$ and at $x = \frac{3}{2}$.

$\lim_{x \to -\frac{3}{2}^-} g(x) = \infty$

$\lim_{x \to -\frac{3}{2}^+} g(x) = -\infty$

$\lim_{x \to \frac{3}{2}^-} g(x) = -\infty$

$\lim_{x \to -\frac{3}{2}^-} g(x) = \infty$

Hence, $x = -\frac{3}{2}$ and $x = \frac{3}{2}$ are vertical asymptotes.
The y-intercept is $-\frac{1}{9}$.

$$g'(x) = \frac{2x(4x^2 - 9) - (x^2 + 1)(8x)}{(4x^2 - 9)^2} = \frac{-26x}{(4x^2 - 9)^2}$$

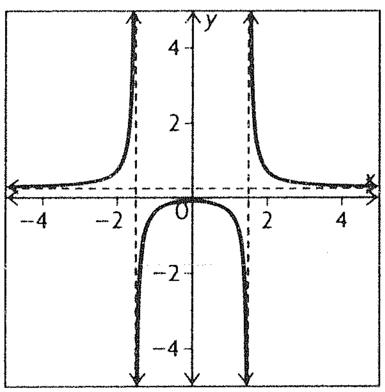
$g'(x) = 0$ when $x = 0$.

Interval	$x < -\frac{3}{2}$	$-\frac{3}{2} < x < 0$	$x = 0$	$0 < x < \frac{3}{2}$	$x > \frac{3}{2}$
$g'(x)$	> 0	> 0	$= 0$	< 0	< 0
Graph $g(x)$	Increasing	Increasing	Local Max	Decreasing	Decreasing

There is a local maximum at $\left(0, -\frac{1}{9}\right)$.

$$\lim_{x \to \infty} g(x) = \lim_{x \to \infty} \frac{1 + \frac{1}{x^2}}{4 - \frac{9}{x^2}} = \frac{1}{4} \text{ and } \lim_{x \to \infty} g(x) = \frac{1}{4}$$

Hence, $y = \frac{1}{4}$ is a horizontal asymptote.



d) $y = x(x - 4)^3$
This is a polynomial function, so there are no discontinuities and no asymptotes. The domain is $\{x \in \mathbf{R}\}$.
x-intercepts at $x = 0$ and $x = 4$
y-intercepts at $y = 0$
$y' = (x - 4)^3 + 3x(x - 4)^2$
$y' = (x - 4)^2(x - 4 + 3x)$
$y' = 4(x - 4)^2(x - 1)$
Let $y' = 0$:
$4(x - 4)^2(x - 1) = 0$
$x = 4$ or $x = 1$
The critical numbers are $(1, -27)$ and $(4, 0)$.

x	$x < 1$	1	$1 < x < 4$	4	$x > 4$
$\frac{dy}{dx}$	−	0	+	0	+
Graph	Dec	Local Min	Inc		Inc

Local minimum at $(1, -27)$
$(4, 0)$ is not a local extremum
$y'' = 4(2(x - 4)(x - 1) + (x - 4)^2)$

$$y'' = 4\left(2(x - 4)\left(x - 1 + \frac{x - 4}{2}\right)\right)$$

$$y'' = 8(x - 4)\left(\frac{3}{2}x - 3\right)$$

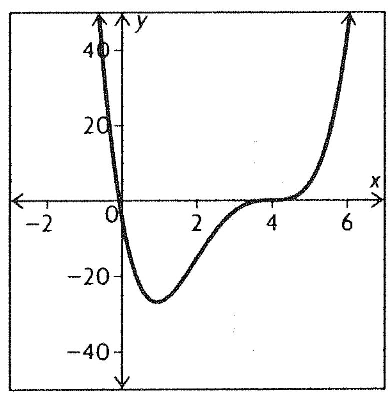
Let $y'' = 0$:

$$8(x - 4)\left(\frac{3}{2}x - 3\right) = 0$$

$x = 4$ or $x = 2$
The points of inflection are $(2, -16)$ and $(4, 0)$.

x	$x < 2$	2	$2 < x < 4$	4	$x > 4$
$\frac{dy}{dx}$	+	0	−	0	+
Graph	c. up	point of inflection	c. down	point of inflection	c. up

The graph has a local minimum at $(1, -27)$ and points of inflection at $(2, -16)$ and $(4, 0)$, with x-intercepts of 0 and 4 and a y-intercept of 0.



e. $h(x) = \dfrac{x}{x^2 - 4x + 4}$

$$= \frac{x}{(x - 2)^2} = x(x - 2)^{-2}$$

There is a discontinuity at $x = 2$
$\lim_{x \to 2^-} h(x) = \infty = \lim_{x \to 2^+} h(x)$

Thus, $x = 2$ is a vertical asymptote. The y-intercept is 0.

$$h'(x) = (x-2)^{-2} + x(-2)(x-2)^{-3}(1)$$
$$= \frac{x-2-2x}{(x-2)^3}$$
$$= \frac{-2-x}{(x-2)^3}$$

$h'(x) = 0$ when $x = -2$.

Interval	$x < -2$	$x = -2$	$-2 < x < 2$	$x > 2$
$h'(x)$	< 0	$= 0$	> 0	< 0
Graph of $h(x)$	Decreasing	Local Min	Increasing	Decreasing

There is a local minimum at $\left(-2, -\frac{1}{8}\right)$.

$$\lim_{x\to\infty} h(x) = \lim_{x\to\infty} = \frac{\frac{1}{x}}{1 - \frac{4}{x} + \frac{4}{x^2}} = 0$$

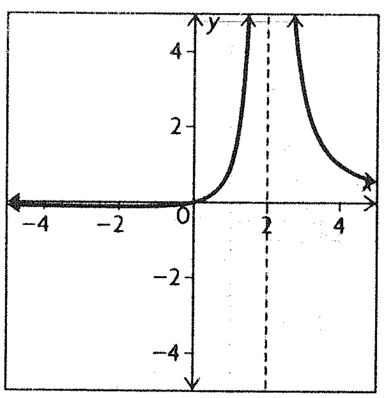
Similarly, $\lim_{x\to-\infty} h(x) = 0$

The x-axis is a horizontal asymptote.

$$h''(x) = -2(x-2)^{-3} - 2(x-2)^{-3} + 6x(x-2)^{-4}$$
$$= -4(x-2)^{-3} + 6x(x-2)^{-4}$$
$$= \frac{2x+8}{(x-2)^4}$$

$h''(x) = 0$ when $x = -4$

The second derivative changes signs on opposite sides $x = -4$, Hence $\left(-4, -\frac{1}{9}\right)$ is a point of inflection.



f. $$f(t) = \frac{t^2 - 3t + 2}{t - 3}$$
$$= t + \frac{2}{t-3}$$

Thus, $f(t) = t$ is an oblique asymptote. There is a discontinuity at $t = 3$.

$\lim_{t\to3^-} f(t) = -\infty$ and $\lim_{t\to3^+} f(t) = \infty$

Therefore, $x = 3$ is a vertical asymptote.

The y-intercept is $-\frac{2}{3}$.

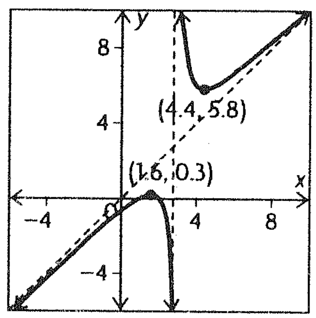
The x-intercepts are $t = 1$ and $t = 2$.

$$f'(t) = 1 - \frac{2}{(t-3)^2}$$

$$f'(t) = 0 \text{ when } 1 - \frac{2}{(t-3)^2} = 0$$
$$(t-3)^2 = 2$$
$$t - 3 = \pm\sqrt{2}$$
$$t = 3 \pm \sqrt{2}.$$

Interval	$t < 3 - \sqrt{2}$	$t = 3 - \sqrt{2}$	$3 - \sqrt{2} < t < 3$	$3 < t \le 3 + \sqrt{2}$	$t = 3 + \sqrt{2}$	$t > 3 + \sqrt{2}$
$f'(t)$	> 0	$= 0$	< 0	< 0	$= 0$	> 0
Graph of $f(t)$	Increasing	Local Max	Decreasing	Decreasing	Local Min	Increasing

(1.6, 0.2) is a local maximum and (4.4, 5.8) is a local minimum.



11. a. $$f(x) = \frac{2x+4}{x^2 - k^2}$$

$$f'(x) = \frac{2(x^2 - k^2) - (2x+4)(2x)}{(x^2 - k^2)^2}$$
$$= -\frac{2x^2 + 8x + 2k^2}{(x^2 - k^2)^2}$$

For critical values, $f'(x) = 0$ and $x \neq \pm k$:

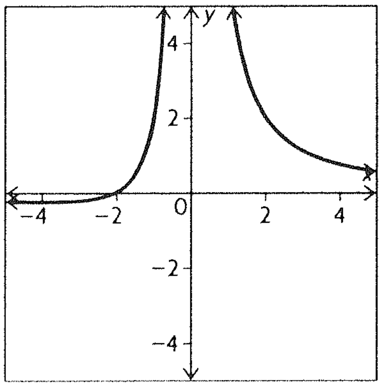
$$x^2 + 4x + k^2 = 0$$
$$x = \frac{-4 \pm \sqrt{16 - 4k^2}}{2}.$$

For real roots, $16 - 4k^2 \geq 0$
$$-2 \leq k \leq 2.$$

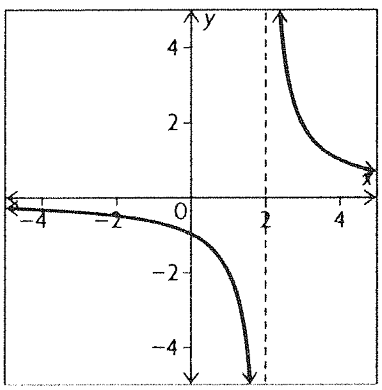
The conditions for critical points to exist are $-2 \leq k \leq 2$ and $x \neq \pm k$.

b. There are three different graphs that results for values of k chosen.

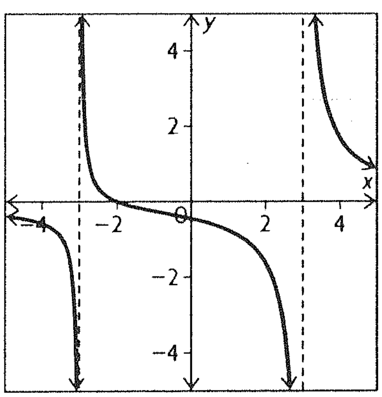
$k = 0$



$k = 2$



For all other values of k, the graph will be similar to that of 1(*i*) in Exercise 9.5.



12. a. $f(x) = \dfrac{2x^2 - 7x + 5}{2x - 1}$

$$f(x) = x - 3 + \frac{2}{2x - 1}$$

The equation of the oblique asymptote is $y = x - 3$.

$$\begin{array}{r} x - 3 \\ 2x - 1\overline{)2x^2 - 7x + 5} \\ \underline{2x^2 - x} \\ -6x + 5 \\ \underline{-6x + 3} \\ 2 \end{array}$$

$$\lim_{x\to\infty}[y - f(x)] = \lim_{x\to\infty}\left[x - 3 - \left(x - 3 + \frac{2}{2x - 1}\right)\right]$$
$$= \lim_{x\to\infty}\left[-\frac{2}{2x - 1}\right] = 0$$

b. $f(x) = \dfrac{4x^3 - x^2 - 15x - 50}{x^2 - 3x}$

$$f(x) = 4x + 11 + \frac{18x - 50}{x^2 - 3x}$$

$$\begin{array}{r} 4x + 11 \\ x^2 - 3x\overline{)4x^3 - x^2 - 15x - 50} \\ \underline{4x^3 - 12x^2} \\ 11x^2 - 15x \\ \underline{11x^2 - 33x} \\ 18x - 50 \end{array}$$

$$\lim_{x\to\infty}[y - f(x)]$$
$$= \lim_{x\to\infty}\left[4x + 11 - \left(4x + 11 + \frac{18x - 50}{x^2 - 3x}\right)\right]$$
$$= \lim_{x\to\infty}\left[\frac{\frac{18}{x} - \frac{50}{x^2}}{1 - \frac{3}{x}}\right]$$
$$= 0$$

13. $g(x) = (x^2 - 4)^2$

$g(x) = (x^2 - 4)(x^2 - 4)$

$g'(x) = 2x(x^2 - 4) + 2x(x^2 - 4)$

$g'(x) = 4x(x^2 - 4)$

$g'(x) = 4x(x - 2)(x + 2)$

Set $g'(x) = 0$

$0 = 4x(x - 2)(x + 2)$

$x = -2$ or $x = 0$ or $x = 2$

	$x < -2$	$-2 < x < 0$	$0 < x < 2$	$x > 2$
4x	−	−	+	+
x − 2	−	−	−	+
x + 2	−	+	+	+
Sign of g′(x)	(−)(−)(−) = −	(−)(−)(+) = +	(+)(−)(+) = −	(+)(+)(+) = +
Behaviour of g(x)	decreasing	increasing	decreasing	increasing

14. $f(x) = x^3 + \dfrac{3}{2}x^2 - 7x + 5, -4 \le x \le 3$

$f'(x) = 3x^2 + 3x - 7$

Set $f'(x) = 0$

$0 = 3x^2 + 3x - 7$

$$x = \frac{-3 \pm \sqrt{(3)^2 - 4(3)(-7)}}{2(3)}$$

$$x = \frac{-3 \pm \sqrt{93}}{6}$$

$x \doteq -2.107$ or $x \doteq 1.107$

$f'(x) = 3x^2 + 3x - 7$

$f''(x) = 6x + 3$

When $x = -2.107$,

$f''(-2.107) = 6(-2.107) + 3$

$f''(-2.107) = -9.642$

Since $f''(-2.107) < 0$, a local maximum occurs when $x = -2.107$.

when $x = 1.107$.

$f''(1.107) = 6(1.107) + 3$

$f''(1.107) = 9.642$

Since $f''(1.107) > 0$, a local minimum occurs when $x = (1.107)$.

when $x = -4$,

$$f(-4) = (-4)^3 + \frac{3}{2}(-4)^2 - 7(-4) + 5$$

$f(-4) = -64 + 24 + 28 + 5$

$f(-4) = -7$

when $x = -2.107$,

$$f(-2.107) = (-2.107)^3 + \frac{3}{2}(-2.107)^2 - 7(-2.107) + 5$$

$$f(-2.107) \doteq -9.353\,919 + 6.659\,173\,5 + 14.749 + 5$$

when $x = 1.107$.

$$f(1.107) = (1.107)^3 + \frac{3}{2}(1.107)^2 - 7(1.107) + 5$$

$f(1.107) \doteq 1.356\,572 + 1.838\,173\,5 - 7.749 + 5$

$f(1.107) \doteq 0.446$

when $x = 3$,

$$f(3) = (3)^3 + \frac{3}{2}(3)^2 - 7(3) + 5$$

$f(3) = 27 + 13.5 - 21 + 5$

$f(3) = 24.5$

Local Maximum: $(-2.107, 17.054)$

Local Minimum: $(1.107, 0.446)$

Absolute Maximum: $(3, 24.5)$

Absolute Minimum: $(-4, -7)$

15. $f(x) = 4x^3 + 6x^2 - 24x - 2$

Evaluate $y = 4(0)^3 + 6(0)^2 - 24(0) - 2$

$y = -2$

$f(x) = 4x^3 + 6x^2 - 24x - 2$

$f'(x) = 12x^2 + 12x - 24$

Set $f'(x) = 0$

$0 = 12x^2 + 12x - 24$

$0 = 12(x^2 + x - 2)$

$0 = 12(x - 1)(x + 2)$

$x = -2$ or $x = 1$

	$x < -2$	$-2 < x < 1$	$x > 1$
$12(x - 1)$	−	−	+
$x + 2$	−	+	+
Sign of $f'(x)$	$(-)(-) = +$	$(-)(+) = -$	$(+)(+) = +$
Behaviour of $f(x)$	increasing	decreasing	increasing
	maximum at $x = -2$		minimum at $x = 1$

when $x = -2$,

$f(-2) = 4(-2)^3 + 6(-2)^2 - 24(-2) - 2$

$f(-2) = -32 + 24 + 48 - 2$

$f(-2) = 38$

when $x = 1$,

$f(1) = 4(1)^3 + 6(1)^2 - 24(1) - 2$

$f(1) = 4 + 6 - 24 - 2$

$f(1) = -16$

Maximum: $(-2, 38)$ Minimum: $(1, -16)$

$f'(x) = 12x^2 + 12x - 24$

$f''(x) = 24x + 12$

Set $f''(x) = 0$

$0 = 24x + 12$

$x = -0.5$

	$x < -0.5$	$x > -0.5$
$f''(x) = 24x + 12$	−	+
$f(x)$	concave down	concave up
	point of inflection at $x = -0.5$	

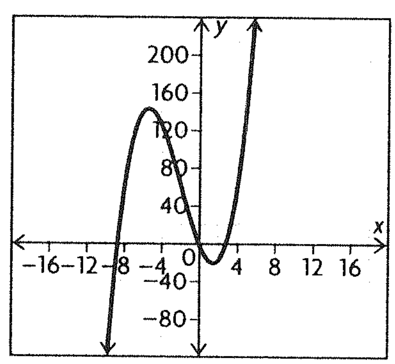
when $x = -0.5$,

$f(-0.5) = 4(-0.5)^3 + 6(-0.5)^2 - 24(-0.5) - 2$

$f(-0.5) = -0.5 + 1.5 + 12 - 2$

$f(-0.5) = 11$

Point of inflection: $(-0.5, 11)$



16. a. $p(x)$: oblique asymptote, because the highest degree of x in the numerator is exactly one degree higher than the highest degree of x in the denominator.
$q(x)$: vertical asymptotes at $x = -1$ and $x = 3$; horizontal asymptote at $y = 0$
$r(x)$: vertical asymptotes at $x = -1$ and $x = 1$; horizontal asymptote at $y = 1$
$s(x)$: vertical asymptote at $y = 2$.

b. $r(x) = \dfrac{x^2 - 2x - 8}{x^2 - 1}$
$= \dfrac{(x - 4)(x + 2)}{(x - 1)(x + 1)}$

The domain is $\{x | x \neq -1, 1, x \in \mathbf{R}\}$.
x-intercepts: $-2, 4$; y-intercept: 8
r has vertical asymptotes at $x = -1$ and $x = 1$.
$r(-1.001) = -2496.75$, so as $x \to -1^-$, $r(x) \to -\infty$
$r(-0.999) = 2503.25$, so as $x \to -1^+$, $r(x) \to \infty$
$r(0.999) = 4502.25$, so as $x \to 1^-$, $r(x) \to \infty$
$r(1.001) = -4497.75$, so as $x \to 1^+$, $r(x) \to -\infty$

$$\lim_{x \to -\infty} \frac{x^2 - 2x - 8}{x^2 - 1} = \lim_{x \to -\infty} \frac{\frac{x^2}{x^2} - \frac{2x}{x^2} - \frac{8}{x^2}}{\frac{x^2}{x^2} - \frac{1}{x^2}}$$
$$= \lim_{x \to -\infty} \frac{1 - \frac{2}{x} - \frac{8}{x^2}}{1 - \frac{1}{x^2}}$$
$$= \frac{1 - 0 - 0}{1 - 0}$$
$$= 1$$

So, $y = 1$ is a horizontal asymptote on the left.

$$\lim_{x \to \infty} \frac{x^2 - 2x - 8}{x^2 - 1} = \lim_{x \to \infty} \frac{\frac{x^2}{x^2} - \frac{2x}{x^2} - \frac{8}{x^2}}{\frac{x^2}{x^2} - \frac{1}{x^2}}$$
$$= \lim_{x \to \infty} \frac{1 - \frac{2}{x} - \frac{8}{x^2}}{1 - \frac{1}{x^2}}$$
$$= \frac{1 - 0 - 0}{1 - 0}$$
$$= 1$$

So, $y = 1$ is a horizontal asymptote on the right.

$$r'(x) = \frac{(x^2 - 1)(2x - 2) - (x^2 - 2x - 8)(2x)}{(x^2 - 1)^2}$$
$$= \frac{2x^3 - 2x^2 - 2x + 2 - (2x^3 - 4x^2 - 16x)}{(x^2 - 1)^2}$$
$$= \frac{2x^2 + 14x + 2}{(x^2 - 1)^2}$$
$$= \frac{2(x^2 + 7x + 1)}{(x^2 - 1)^2}$$

r' is defined for all values of x in the domain of r.
$r'(x) = 0$ for $x \doteq -0.15$ and $x \doteq -6.85$. $r'(1)$ and $r'(-1)$ do not exist.

	$x < -6.85$	$x = -6.85$	$-6.85 < x < -1$
$x^2 + 7x + 1$	+	0	−
$r'(x)$	+	0	−
	$x = -1$	$-1 < x < -0.15$	$x = -0.15$
$x^2 + 7x + 1$	−	−	0
$r'(x)$	undefined	−	0
	$-0.15 < x < 1$	$x = 1$	$x > 1$
$x^2 + 7x + 1$	+	+	+
$r'(x)$	+	undefined	+

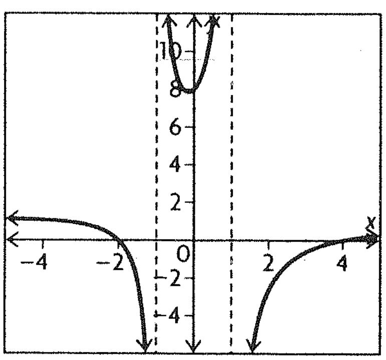
r is increasing when $x < -6.85$, $-0.15 < x < 1$, and $x > 1$. r is decreasing when $-6.85 < x < -1$ and $-1 < x < -0.15$. r has a maximum turning point at $x = -6.85$ and a minimum turning point at $x = -0.15$.

$$r''(x) = \frac{(x^2 - 1)^2(4x + 14)}{(x^2 - 1)^4} - \frac{(2x^2 + 14x + 2)[2(x^2 - 1)(2x)]}{(x^2 - 1)^4}$$
$$= \frac{(x^2 - 1)(4x + 14) - 4x(2x^2 + 14x + 2)}{(x^2 - 1)^3}$$
$$= \frac{4x^3 + 14x^2 - 4x - 14 - 8x^3 - 56x^2 - 8x}{(x^2 - 1)^3}$$
$$= \frac{-4x^3 - 42x^2 - 12x - 14}{(x^2 - 1)^3}$$
$$= \frac{-2(2x^3 + 21x^2 + 6x + 7)}{(x^2 - 1)^3}$$

r'' is defined for all values of x in the domain of r.
$r''(x) = 0$ for $x \doteq -10.24$. This is a possible point of inflection. $r''(1)$ and $r''(-1)$ do not exist.

	$x < -10.24$	$x = 10.24$
$-2(2x^3 + 21x^2 + 6x + 7)$	+	0
$(x^2 - 1)^3$	+	+
$r''(x)$	+	0
	$-10.24 < x < -1$	$x = -1$
$-2(2x^3 + 21x^2 + 6x + 7)$	−	−
$(x^2 - 1)^3$	+	0
$r''(x)$	−	undefined
	$-1 < x < 1$	$x = 1$
$-2(2x^3 + 21x^2 + 6x + 7)$	−	−
$(x^2 - 1)^3$	−	0
$r''(x)$	+	undefined
	$x > 1$	
$-2(2x^3 + 21x^2 + 6x + 7)$	−	
$(x^2 - 1)^3$	+	
$r''(x)$	−	

The graph is concave up for $x < -10.24$ and $-1 < x < 1$. The graph is concave down for $-10.24 < x < -1$ and $x > 1$. The graph changes concavity at $x = -10.24$. This is a point of inflection with coordinates $(-10.24, 1.13)$.
$r(-6.85) = 1.15$ and $r(-0.15) = 7.85$. The graph has a local maximum point at $(-6.85, 1.15)$ and a local minimum point at $(-0.15) = 7.85$.



17. The domain is $\{x | x \neq 0, x \in \mathbf{R}\}$: x-intercept: -2, y-intercept: 8; f has a vertical asymptote at $x = 0$. $f(-0.001) = -7999.99$, so $f(x) \to -\infty$ as $x \to 0^-$. $f(0.001) = 8000.00$, so $f(x) \to \infty$ as $x \to 0^+$. There are no horizontal asymptotes.

$$f'(x) = \frac{x(3x^2) - (x^3 + 8)(1)}{x^2}$$
$$= \frac{3x^3 - x^3 - 8}{x^2}$$
$$= \frac{2x^3 - 8}{x^2}$$

$f'(x)$ is defined for all values of x in the domain of f. $f'(x) = 0$ when $x = 1.59$. $f'(0)$ does not exist.

	$x < 0$	$x = 0$	$0 < x < 1.59$
$2x^3 - 8$	−	−	−
x^2	+	0	+
$f'(x)$	−	undefined	−
	$x = 1.59$	$x > 1.59$	
$2x^3 - 8$	0	+	
x^2	+	+	
$f'(x)$	0	+	

f is increasing for $x > 1.59$ and decreasing for $x < 0$ and $0 < x < 1.59$. f has a minimum turning point at $x = 1.59$.

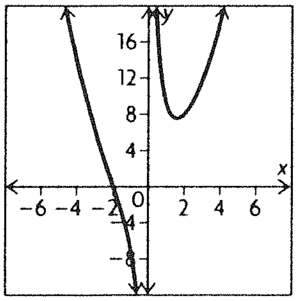
$$f''(x) = \frac{x^2(6x^2) - (2x^3 - 8)(2x)}{x^4}$$
$$= \frac{x(6x^2) - (2x^3 - 8)2}{x^3}$$
$$= \frac{6x^3 - 4x^3 + 16}{x^3}$$
$$= \frac{2x^3 + 16}{x^3}$$

f'' is defined for all values of x in the domain of f. $f''(x) = 0$ when $x = -2$. This is a possible point of inflection. $f(0)$ does not exist.

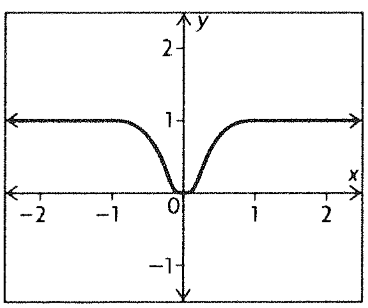
	$x < -2$	$x = -2$	$-2 < x < 0$
$2x^3 + 16$	−	0	+
x^3	−	−	−
$f''(x)$	+	0	−
	$x = 0$	$x > 0$	
$2x^3 + 16$	+	+	
x^3	0	+	
$f''(x)$	undefined	+	

f is concave up when $x < -2$ and $x > 0$. f is concave down when $-2 < x < 0$. The graph changes

concavity where $x = -2$. This is point of inflection with coordinates $(-2, 0)$.
$f(1.59) \doteq 7.56$. The graph has a local minimum at $(1.59, 7.56)$.



18. If $f(x)$ is increasing, then $f'(x) > 0$. From the graph of f', $f'(x) > 0$ for $x > 0$. If $f(x)$ is decreasing, then $f'(x) < 0$. From the graph of f', $f'(x) < 0$ for $x < 0$. At a stationary point, $f'(x) = 0$. From the graph, the zero for $f'(x)$ occurs at $x = 0$. At $x = 0$, $f'(x)$ changes from negative to positive, so f has a local minimum point there.
If the graph of f is concave up, then f'' is positive. From the slope of f', the graph of f is concave up for $-0.6 < x < 0.6$. If the graph of f is concave down, then f'' is negative. From the slope of f', the graph of f is concave down for $x < -0.6$ and $x > 0.6$. Graphs will vary slightly.



19. $f'(x) = \dfrac{(x-1)^2(5) - 5x(2)(x-1)(1)}{(x-1)^4}$

$= \dfrac{5(x-1) - 10x}{(x-1)^3}$

$= \dfrac{-5x - 5}{(x-1)^3}$

$= \dfrac{-5(x+1)}{(x-1)^3}$

$f''(x) = \dfrac{(x-1)^3(-5)}{(x-1)^6}$

$- \dfrac{(-5x-5)(3)(x-1)^2(1)}{(x-1)^6}$

$= \dfrac{(x-1)(-5) - 3(-5x-5)}{(x-1)^4}$

$= \dfrac{10x - 20}{(x-1)^4}$

$= \dfrac{10(x-2)}{(x-1)^4}$

The domain is $\{x|x \neq 1, x \in \mathbf{R}\}$. The x- and y-intercepts are both 0. f has a vertical asymptote at $x = 1$.
$f(0.999) = 4\,995\,000$ so as $x \to 1^-$, $f(x) \to \infty$
$f(1.001) = 5\,005\,000$ so as $x \to 1^+$, $f(x) \to \infty$

$\lim\limits_{x \to \infty} \dfrac{5x}{x^2 - 2x + 1} = 0$ $\qquad$ $\lim\limits_{x \to -\infty} \dfrac{5x}{x^2 - 2x + 1} = 0$

$y = 0$ is a horizontal asymptote on the right. $\qquad$ $y = 0$ is a horizontal asymptote on the left.

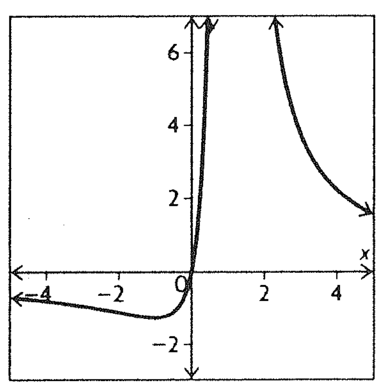
$f'(x)$ is defined for all values of x in the domain of f. $f'(x) = 0$ when $x = -1$. $f(1)$ does not exist.

	$x < -1$	$x = -1$	$-1 < x < 1$	$x = 1$	$x > 1$
$-5(x+1)$	+	0	−	−	−
$(x-1)^3$	−	−	−	0	+
$f'(x)$	−	0	+	undefined	−

f is decreasing when $x < -1$ and $x > 1$. f is increasing when $-1 < x < 1$. f has a minimum turning point at $x = -1$.
$f''(x)$ is defined for all values of x in the domain of f. $f''(x) = 0$ when $x = -3$. This is a possible point of inflection.
$f(1)$ does not exist.

	$x < -2$	$x = -2$	$-2 < x < 1$	$x = 1$	$x > 1$
$x + 2$	−	0	+	+	+
$f''(x)$	−	0	+	undefined	+

The graph is concave down for $x < -2$ and concave up when $-2 < x < 1$ and $x > 1$. It changes concavity at $x = -2$. f has an inflection point at $x = -2$ with coordinates $(-2, -1.11)$.
$f(-1) = -1.25$. f has a local minimum at $(-1, -1.25)$.



20. a. Graph A is f, graph C is f', and graph B is f''. We know this because when you take the derivative, the degree of the denominator increases by one. Graph A has a squared term in the denominator, graph C has a cubic term in the denominator, and graph B has a term to the power of four in the denominator.

b. Graph F is f, graph E is f' and graph D is f''. We know this because the degree of the denominator increases by one degree when the derivative is taken.

Chapter 4 Test, p. 220

1. a. $x < -9$ or $-6 < x < -3$ or $0 < x < 4$ or $x > 8$

b. $-9 < x < -6$ or $-3 < x < 0$ or $4 < x < 8$

c. $(-9, 1)$, $(-6, -2)$, $(0, 1)$, $(8, -2)$

d. $x = -3$, $x = 4$

e. $f''(x) > 0$

f. $-3 < x < 0$ or $4 < x < 8$

g. $(-8, 0)$, $(10, -3)$

2. a. $g(x) = 2x^4 - 8x^3 - x^2 + 6x$

$g'(x) = 8x^3 - 24x^2 - 2x + 6$

To find the critical points, we solve $g'(x) = 0$:

$8x^3 - 24x^2 - 2x + 6 = 0$

$4x^3 - 12x^2 - x + 3 = 0$

Since $g'(3) = 0$, $(x - 3)$ is a factor.

$(x - 3)(4x^2 - 1) = 0$

$x = 3$ or $x = -\frac{1}{2}$ or $x = \frac{1}{2}$.

Note: We could also group to get

$4x^2(x - 3) - (x - 3) = 0$.

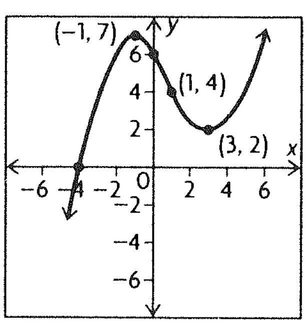
b. $g''(x) = 24x^2 - 48x - 2$

Since $g''\left(-\frac{1}{2}\right) = 28 > 0$, $\left(-\frac{1}{2}, -\frac{17}{8}\right)$ is a local maximum.

Since $g''\left(-\frac{1}{2}\right) = -20 < 0$, $\left(\frac{1}{2}, \frac{15}{8}\right)$ is a local maximum.

Since $g''(3) = 70 > 0$, $(3, -45)$ is a local minimum.

3.



4. $g(x) = \dfrac{x^2 + 7x + 10}{(x - 3)(x + 2)}$

The function $g(x)$ is not defined at $x = -2$ or $x = 3$. At $x = -2$, the value of the numerator is 0. Thus, there is a discontinuity at $x = -2$, but $x = -2$ is not a vertical asymptote.

At $x = 3$, the value of the numerator is 40. $x = 3$ is a vertical asymptote.

$$g(x) = \frac{(x + 2)(x + 5)}{(x - 3)(x + 2)} = \frac{x + 5}{x - 3}, x \neq -2$$

$$\lim_{x \to -2} g(x) = \lim_{x \to -2} \left(\frac{x + 5}{x - 3}\right) = -\frac{3}{5}$$

$$\lim_{x \to -2^-} g(x) = \lim_{x \to -2^-} \left(\frac{x + 5}{x - 3}\right) = -\frac{3}{5}$$

There is a hole in the graph of $g(x)$ at $\left(-2, -\frac{3}{5}\right)$.

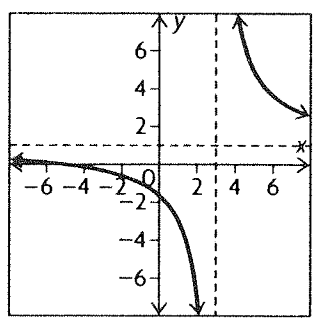
$$\lim_{x \to 3} g(x) = \lim_{x \to 3^-} \left(\frac{x + 5}{x - 3}\right) = -\infty$$

$$\lim_{x \to 3^+} g(x) = \lim_{x \to 3^+} \left(\frac{x + 5}{x - 3}\right) = \infty$$

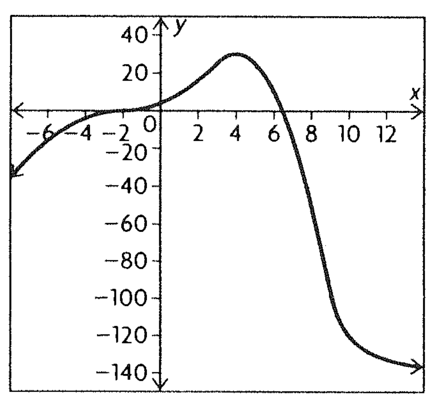
There is a vertical asymptote at $x = 3$.

Also, $\lim_{x \to \infty} g(x) = \lim_{x \to -\infty} g(x) = 1$.

Thus, $y = 1$ is a horizontal asymptote.



5.



6. $f(x) = \dfrac{2x + 10}{x^2 - 9}$

$= \dfrac{2x + 10}{(x - 3)(x + 3)}$

There are discontinuities at $x = -3$ and at $x = 3$.

$\left.\begin{aligned}\lim_{x\to -3^-} f(x) &= \infty \\ \lim_{x\to -3^+} f(x) &= -\infty\end{aligned}\right\} x = -3$ is a vertical asymptote.

$\left.\begin{aligned}\lim_{x\to 3^-} f(x) &= -\infty \\ \lim_{x\to 3^+} f(x) &= \infty\end{aligned}\right\} x = 3$ is a vertical asymptote.

The y-intercept is $-\frac{10}{9}$ and $x = -5$ is an x-intercept.

$f'(x) = \dfrac{2(x^2 - 9) - (2x + 10)(2)}{(x^2 - 9)^2}$

$= \dfrac{-2x^2 - 20x - 18}{(x^2 - 9)^2}$

For critical values, we solve $f'(x) = 0$:

$x^2 + 10x + 9 = 0$

$(x + 1)(x + 9) = 0$

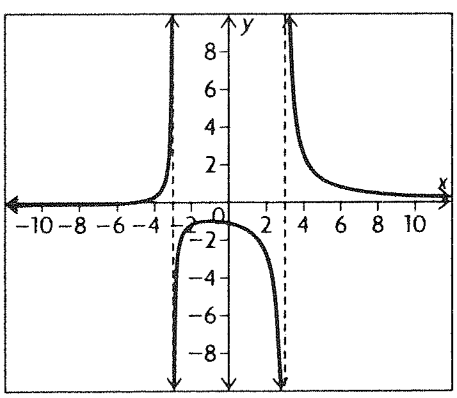
$x = -1$ or $x = -9$.

$\left(-9, -\frac{1}{9}\right)$ is a local minimum and $(-1, -1)$ is a local maximum.

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \dfrac{\frac{2}{x} + \frac{10}{x^2}}{1 - \frac{9}{x^2}} = 0$ and

$\lim_{x\to -\infty} f(x) = \lim_{x\to -\infty} \left(\dfrac{\frac{2}{x} + \frac{10}{x^2}}{1 - \frac{9}{x^2}}\right) = 0$

$y = 0$ is a horizontal asymptote.



7. $f(x) = x^3 + bx^2 + c$

$f'(x) = 3x^2 + 2bx$

Since $f'(-2) = 0$, $12 - 4b = 0$

$b = 3$.

Also, $f(-2) = 6$.

Thus, $-8 + 12 + c = 6$

$c = 2$.

$f'(x) = 3x^2 + 6x$

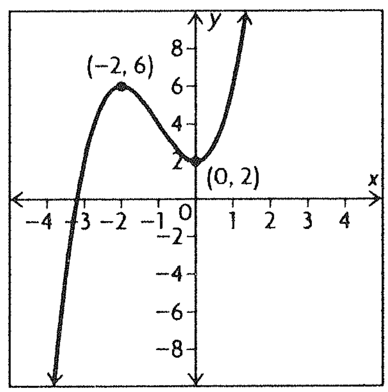
$= 3x(x + 2)$

The critical points are $(-2, 6)$ and $(0, 2)$.

$f''(x) = 6x + 6$

Since $f''(-2) = -6 < 0$, $(-2, 6)$ is a local maximum.

Since $f'(0) = 6 > 0$, $(0, 2)$ is a local minimum.



CHAPTER 5: Derivatives of Exponential and Trigonometric Functions

Review of Prerequisite Skills, pp. 224–225

1. a. $3^{-2} = \frac{1}{3^2}$

$= \frac{1}{9}$

b. $32^{\frac{2}{5}} = (\sqrt[5]{32})^2$

$= 2^2$

$= 4$

c. $27^{-\frac{2}{3}} = \frac{1}{(\sqrt[3]{27})^2}$

$= \frac{1}{3^2}$

$= \frac{1}{9}$

d. $\left(\frac{2}{3}\right)^{-2} = \left(\frac{3}{2}\right)^2$

$= \frac{9}{4}$

2. a. $\log_5 625 = 4$

b. $\log_4 \frac{1}{16} = -2$

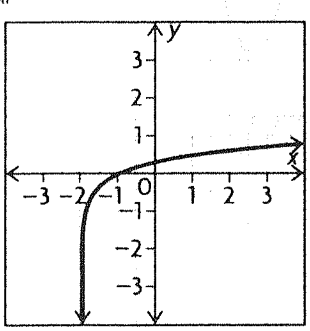
c. $\log_x 3 = 3$

d. $\log_{10} 450 = w$

e. $\log_3 z = 8$

f. $\log_a T = b$

3. a.



The x-intercept occurs where $y = 0$.

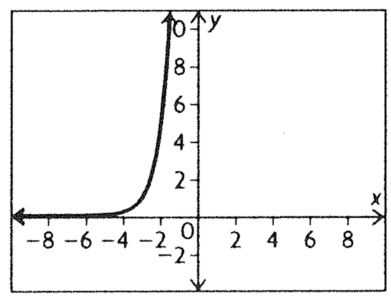
$0 = \log_{10}(x + 2)$

$10^0 = x + 2$

$x = -1$

The x-intercept is $(-1, 0)$.

b.



An exponential function is always positive, so there is no x-intercept.

4. a. $\sin\theta = \frac{y}{r}$

b. $\cos\theta = \frac{x}{r}$

c. $\tan\theta = \frac{y}{x}$

5. To convert to radian measure from degree measure, multiply the degree measure by $\frac{\pi}{180°}$.

a. $360° \times \frac{\pi}{180°} = 2\pi$

b. $45° \times \frac{\pi}{180°} = \frac{\pi}{4}$

c. $-90° \times \frac{\pi}{180°} = -\frac{\pi}{2}$

d. $30° \times \frac{\pi}{180°} = \frac{\pi}{6}$

e. $270° \times \frac{\pi}{180°} = \frac{3\pi}{2}$

f. $-120° \times \frac{\pi}{180°} = -\frac{2\pi}{3}$

g. $225° \times \frac{\pi}{180°} = \frac{5\pi}{4}$

h. $330° \times \frac{\pi}{180°} = \frac{11\pi}{6}$

6. For the unit circle, sine is associated with the y-coordinate of the point where the terminal arm of the angle meets the circle, and cosine is associated with the x-coordinate.

a. $\sin\theta = b$

b. $\tan\theta = \frac{b}{a}$

c. $\cos\theta = a$

d. $\sin\left(\frac{\pi}{2} - \theta\right) = a$

e. $\cos\left(\frac{\pi}{2} - \theta\right) = b$

f. $\sin(-\theta) = -b$

7. a. The angle is in the second quadrant, so cosine and tangent will be negative.

$\cos\theta = -\frac{12}{13}$

$\tan\theta = -\frac{5}{12}$

b. The angle is in the third quadrant, so sine will be negative and tangent will be positive.

$$\sin^2\theta + \cos^2\theta = 1$$
$$\sin^2\theta + \frac{4}{9} = 1$$
$$\sin^2\theta = \frac{5}{9}$$
$$\sin\theta = -\frac{\sqrt{5}}{3}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{5}}{2}$$

c. The angle is in the fourth quadrant, so cosine will be positive and sine will be negative.
Because $\tan\theta = -2$, the point $(1, -2)$ is on the terminal arm of the angle. The reference triangle for this angle has a hypotenuse of $\sqrt{2^2 + 1^2}$ or $\sqrt{5}$.

$\sin\theta = -\frac{2}{\sqrt{5}}$

$\cos\theta = \frac{1}{\sqrt{5}}$

d. The sine is only equal to 1 for one angle between 0 and π, so $\theta = \frac{\pi}{2}$.

$\cos\frac{\pi}{2} = 0$

$\tan\frac{\pi}{2}$ is undefined

8. a. The period is $\frac{2\pi}{2}$ or π. The amplitude is 1.

b. The period is $\frac{2\pi}{\frac{1}{2}}$ or 4π. The amplitude is 2.

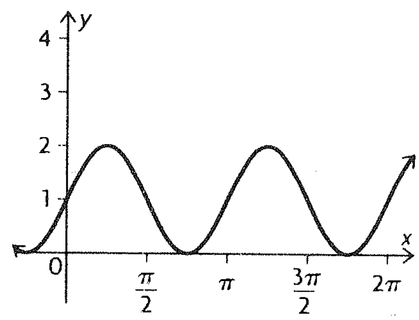
c. The period is $\frac{2\pi}{\pi}$ or 2. The amplitude is 3.

d. The period is $\frac{2\pi}{12}$ or $\frac{\pi}{6}$. The amplitude is $\frac{2}{7}$.

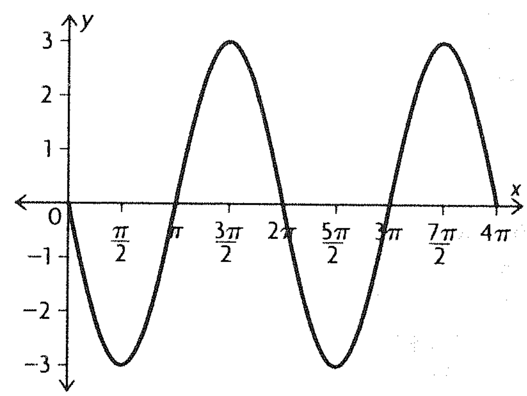
e. The period is 2π. The amplitude is 5.

f. The period is 2π. Because of the absolute value being taken, the amplitude is $\frac{3}{2}$.

9. a. The period is $\frac{2\pi}{2}$ or π. Graph the function from $x = 0$ to $x = 2\pi$.



b. The period is 2π, so graph the function from $x = 0$ to $x = 4\pi$.



10. a. $\tan x + \cot x = \sec x \csc x$

$$\text{LS} = \tan x + \cot x$$
$$= \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}$$
$$= \frac{\sin^2 x + \cos^2 x}{\cos x + \sin x}$$

$$= \frac{1}{\cos x + \sin x}$$
$$\text{RS} = \sec x + \csc x$$
$$= \frac{1}{\cos x} \cdot \frac{1}{\sin x}$$
$$= \frac{1}{\cos x \sin x}$$

Therefore, $\tan x + \cot x = \sec x \csc x$.

b. $\dfrac{\sin x}{1 - \sin^2 x} = \tan x + \sec x$

$$\text{LS} = \frac{\sin x}{1 - \sin^2 x}$$
$$= \frac{\sin x}{\cos^2 x}$$
$$\text{RS} = \tan x \sec x$$
$$= \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x}$$
$$= \frac{\sin x}{\cos^2 x}$$

Therefore, $\dfrac{\sin x}{1 - \sin^2 x} = \tan x \sec x$.

11. a. $3 \sin x = \sin x + 1$
$$2 \sin x = 1$$
$$\sin x = \frac{1}{2}$$
$$x = \frac{\pi}{6}, \frac{5\pi}{6}$$

b. $\cos x - 1 = -\cos x$
$$2 \cos x = 1$$
$$\cos x = \frac{1}{2}$$
$$x = \frac{\pi}{3}, \frac{5\pi}{3}$$

5.1 Derivatives of Exponential Functions, $y = e^x$, pp. 232–234

1. You can only use the power rule when the term containing variables is in the base of the exponential expression. In the case of $y = e^x$, the exponent contains a variable.

2. a. $y = e^{3x}$
$$\frac{dy}{dx} = 3e^{3x}$$

b. $s = e^{3t-5}$
$$\frac{ds}{dt} = 3e^{3t-5}$$

c. $y = 2e^{10t}$
$$\frac{dy}{dt} = 20e^{10t}$$

d. $y = e^{-3x}$
$$\frac{dy}{dx} = -3e^{-3x}$$

e. $y = e^{5-6x+x^2}$
$$\frac{dy}{dx} = (-6 + 2x)e^{5-6x+x^2}$$

f. $y = e^{\sqrt{x}}$
$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}e^{\sqrt{x}}$$

3. a. $y = 2e^{x^3}$
$$\frac{dy}{dx} = 2(3x^2)e^{x^3}$$
$$= 6x^2e^{x^3}$$

b. $\dfrac{dy}{dx} = \dfrac{d(xe^{3x})}{dx}$
$$= (x)(3e^{3x}) + (e^{3x})(1)$$
$$= 3xe^{3x} + e^{3x}$$
$$= e^{3x}(3x + 1)$$

c. $f(x) = \dfrac{e^{-x^3}}{x}$
$$f'(x) = \frac{-3x^2e^{-x^3}(x) - e^{-x^3}}{x^2}$$

d. $f(x) = \sqrt{x}e^x$
$$f'(x) = \sqrt{x}e^x + e^x\left(\frac{1}{2\sqrt{x}}\right)$$

e. $h(t) = e^{t^2} + 3e^{-t}$
$$h'(t) = 2te^{t^2} - 3e^{-t}$$

f. $g(t) = \dfrac{e^{2t}}{1 + e^{2t}}$
$$g'(t) = \frac{2e^{2t}(1 + e^{2t}) - 2e^{2t}(e^{2t})}{(1 + e^{2t})^2}$$
$$= \frac{2e^{2t}}{(1 + e^{2t})^2}$$

4. a. $f'(x) = \dfrac{1}{3}(3e^{3x} - 3e^{-3x})$
$$= e^{3x} - e^{-3x}$$
$$f'(1) = e^3 - e^{-3}$$

b. $f(x) = e^{-\frac{1}{x+1}}$
$$f'(x) = e^{-\frac{1}{x+1}}\left(\frac{1}{(x + 1)^2}\right)$$
$$f'(0) = e^{-1}(1)$$
$$= \frac{1}{e}$$

c. $h'(z) = 2z(1 + e^{-z}) + z^2(-e^{-z})$

$h'(-1) = 2(-1)(1 + e) + (-1)^2(-e^1)$

$= -2 - 2e - e$

$= -2 - 3e$

5. a. $y = \dfrac{2e^x}{1 + e^x}$

$\dfrac{dy}{dx} = \dfrac{(1 + e^x)2e^x - 2e^x(e^x)}{(1 + e^x)^2}$

$\dfrac{dy}{dx} = \dfrac{2(2) - 2(1)(1)}{2^2}$

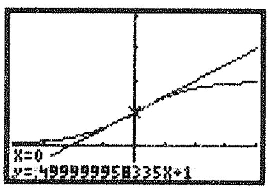
$= \dfrac{1}{2}$

When $x = 0$,

the slope of the tangent is $\frac{1}{2}$.

The equation of the tangent is $y = \frac{1}{2}x + 1$, since the y-intercept was given as $(0, 1)$.

b.



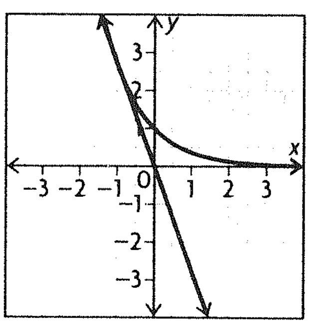
c. The answers agree very well; the calculator does not show a slope of exactly 0.5, due to internal rounding.

6. $y = e^{-x}$

$\dfrac{dy}{dx} = -e^{-x}$

When $x = -1$, $\dfrac{dy}{dx} = -e$. And when $x = -1$, $y = e$.

The equation of the tangent is $y - e = -e(x + 1)$ or $ex + y = 0$.



7. The slope of the tangent line at any point is given by

$\dfrac{dy}{dx} = (1)(e^{-x}) + x(-e^{-x})$

$= e^{-x}(1 - x)$.

At the point $(1, e^{-1})$, the slope is $e^{-1}(0) = 0$. The equation of the tangent line at the point A is

$y - e^{-1} = 0(x - 1)$ or $y = \dfrac{1}{e}$.

8. The slope of the tangent line at any point on the curve is $\dfrac{dy}{dx} = 2xe^{-x} + x^2(e^{-x})$

$= (2x - x^2)(e^{-x})$

$= \dfrac{2x - x^2}{e^x}$.

Horizontal lines have slope equal to 0.

We solve $\dfrac{dy}{dx} = 0$

$\dfrac{x(2 - x)}{e^x} = 0$.

Since $e^x > 0$ for all x, the solutions are $x = 0$ and $x = 2$. The points on the curve at which the tangents are horizontal are $(0, 0)$ and $\left(2, \frac{4}{e^2}\right)$.

9. If $y = \dfrac{5}{2}(e^{\frac{x}{5}} + e^{-\frac{x}{5}})$, then

$y' = \dfrac{5}{2}\left(\dfrac{1}{5}e^{\frac{x}{5}} - \dfrac{1}{5}e^{-\frac{x}{5}}\right)$, and

$y'' = \dfrac{5}{2}\left(\dfrac{1}{25}e^{\frac{x}{5}} + \dfrac{1}{25}e^{-\frac{x}{5}}\right)$

$= \dfrac{1}{25}\left[\dfrac{5}{2}(e^{\frac{x}{5}} + e^{-\frac{x}{5}})\right]$

$= \dfrac{1}{25}y$.

10. a. $y = e^{-3x}$

$\dfrac{dy}{dx} = -3e^{-3x}$

$\dfrac{d^2y}{dx^2} = 9e^{-3x}$

$\dfrac{d^3y}{dx^3} = -27e^{-3x}$

b. $\dfrac{d^ny}{dx^n} = (-1)^n(3^n)e^{-3x}$

11. a. $\dfrac{dy}{dx} = \dfrac{d(-3e^x)}{dx}$

$= -3e^x$

$\dfrac{d^2y}{dx^2} = -3e^x$

b. $\dfrac{dy}{dx} = \dfrac{d(xe^{2x})}{dx}$

$= (x)(2e^{2x}) + (e^{2x})(1)$

$= 2xe^{2x} + e^{2x}$

$= e^{2x}(2x + 1)$

$$\frac{d^2y}{dx^2} = e^{2x}(2) + (2x + 1)(2e^{2x})$$
$$= 4xe^{2x} + 4e^{2x}$$

c. $\frac{dy}{dx} = \frac{d(e^x(4 - x))}{dx}$
$$= (e^x)(-1) + (4 - x)(e^x)$$
$$= -e^x + 4e^x - xe^x$$
$$= 3e^x - xe^x$$
$$= e^x(3 - x)$$
$$\frac{d^2y}{dx^2} = e^x(-1) + (3 - x)(e^x)$$
$$= 2e^x - xe^x$$
$$= e^x(2 - x)$$

12. a. When $t = 0$, $N = 1000[30 + e^0] = 31\,000$.

b. $\frac{dN}{dt} = 1000\left[0 - \frac{1}{30}e^{-\frac{t}{30}}\right] = -\frac{100}{3}e^{-\frac{t}{30}}$

c. When $t = 20h$, $\frac{dN}{dt} = -\frac{100}{3}e^{-\frac{2}{3}} \doteq -17$ bacteria/h.

d. Since $e^{-\frac{t}{30}} > 0$ for all t, there is no solution to $\frac{dN}{dt} = 0$.
Hence, the maximum number of bacteria in the culture occurs at an endpoint of the interval of domain.
When $t = 50$, $N = 1000[30 + e^{-\frac{5}{3}}] \doteq 30\,189$.
The largest number of bacteria in the culture is 31 000 at time $t = 0$.

e. The number of bacteria is constantly decreasing as time passes.

13. a. $v = \frac{ds}{dt} = 160\left(\frac{1}{4} - \frac{1}{4}e^{-\frac{t}{4}}\right)$
$$= 40(1 - e^{-\frac{t}{4}})$$

b. $a = \frac{dv}{dt} = 40\left(\frac{1}{4}e^{-\frac{t}{4}}\right) = 10e^{-\frac{t}{4}}$

From **a.**, $v = 40(1 - e^{-\frac{t}{4}})$, which gives $e^{-\frac{t}{4}} = 1 - \frac{v}{40}$.

Thus, $a = 10\left(1 - \frac{v}{40}\right) = 10 - \frac{1}{4}v$.

c. $v_T = \lim_{t\to\infty} v$
$$v_T = \lim_{t\to\infty} 40(1 - e^{-\frac{t}{4}})$$
$$= 40 \lim_{t\to\infty}\left(1 - \frac{1}{e^{\frac{t}{4}}}\right)$$
$$= 40(1), \text{ since } \lim_{t\to\infty}\frac{1}{e^{\frac{t}{4}}} = 0$$

The terminal velocity of the skydiver is 40 m/s.

d. 95% of the terminal velocity is
$\frac{95}{100}(40) = 38$ m/s.
To determine when this velocity occurs, we solve
$$40(1 - e^{-\frac{t}{4}}) = 38$$
$$1 - e^{-\frac{t}{4}} = \frac{38}{40}$$
$$e^{-\frac{t}{4}} = \frac{1}{20}$$
$$e^{\frac{t}{4}} = 20$$
and $\frac{t}{4} = \ln 20$,
which gives $t = 4\ln 20 \doteq 12$ s.
The skydiver's velocity is 38 m/s, 12 s after jumping.
The distance she has fallen at this time is
$$S = 160(\ln 20 - 1 + e^{-20})$$
$$= 160\left(\ln 20 - 1 + \frac{1}{20}\right)$$
$$\doteq 327.3 \text{ m}.$$

14. a. i. Let $f(x) = \left(1 + \frac{1}{x}\right)^x$. Then,

x	f(x)
1	2
10	2.5937
100	2.7048
1000	2.7169
10 000	2.7181

So, from the table one can see that $\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = e$.

ii. Let $f(x) = (1 + x)^{\frac{1}{x}}$.

x	f(x)
−0.1	2.8680
−0.01	2.7320
−0.001	2.7196
−0.0001	2.7184
?	?
0.0001	2.7181
0.001	2.7169
0.01	2.7048
0.1	2.5937

So, from the table one can see that $\lim_{x\to 0}(1 + x)^{\frac{1}{x}} = e$.
That is, the limit approaches the value of
$e = 2.718\,281\,828\ldots$

b. The limits have the same value because as $x \to \infty$, $\frac{1}{x} \to 0$.

15. a. The given limit can be rewritten as

$$\lim_{h\to 0}\frac{e^h - 1}{h} = \lim_{h\to 0}\frac{e^{0+h} - e^0}{h}$$

This expression is the limit definition of the derivative at $x = 0$ for $f(x) = e^x$.

$$\left[f'(0) = \lim_{h\to 0}\frac{e^{0+h} - e^0}{h}\right]$$

Since $f'(x) = \frac{de^x}{dx} = e^x$, the value of the given limit is $e^0 = 1$.

b. Again, $\lim_{h\to 0}\frac{e^{2+h} - e^2}{h}$ is the derivative of e^x at $x = 2$.

Thus, $\lim_{h\to 0}\frac{e^{2+h} - e^2}{h} = e^2$.

16. For $y = Ae^{ex}$, $\frac{dy}{dt} = Ame^{ex}$ and $\frac{d^2y}{dt^2} = Am^2e^{ex}$.

Substituting in the differential equation gives

$$Am^2e^{ex} + Ame^{ex} - 6Ae^{ex} = 0$$
$$Ae^{ex}(m^2 + m - 6) = 0.$$

Since $Ae^{ex} \neq 0$, $m^2 + m - 6 = 0$

$$(m + 3)(m - 2) = 0$$
$$m = -3 \text{ or } m = 2.$$

17. a. $\frac{d}{dx}\sinh x = \frac{d}{dx}\left[\frac{1}{2}(e^x - e^{-x})\right]$

$$= \frac{1}{2}(e^x + e^{-x})$$
$$= \cosh x$$

b. $\frac{d}{dx}\cosh x = \frac{1}{2}(e^x - e^{-x})$

$$= \sinh x$$

c. Since $\tanh x = \frac{\sinh x}{\cosh x}$,

$\frac{d}{dx}\tanh x$

$$= \frac{\left(\frac{d}{dx}\sinh x\right)(\cosh x) - (\sinh x)\left(\frac{d}{dx}\cosh x\right)}{(\cosh x)^2}$$

$$= \frac{\frac{1}{2}(e^x + e^{-x})\left(\frac{1}{2}\right)(\cosh x)^2(e^x + e^{-x})}{(\cosh x)^2}$$

$$- \frac{\frac{1}{2}(e^x - e^{-x})\left(\frac{1}{2}\right)(e^x - e^{-x})}{(\cosh x)^2}$$

$$= \frac{\frac{1}{4}\left[(e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})\right]}{(\cosh x)^2}$$

$$= \frac{\frac{1}{4}(4)}{(\cosh x)^2}$$

$$= \frac{1}{(\cosh x)^2}$$

18. a. Four terms:

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} = 2.666\,66\overline{6}$$

Five terms:

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} = 2.708\,33\overline{3}$$

Six terms:

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} = 2.716\,66\overline{6}$$

Seven terms:

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} = 2.718\,05\overline{5}$$

b. The expression for e in part **a.** is a special case of $e^x = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$ in that it is the case when $x = 1$. Then $e^x = e^1 = e$ is in fact $e^1 = e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots$. The value of x is 1.

5.2 Derivatives of the General Exponential Function, $y = b^x$, p. 240

1. a. $\frac{dy}{dx} = \frac{d(2^{3x})}{dx}$

$$= 3(2^{3x})\ln 2$$

b. $\frac{dy}{dx} = \frac{d(3.1^x + x^3)}{dx}$

$$= \ln 3.1(3.1)^x + 3x^2$$

c. $\frac{ds}{dt} = \frac{d(10^{3t-5})}{dt}$

$$= 3(10^{3t-5})\ln 10$$

d. $\frac{dw}{dn} = \frac{d(10^{5-6n+n^2})}{dn}$

$$= (-6 + 2n)(10^{5-6n+n^2})\ln 10$$

e. $\frac{dy}{dx} = \frac{d(3^{x^2+2})}{dx}$

$$= 2x(3^{x^2+2})\ln 3$$

f. $\frac{dy}{dx} = \frac{d(400(2)^{x+3})}{dx}$

$$= 400(2)^{x+3}\ln 2$$

2. a. $\frac{dy}{dx} = \frac{d(x^5 \times (5)^x)}{dx}$

$= (x^5)((5)^x(\ln 5)) + ((5)^x)(5x^4)$

$= 5^x[(x^5 \times \ln 5) + 5x^4]$

b. $\frac{dy}{dx} = \frac{d(x(3)^{x^2})}{dx}$

$= (x)(2x(3)^{x^2} \ln 3) + (3)^{x^2}(1)$

$= (3)^{x^2}[(2x^2 \ln 3) + 1]$

c. $v = (2^t)(t^{-1})$

$\frac{dv}{dt} = \frac{d((2^t)(t^{-1}))}{dx}$

$= (2^t)(-1t^{-2}) + (t^{-1})(2^t \ln 2)$

$= -\frac{2^t}{t^2} + \frac{2^t \ln 2}{t}$

d. $f(x) = \frac{3^{\frac{x}{2}}}{x^2}$

$f'(x) = \frac{\frac{1}{2} \ln 3(3^{\frac{x}{2}})(x^2) - 2x(3^{\frac{x}{2}})}{x^4}$

$= \frac{x \ln 3(3^{\frac{x}{2}}) - 4(3^{\frac{x}{2}})}{x^4}$

$= \frac{3^{\frac{x}{2}}[x \ln 3 - 4]}{x^3}$

3. $f(t) = 10^{3t-5} \cdot e^{2t^2}$

$f'(t) = (10^{3t-5})(4te^{2t^2}) + (e^{2t^2})(3(10)^{3t-5} \ln 10)$

$= 10^{3t-5}e^{2t^2}(4t + 3 \ln 10)$

Now, set $f'(t) = 0$.

So, $f'(t) = 0 = 10^{3t-5}e^{2t^2}(4t + 3\ln 10)$

So $10^{3t-5}e^{2t^2} = 0$ and $4t + 3 \ln 10 = 0$.

The first equation never equals zero because solving would force one to take the natural log of both sides, but ln 0 is undefined. So the first equation does not produce any values for which $f'(t) = 0$. The second equation does give one value.

$4t + 3 \ln 10 = 0$

$4t = -3 \ln 10$

$t = -\frac{3 \ln 10}{4}$

4. When $x = 3$, the function $y = f(x)$ evaluated at 3 is $f(3) = 3(2^3) = 3(8) = 24$. Also,

$\frac{dy}{dx} = \frac{d(3(2)^x)}{dx}$

$= 3(2^x)\ln 2$

So, at $x = 3$,

$\frac{dy}{dx} = 3(2^3)(\ln 2) = 24(\ln 2) \doteq 16.64$

Therefore, $y - 24 = 16.64(x - 3)$

$y - 24 = 16.64x - 49.92$

$-16.64x + y + 25.92 = 0$

5. $\frac{dy}{dx} = \frac{d(10^x)}{dx}$

$= 10^x \ln 10$

So, at $x = 1$,

$\frac{dy}{dx} = 10^1 \ln 10 = 10(\ln 10) \doteq 23.03$

Therefore, $y - 10 = 23.03(x - 1)$

$y - 10 = 23.03x - 23.03$

$-23.03x + y + 13.03 = 0$

6. a. The half-life of the substance is the time required for half of the substance to decay. That is, it is when 50% of the substance is left, so $P(t) = 50$.

$50 = 100(1.2)^{-t}$

$\frac{1}{2} = (1.2)^{-t}$

$\frac{1}{2} = \frac{1}{(1.2)^t}$

$(1.2)^t = 2$

$t(\ln 1.2) = \ln 2$

$t = \frac{\ln 1.2}{\ln 2}$

$t \doteq 3.80$ years

Therefore, the half-life of the substance is about 3.80 years.

b. The problem asks for the rate of change when $t \doteq 3.80$ years.

$P'(t) = -100(1.2)^{-t}(\ln 1.2)$

$P'(3.80) = -100(1.2)^{-(3.80)}(\ln 1.2)$

$\doteq -9.12$

So, the substance is decaying at a rate of about -9.12 percent/year at the time 3.80 years where the half-life is reached.

7. $P = 0.5(10^9)e^{0.200\,15t}$

a. $\frac{dP}{dt} = 0.5(10^9)(0.200\,15)e^{0.200\,15t}$

In 1968, $t = 1$ and $\frac{dP}{dt} = 0.5(10^9)(0.200\,15)e^{0.200\,15} \doteq 0.122\,25 \times 10^9$ dollars/annum

In 1978, $t = 11$ and

$\frac{dP}{dt} = 0.5(10^9)(0.200\,15)e^{11 \times 0.200\,15}$

$\doteq 0.904\,67 \times 10^9$ dollars/annum.

In 1978, the rate of increase of debt payments was \$904 670 000/annum compared to \$122 250 000/annum in 1968. As a ratio,

$\frac{\text{Rate in 1978}}{\text{Rate in 1968}} = \frac{7.4}{1}$. The rate of increase for 1978 is 7.4 times larger than that for 1968.

b. In 1988, $t = 21$ and

$$\frac{dP}{dt} = 0.5(10^9)(0.20015)e^{21 \times 0.200\,15}$$
$$\doteq 6.694\,69 \times 10^9 \text{ dollars/annum}$$

In 1998, $t = 31$ and

$$\frac{dP}{dt} = 0.5(10^9)(0.20015)e^{31 \times 0.200\,15}$$
$$\doteq 49.541\,69 \times 10^9 \text{ dollars/annum}$$

As a ratio, $\frac{\text{Rate in 1998}}{\text{Rate in 1988}} = \frac{7.4}{1}$. The rate of increase for 1998 is 7.4 times larger than that for 1988.

c. Answers may vary. For example, data from the past are not necessarily good indicators of what will happen in the future. Interest rates change, borrowing may decrease, principal may be paid off early.

8. When $x = 0$, the function $y = f(x)$ evaluated at 0 is $f(0) = 2^{-0^2} = 2^0 = 1$. Also,

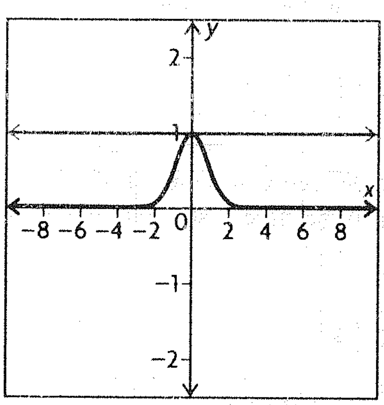
$$\frac{dy}{dx} = \frac{d(2^{-x^2})}{dx}$$
$$= -2x(2^{-x^2})\ln 2$$

So, at $x = 0$,

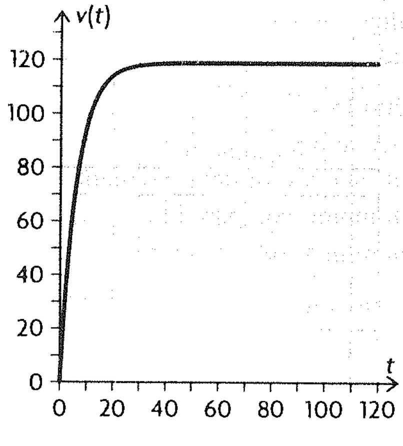
$$\frac{dy}{dx} = -2(0)(2^{-0^2})\ln 2 = 0$$

Therefore, $y - 1 = 0(x - 0)$

So, $y - 1 = 0$ or $y = 1$.



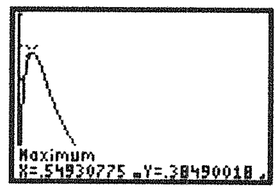
9.

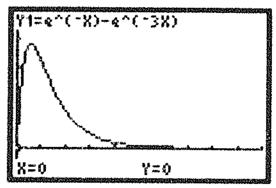


From the graph, one can notice that the values of $v(t)$ quickly rise in the range of about $0 \leq t \leq 15$. The slope for these values is positive and steep. Then as the graph nears $t = 20$ the steepness of the slope decreases and seems to get very close to 0. One can reason that the car quickly accelerates for the first 20 units of time. Then, it seems to maintain a constant acceleration for the rest of the time. To verify this, one could differentiate and look at values where $v'(t)$ is increasing.

5.3 Optimization Problems Involving Exponential Functions, pp. 245–247

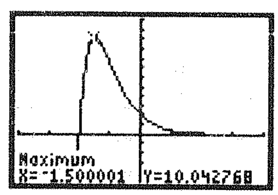
1. a.

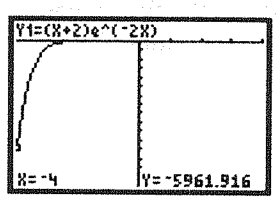




The maximum value is about 0.3849. The minimum value is 0.

b.





The maximum value is about 10.043. The minimum value is about -5961.916.

2. a. $f(x) = e^{-x} - e^{-3x}$ on $0 \leq x \leq 10$

$$f'(x) = -e^{-x} + 3e^{-3x}$$

Let $f'(x) = 0$, therefore $e^{-x} + 3e^{-3x} = 0$.

Let $e^{-x} = w$, when $-w + 3w^3 = 0$.

$$w(-1 + 3w^2) = 0.$$

Therefore, $w = 0$ or $w^2 = \frac{1}{3}$

$$w = \pm\frac{1}{\sqrt{3}}.$$

But $w \geq 0$, $w = +\frac{1}{\sqrt{3}}$.

When $w = \frac{1}{\sqrt{3}}$, $e^{-x} = \frac{1}{\sqrt{3}}$,

$$-x \ln e = \ln 1 - \ln \sqrt{3}$$
$$x = \frac{\ln \sqrt{3} - \ln 1}{1}$$
$$= \ln \sqrt{3}$$
$$\doteq 0.55.$$

$$f(0) = e^0 - e^0$$
$$= 0$$
$$f(0.55) \doteq 0.3849$$
$$f(10) = e^{-10} - e^{-30} \doteq 0.00005$$

Absolute maximum is about 0.3849 and absolute minimum is 0.

$m(x) = (x + 2)e^{-2x}$ on $-4 \le x \le 4$

$m'(x) = e^{-2x} + (-2)(x + 2)e^{-2x}$

Let $m'(x) = 0$.

$e^{-2x} \neq 0$, therefore, $1 + (-2)(x + 2) = 0$

$$x = \frac{-3}{2}$$
$$= -1.5.$$

$$m(-4) = -2e^8 \doteq -5961$$
$$m(-1.5) = 0.5e^3 \doteq 10$$
$$m(4) = 6e^{-8} \doteq 0.0002$$

The maximum value is about 10 and the minimum value is about -5961.

b. The graphing approach seems to be easier to use for the functions. It is quicker and it gives the graphs of the functions in a good viewing rectangle. The only problem may come in the second function, $m(x)$, because for $x < 1.5$ the function quickly approaches values in the negative thousands.

3. a. $P(t) = \frac{20}{1 + 3e^{-0.02t}}$

$$P(0) = \frac{20}{1 + 3e^{-0.02(0)}}$$
$$= \frac{20}{1 + 3e^0}$$
$$= \frac{20}{4}$$
$$= 5$$

So, the population at the start of the study when $t = 0$ is 500 squirrels.

b. The question asks for $\lim_{t\to\infty} P(t)$.

As t approaches ∞, $e^{-0.02t} = \frac{1}{e^{0.02t}}$ approaches 0.

So, $\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{20}{1 + 3e^{-0.02t}}$

$$= \frac{20}{1 + 3(0)}$$
$$= 20.$$

Therefore, the largest population of squirrels that the forest can sustain is 2000 squirrels.

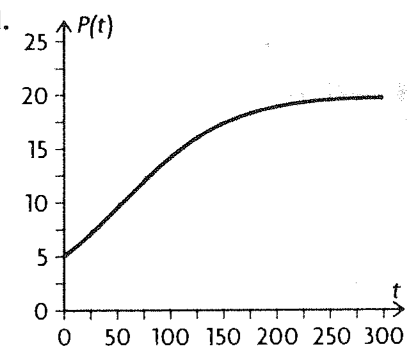
c. A point of inflection can only occur when $P''(t) = 0$ and concavity changes around the point.

$$P(t) = \frac{20}{1 + 3e^{-0.02t}}$$
$$P(t) = 20(1 + 3e^{-0.02t})^{-1}$$
$$P'(t) = 20(-(1 + 3e^{-0.02t})^{-2}(-0.06e^{-0.02t}))$$
$$= (1.2e^{-0.02t})(1 + 3e^{-0.02t})^{-2}$$
$$P''(t) = [(1.2e^{-0.02t})(-2(1 + 3e^{-0.02t})^{-3}(-0.06e^{-0.02t})]$$
$$+ (1 + 3e^{-0.02t})^{-2}(-0.024e^{-0.02t})$$
$$= \frac{0.144e^{-0.04t}}{(1 + 3e^{-0.02t})^3} - \frac{0.024e^{-0.02t}}{(1 + 3e^{-0.02t})^2}$$

$P''(0)$ when $\frac{0.144e^{-0.04t}}{(1 + 3e^{-0.02t})^3} - \frac{0.024e^{-0.02t}}{(1 + 3e^{-0.02t})^2} = 0$

Solving for t after setting the second derivative equal to 0 is very tedious. Use a graphing calculator to determine the value of t for which the second derivative is 0, 54.9. Evaluate $P(54.9)$. The point of inflection is (54.9, 10).

d.



e. P grows exponentially until the point of inflection, then the growth rate decreases and the curve becomes concave down.

4. a. $P(x) = 10^6[1 + (x - 1)e^{-0.001x}]$, $0 \le x \le 2000$

Using the Algorithm for Extreme Values, we have

$$P(0) = 10^6[1 - 1] = 0$$
$$P(2000) = 10^6[1 + 1999e^{-2}] \doteq 271.5 \times 10^6.$$

Now,

$$P'(x) = 10^6[(1)e^{-0.001x} + (x - 1)(-0.001)e^{-0.001x}]$$
$$= 10^6e^{-0.001x}(1 - 0.001x + 0.001)$$

Since $e^{-0.001x} > -$ for all x,
$P'(x) = 0$ when $1.001 - 0.001x = 0$

$$x = \frac{1.001}{0.001} = 1001.$$

$P(1001) = 10^6[1 + 1000e^{-1.001}] \doteq 368.5 \times 10^6$
The maximum monthly profit will be 368.5 × 106 dollars when 1001 items are produced and sold.
b. The domain for $P(x)$ becomes $0 \le x \le 500$.
$P(500) = 10^6[1 + 499e^{-0.5}] = 303.7 \times 10^6$
Since there are no critical values in the domain, the maximum occurs at an endpoint. The maximum monthly profit when 500 items are produced and sold is 303.7×10^6 dollars.
5. $R(x) = 40x^2e^{-0.4x} + 30, 0 \le x \le 8$
We use the Algorithm for Extreme Values:
$R'(x) = 80xe^{-0.4x} + 40x^2(-0.4)e^{-0.4x}$
$= 40xe^{-0.4x}(2 - 0.4x)$
Since $e^{-0.4x} > 0$ for all x, $R'(x) = 0$ when
$x = 0$ or $2 - 0.4x = 0$
$x = 5$.
$R(0) = 30$
$R(5) \doteq 165.3$
$R(8) \doteq 134.4$
The maximum monthly revenue of 165.3 thousand dollars is achieved when 500 units are produced and sold.
6. $P(t) = 100(e^{-t} - e^{-4t}), 0 \le t \le 3$
$P'(t) = 100(-e^{-t} + 4e^{-4t})$
$= 100e^{-t}(-1 + 4e^{-3t})$
Since $e^{-t} > 0$ for all t, $P'(t) = 0$ when
$4e^{-3t} = 1$

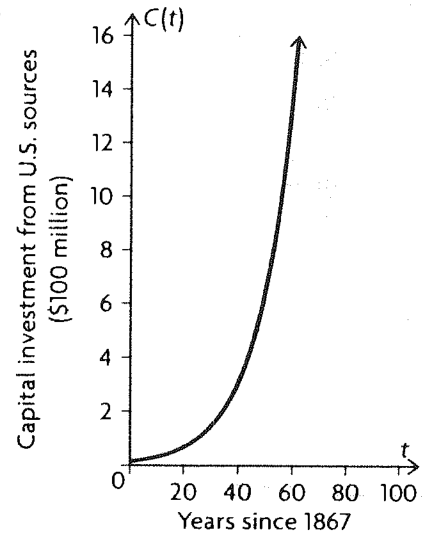
$$e^{-3t} = \frac{1}{4}$$

$-3t = \ln(0.25)$

$$t = \frac{-\ln(0.25)}{3}$$

$= 0.462$.
$P(0) = 0$
$P(0.462) \doteq 47.2$
$P(3) \doteq 4.98$
The highest percentage of people spreading the rumour is 47.2% and occurs at the 0.462 h point.

7. $C = 0.015 \times 10^9e^{0.075\,33t}, 0 \le t \le 100$
a.



b. $\frac{dC}{dt} = 0.015 \times 10^9 \times 0.075\,33e^{0.075\,33t}$
In 1947, $t = 80$ and the growth rate was

$$\frac{dC}{dt} = 0.468\,05 \times 10^9 \text{ dollars/year.}$$

In 1967, $t = 100$ and the growth rate was

$$\frac{dC}{dt} = 2.1115 \times 10^9 \text{ dollars/year.}$$

The ratio of growth rates of 1967 to that of 1947 is

$$\frac{2.1115 \times 10^9}{0.468\,05 \times 10^9} = \frac{4.511}{1}.$$

The growth rate of capital investment grew from 468 million dollars per year in 1947 to 2.112 billion dollars per year in 1967.
c. In 1967, the growth rate of investment as a percentage of the amount invested is

$$\frac{2.1115 \times 10^9}{28.0305 \times 10^9} \times 100 = 7.5\%.$$

d. In 1977, $t = 110$
$C = 59.537 \times 10^9$ dollars

$$\frac{dC}{dt} = 4.4849 \times 10^9 \text{ dollars/year.}$$

e. Statistics Canada data shows the actual amount of U.S. investment in 1977 was 62.5×10^9 dollars. The error in the model is 3.5%.
f. In 2007, $t = 140$.
The expected investment and growth rates are
$C = 570.490 \times 10^9$ dollars and $\frac{dC}{dt} = 42.975 \times 10^9$ dollars/year.

8. a. The growth function is $N = 2^{\frac{t}{5}}$.
The number killed is given by $K = e^{\frac{t}{3}}$.
After 60 minutes, $N = 2^{12}$.
Let T be the number of minutes after 60 minutes. The population of the colony at any time, T after the first 60 minutes is

$$P = N - k$$
$$= 2^{\frac{60+T}{5}} - e^{\frac{T}{3}}, T \geq 0$$
$$\frac{dP}{dt} = 2^{\frac{60+T}{5}}\left(\frac{1}{5}\right)\ln 2 - \frac{1}{3}e^{\frac{T}{3}}$$
$$= 2^{\frac{12+T}{5}}\left(\frac{\ln 2}{5}\right) - \frac{1}{3}e^{\frac{T}{3}}$$
$$= 2^{12} \cdot 2^{\frac{T}{5}}\left(\frac{\ln 2}{5}\right) - \frac{1}{3}e^{\frac{T}{3}}$$
$$\frac{dP}{dt} = 0 \text{ when } 2^{12}\frac{\ln 2}{5}2^{\frac{T}{5}} = \frac{1}{3}e^{\frac{T}{3}} \text{ or}$$
$$3\frac{\ln 2}{5} \cdot 2^{12}\, 2^{\frac{T}{5}} = e^{\frac{T}{3}}.$$

We take the natural logarithm of both sides:

$$\ln\left(3.2^{12}\frac{\ln 2}{5}\right) + \frac{T}{5}\ln 2 = \frac{T}{3}$$
$$7.4404 = T\left(\frac{1}{3} - \frac{\ln 2}{5}\right)$$
$$T = \frac{7.4404}{0.1947} = 38.2 \text{ min.}$$

At $T = 0$, $P = 2^{12} = 4096$.
At $T = 38.2$, $P = 478\,158$.

For $T > 38.2$, $\frac{dP}{dt}$ is always negative.
The maximum number of bacteria in the colony occurs 38.2 min after the drug was introduced. At this time the population numbers 478 158.

b. $P = 0$ when $2^{\frac{60+T}{5}} = e^{\frac{T}{3}}$

$$\frac{60 + T}{5}\ln 2 = \frac{T}{3}$$
$$12\ln 2 = T\left(\frac{1}{3} - \frac{\ln 2}{5}\right)$$
$$T = 42.72$$

The colony will be obliterated 42.72 minutes after the drug was introduced.

9. Let t be the number of minutes assigned to study for the first exam and $30 - t$ minutes assigned to study for the second exam. The measure of study effectiveness for the two exams is given by

$$E(t) = E_1(t) + E_2(30 - t), 0 \leq t \leq 30$$
$$= 0.5\left(10 + te^{-\frac{t}{10}}\right) + 0.6\left(9 + (30 - t)e^{-\frac{30-t}{20}}\right)$$
$$E'(t) = 0.5\left(e^{-\frac{t}{10}} - \frac{1}{10}te^{-\frac{t}{10}}\right)$$
$$+ 0.6\left(-e^{-\frac{30-t}{20}} + \frac{1}{20}(30 - t)e^{-\frac{30-t}{20}}\right)$$
$$= 0.05e^{-\frac{t}{10}}(10 - t) + 0.03e^{-\frac{30-t}{20}}$$
$$(-20 + 30 - t)$$
$$= \left(0.05e^{-\frac{t}{10}} + 0.03e^{-\frac{30-t}{20}}\right)(10 - t)$$

$E'(t) = 0$ when $10 - t = 0$
$t = 10$ (The first factor is always a positive number.)
$E(0) = 5 + 5.4 + 18e^{-\frac{3}{2}} = 14.42$
$E(10) = 16.65$
$E(30) = 11.15$
For maximum study effectiveness, 10 h of study should be assigned to the firs exam and 20 h of study for the second exam.

10. Use the algorithm for finding extreme values. First, find the derivative $f'(x)$. Then, find any critical points by setting $f'(x) = 0$ and solving for x. Also, find the values of x for which $f'(x)$ is undefined. Together these are the critical values. Now, evaluate $f(x)$ for the critical values and the endpoints 2 and -2. The highest value will be the absolute maximum on the interval and the lowest value will be the absolute minimum on the interval.

11. a. $f'(x) = (x^2)(e^x) + (e^x)(2x)$
$= e^x(x^2 + 2x)$
The function is increasing when $f'(x) > 0$ and decreasing when $f'(x) < 0$. First, find the critical values of $f'(x)$. Solve $e^x = 0$ and $(x^2 + 2x) = 0$
e^x is never equal to zero.
$x^2 + 2x = 0$
$x(x + 2) = 0$.
So, the critical values are 0 and -2.

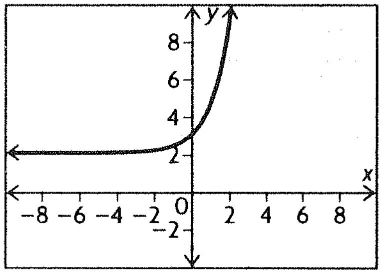
Interval	$e^x(x^2 + 2x)$
$x < -2$	+
$-2 < x < 0$	−
$0 < x$	+

So, $f(x)$ is increasing on the intervals $(-\infty, -2)$ and $(0, \infty)$.
Also, $f(x)$ is decreasing on the interval $(-2, 0)$.
b. At $x = 0$, $f'(x)$ switches from decreasing on the left of zero to increasing on the right of zero. So, $x = 0$ is a minimum. Since it is the only critical point that is a minimum, it is the x-coordinate of the

absolute minimum value of $f(x)$. The absolute minimum value is $f(0) = 0$.

12. a. $y' = e^x$

Setting $e^x = 0$ yields no solutions for x. e^x is a function that is always increasing. So, there is no maximum or minimum value for $y = e^x + 2$.



b. $y' = (x)(e^x) + (e^x)(1)$

$= e^x(x + 1)$

Solve $e^x = 0$ and $(x + 1) = 0$

e^x is never equal to zero.

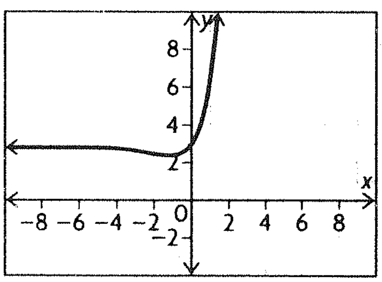
$x + 1 = 0$

$x = -1$.

So there is one critical point: $x = -1$.

Interval	$e^x(x + 1)$
$x < -1$	−
$x > -1$	+

So y is decreasing on the left of $x = -1$ and increasing on the right of $x = -1$. So $x = -1$ is the x-coordinate of the minimum of y. The minimum value is $-e^{-1} + 3 \doteq 2.63$. There is no maximum value.



c. $y' = (2x)(2e^{2x}) + (e^{2x})(2)$

$= 2e^{2x}(2x + 1)$

Solve $2e^{2x} = 0$ and $(2x + 1) = 0$

$2e^{2x}$ is never equal to zero.

$2x + 1 = 0$

$$x = -\frac{1}{2}$$

So there is one critical point: $x = -\frac{1}{2}$.

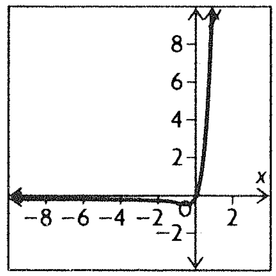
Interval	$2e^{2x}(2x + 1)$
$x < -\frac{1}{2}$	−
$x > -\frac{1}{2}$	+

So y is decreasing on the left of $x = -\frac{1}{2}$ and increasing on the right of $x = -\frac{1}{2}$. So $x = -\frac{1}{2}$ is the x-coordinate of the minimum of y. The minimum value is

$$2\left(-\frac{1}{2}\right)(e^{2(-\frac{1}{2})})$$

$= -e^{-1}$

$\doteq -0.37$. There is no maximum value.

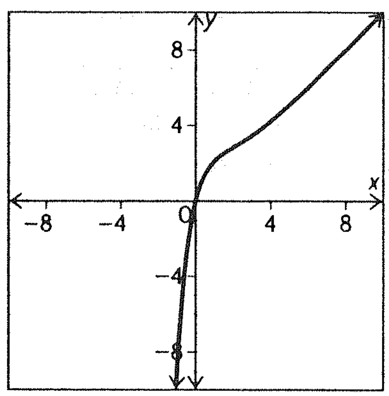


d. $y' = (3x)(-e^{-x}) + (e^{-x})(3) + 1$

$= 3e^{-x}(1 - x) + 1$

Solve $3e^{-x}(1 - x) + 1 = 0$.

This gives no real solutions. By looking at the graph of $y = f(x)$, one can see that the function is always increasing. So, there is no maximum or minimum value for $y = 3xe^{-x} + x$.



13. $P'(x) = (x)(-xe^{-0.5x^2}) + (e^{-0.5x^2})(1)$

$= e^{-0.5x^2}(-x^2 + 1)$

Solve $e^{-0.5x^2} = 0$ and $(1 - x^2) = 0$.

$e^{-0.5x^2}$ gives no critical points.

$1 - x^2 = 0$

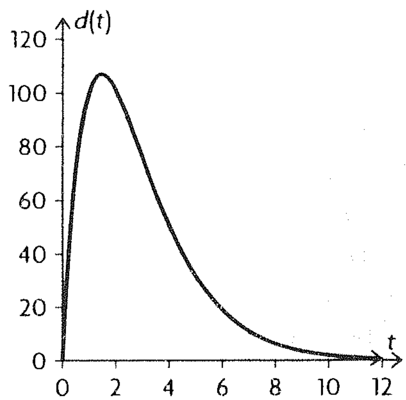
$(1 - x)(1 + x) = 0$

So $x = 1$ and $x = -1$ are the critical points.
So $P(x)$ is decreasing on the left of $x = -1$ and on

Interval	$e^{-0.5x^2}(-x^2 + 1)$
$x < -1$	$-$
$-1 < x < 1$	$+$
$1 < x$	$-$

the right of $x = 1$ and it is increasing between $x = -1$ and $x = 1$. So $x = -1$ is the x-coordinate of the minimum of $P(x)$. Also, $x = 1$ is the x-coordinate of the maximum of $P(x)$. The minimum value is
$P(-1) = (-1)(e^{-0.5(-1)^2}) = -e^{-0.5} \doteq -0.61$.
The maximum value is
$P(1) = (1)(e^{-0.5(1)^2}) = e^{-0.5} \doteq 0.61$.

14. a.



b. The speed is increasing when $d'(t) > 0$ and the speed is decreasing when $d'(t) < 0$.

$$d'(t) = (200t)(-2^{-t})(\ln 2) + (2^{-t})(200)$$
$$= 200(2)^{-t}(-t\ln 2 + 1)$$

Solve $200(2)^{-t} = 0$ and $-t\ln 2 + 1 = 0$.
$200(2)^{-t}$ gives no critical points.
$-t\ln 2 + 1 = 0$

$$t = \frac{1}{\ln 2} \doteq 1.44$$

So $t = \frac{1}{\ln 2}$ is the critical point.

Interval	$200(2)^{-t}(-t\ln 2 + 1)$
$t < \frac{1}{\ln 2}$	$+$
$t > \frac{1}{\ln 2}$	$-$

So the speed of the closing door is increasing when $0 < t < \frac{1}{\ln 2}$ and decreasing when $t > \frac{1}{\ln 2}$.

c. There is a maximum at $t = \frac{1}{\ln 2}$ since $d'(t) < 0$ for $t < \frac{1}{\ln 2}$ and $d'(t) > 0$ for $t > \frac{1}{\ln 2}$.

The maximum speed is

$$d\left(\frac{1}{\ln 2}\right) = 200\left(\frac{1}{\ln 2}\right)(2)^{-\frac{1}{\ln 2}} \doteq 106.15 \text{ degrees/s}$$

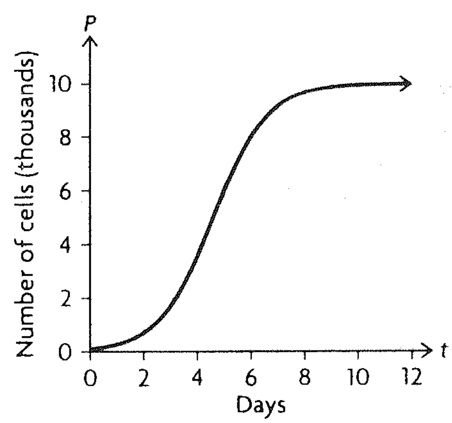
d. The door seems to be closed for $t > 10$ s.

15. The solution starts in a similar way to that of 9. The effectiveness function is
$E(t) = 0.5\left(10 + te^{-\frac{t}{10}}\right) + 0.6\left(9 + (25 - t)e^{-\frac{25-t}{20}}\right)$.
The derivative simplifies to
$E'(t) = 0.05e^{-\frac{t}{10}}(10 - t) + 0.03e^{-\frac{25-t}{20}}(5 - t)$.
This expression is very difficult to solve analytically. By calculation on a graphing calculator, we can determine the maximum effectiveness occurs when $t = 8.16$ hours.

16. $P = \dfrac{aL}{a + (L - a)e^{-kLt}}$

a. We are given $a = 100$, $L = 10\,000$, $k = 0.0001$.

$$P = \frac{10^6}{100 + 9900e^{-t}} = \frac{10^4}{1 + 99e^{-t}}$$
$$= 10^4(1 + 99e^{-t})^{-1}$$



b. We need to determine when the derivative of the growth rate $\left(\frac{dP}{dt}\right)$ is zero, i.e., when $\frac{d^2P}{dt^2} = 0$.

$$\frac{dP}{dt} = \frac{-10^4(-99e^{-t})}{(1 + 99e^{-t})^2} = \frac{990000e^{-t}}{(1 + 99e^{-t})^2}$$

$$\frac{d^2P}{dt^2} = \frac{-990000e^{-t}(1 + 99e^{-t})^2 - 990000e^{-t}}{(1 + 99e^{-t})^4}$$
$$\times \frac{(2)(1 + 99e^{-t})(-99e^{-t})}{(1 + 99e^{-t})^4}$$
$$= \frac{-990000e^{-t}(1 + 99e^{-t}) + 198(990000)e^{-2t}}{(1 + 99e^{-t})^3}$$

$\frac{d^2P}{dt^2} = 0$ when

$990000e^{-t}(-1 - 99e^{-t} + 198e^{-t}) = 0$

$99e^{-t} = 1$

$e^t = 99$

$t = \ln 99$

$\doteq 4.6$

After 4.6 days, the rate of change of the growth rate is zero. At this time the population numbers 5012.

c. When $t = 3$, $\frac{dP}{dt} = \frac{990\,000e^{-3}}{(1 + 99e^{-3})^2} \doteq 1402$ cells/day.

When $t = 8$, $\frac{dP}{dt} = \frac{990\,000e^{-8}}{(1 + 99e^{-8})^2} \doteq 311$ cells/day.

The rate of growth is slowing down as the colony is getting closer to its limiting value.

Mid-Chapter Review, pp. 248–249

1. a. $\frac{dy}{dx} = \frac{d(5e^{-3x})}{dx}$

$= (5e^{-3x})(-3x)'$

$= (5e^{-3x})(-3)$

$= -15e^{-3x}$

b. $\frac{dy}{dx} = \frac{d(7e^{\frac{1}{7}x})}{dx}$

$= (7e^{\frac{1}{7}x})\left(\frac{1}{7}x\right)'$

$= (7e^{\frac{1}{7}x})\left(\frac{1}{7}\right)$

$= e^{\frac{1}{7}x}$

c. $\frac{dy}{dx} = (x^3)(e^{-2x})' + (x^3)'(e^{-2x})$

$= (x^3)((e^{-2x})(-2x)') + (3x^2)(e^{-2x})$

$= (x^3)((e^{-2x}))(-2) + 3x^2e^{-2x}$

$= -2x^3e^{-2x} + 3x^2e^{-2x}$

$= e^{-2x}(-2x^3 + 3x^2)$

d. $\frac{dy}{dx} = (x - 1)^2(e^x)' + ((x - 1)^2)'(e^x)$

$= (x - 1)^2(e^x) + (2(x - 1))(e^x)$

$= (x^2 - 2x + 1)(e^x) + (2x - 2)(e^x)$

$= (e^x)(x^2 - 2x + 1 + 2x - 2)$

$= (e^x)(x^2 - 1)$

e. $\frac{dy}{dx} = 2(x - e^{-x})(x - e^{-x})'$

$= 2(x - e^{-x})(1 - (e^{-x})(-x)')$

$= 2(x - e^{-x})(1 - (e^{-x})(-1))$

$= 2(x - e^{-x})(1 + e^{-x})$

$= 2(x + xe^{-x} - e^{-x} - e^{-x+-x})$

$= 2(x + xe^{-x} - e^{-x} - e^{-2x})$

f. $\frac{dy}{dx} = \frac{(e^x + e^{-x})(e^x - e^{-x})'}{(e^x + e^{-x})^2}$

$- \frac{(e^x - e^{-x})(e^x + e^{-x})'}{(e^x + e^{-x})^2}$

$= \frac{(e^x + e^{-x})(e^x - (e^{-x})(-x)')}{(e^x + e^{-x})^2}$

$- \frac{(e^x - e^{-x})(e^x + (e^{-x})(-x)')}{(e^x + e^{-x})^2}$

$= \frac{(e^x + e^{-x})(e^x - (e^{-x})(-1))}{(e^x + e^{-x})^2}$

$- \frac{(e^x - e^{-x})(e^x + (e^{-x})(-1))}{(e^x + e^{-x})^2}$

$= \frac{(e^x + e^{-x})(e^x + e^{-x})}{(e^x + e^{-x})^2}$

$- \frac{(e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2}$

$= \frac{e^{2x} + e^0 + e^0 + e^{-2x}}{(e^x + e^{-x})^2}$

$- \frac{(e^{2x} - e^0 - e^0 + e^{-2x})}{(e^x + e^{-x})^2}$

$= \frac{e^{2x} + e^0 + e^0 + e^{-2x} - e^{2x}}{(e^x + e^{-x})^2}$

$+ \frac{e^0 + e^0 - e^{-2x}}{(e^x + e^{-x})^2}$

$= \frac{4}{(e^x + e^{-x})^2}$

2. a. $\frac{dP}{dt} = 100e^{-5t}(-5t)'$

$= 100e^{-5t}(-5)$

$= -500e^{-5t}$

b. The time is needed for when the sample of the substance is at half of the original amount. So, find t when $P = \frac{1}{2}$.

$P = 100e^{-5t}$

$\frac{1}{2} = 100e^{-5t}$

$\frac{1}{200} = e^{-5t}$

$\ln \frac{1}{200} = -5t$

$\frac{\ln \frac{1}{200}}{-5} = t$

Now, the question asks for $\frac{dP}{dt} = P'$ when

$$t = \frac{\ln \frac{1}{200}}{-5} \doteq 1.06$$

$$P'\left(\frac{\ln \frac{1}{200}}{-5}\right) = -2.5 \text{ (using a calculator)}$$

3. $$\frac{dy}{dx} = (-x)(e^x)' + (e^x)(-x)'$$
$$= (-x)(e^x) + (e^x)(-1)$$
$$= -xe^x - e^x$$

At the point $x = 0$,
$$\frac{dy}{dx} = -0e^0 - e^0 = -1.$$
At the point $x = 0$,
$$y = 2 - 0e^0 = 2$$
So, an equation of the tangent to the curve at the point $x = 0$ is
$$y - 2 = -1(x - 0)$$
$$y - 2 = -x$$
$$y = -x + 2$$
$$x + y - 2 = 0$$

4. a. $$y' = -3(e^x)'$$
$$= -3e^x$$
$$y'' = -3e^x$$

b. $$y' = (x)(e^{2x})' + (e^{2x})(x)'$$
$$= (x)((e^{2x}) + (2x)') + (e^{2x})(1)$$
$$= (x)((e^{2x})(2)) + e^{2x}$$
$$= 2xe^{2x} + e^{2x}$$
$$y'' = (2x)(e^{2x})' + (e^{2x})(2x)' + e^{2x}(2x)'$$
$$= (2x)((e^{2x})(2x)') + (e^{2x})(2) + (e^{2x})(2)$$
$$= (2x)((e^{2x})(2)) + 2e^{2x} + 2e^{2x}$$
$$= 4xe^{2x} + 4e^{2x}$$

c. $$y' = (e^x)(4 - x)' + (4 - x)(e^x)'$$
$$= (e^x)(-1) + (4 - x)(e^x)$$
$$= -e^x + 4e^x - xe^x$$
$$= 3e^x - xe^x$$
$$y'' = (3e^x)' - [(x)(e^x)' + (e^x)(x)']$$
$$= 3e^x - [xe^x + (e^x)(1)]$$
$$= 3e^x - xe^x - e^x$$
$$= 2e^x - xe^x$$

5. a. $$\frac{dy}{dx} = (8^{2x+5})(\ln 8)(2x + 5)'$$
$$= (8^{2x+5})(\ln 8)(2)$$
$$= 2(\ln 8)(8^{2x+5})$$

b. $$\frac{dy}{dx} = 3.2((10)^{0.2x})(\ln 10)(0.2x)'$$
$$= 3.2((10)^{0.2x})(\ln 10)(0.2)$$
$$= 0.64(\ln 10)((10)^{2x})$$

c. $$f'(x) = (x^2)(2^x)' + (2^x)(x^2)'$$
$$= (x^2)(2^x)(\ln 2) + (2^x)(2x)$$
$$= (\ln 2)(x^2 2^x) + 2x2^x$$
$$= 2^x((\ln 2)(x^2) + 2x)$$

d. $$H'(x) = 300((5)^{3x-1})(\ln 5)(3x - 1)'$$
$$= 300((5)^{3x-1})(\ln 5)(3)$$
$$= 900(\ln 5)(5)^{3x-1}$$
$$= 900(\ln 5)(5)^{3x-1}$$

e. $$q'(x) = (1.9)^x \cdot (\ln 1.9) + 1.9(x)^{1.9-1}$$
$$= (1.9)^x \cdot (\ln 1.9) + 1.9(x)^{0.9}$$
$$= (\ln 1.9)(1.9)^x + 1.9x^{0.9}$$

f. $$f'(x) = (x - 2)^2(4^x)' + (4^x)((x - 2)^2)'$$
$$= (x - 2)^2(4^x)(\ln 4) + (4^x)(2(x - 2))$$
$$= (\ln 4)(4^x)(x - 2)^2 + (4^x)(2x - 4)$$
$$= 4^x((\ln 4)(x - 2)^2 + 2x - 4)$$

6. a. The initial number of rabbits in the forest is given by the time $t = 0$.
$$R(0) = 500\left(10 + e^{-\frac{0}{10}}\right)$$
$$= 500(10 + 1)$$
$$= 500(11)$$
$$= 5500$$

b. The rate of change is the derivative, $\frac{dR}{dt}$.
$$R(t) = 5000 + 500\left(e^{-\frac{t}{10}}\right)$$
$$\frac{dR}{dt} = 0 + 500\left(e^{-\frac{t}{10}}\right)\left(-\frac{t}{10}\right)'$$
$$= 500\left(e^{-\frac{t}{10}}\right)\left(-\frac{1}{10}\right)$$
$$= -50\left(e^{-\frac{t}{10}}\right)$$

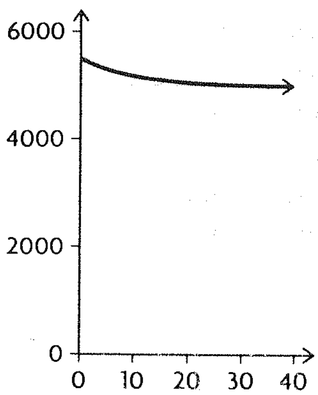
c. 1 year = 12 months

The question asks for $\frac{dR}{dt} = R'$ when $t = 12$.
$$R'(12) = -50\left(e^{-\frac{12}{10}}\right)$$
$$\doteq -15.06$$

d. To find the maximum number of rabbits, optimize the function.
$$R'(t) = -50\left(e^{-\frac{t}{10}}\right)$$
$$0 = -50\left(e^{-\frac{t}{10}}\right)$$
$$0 = e^{-\frac{t}{10}}$$

When solving, the natural log (ln) of both sides must be taken, but (ln 0) does not exist. So there are no solutions to the equation. The function is therefore always decreasing. So, the largest number of rabbits will exist at the earliest time in the interval at time $t = 0$. To check, compare $R(0)$ and $R(36)$. $R(0) = 5500$ and $R(36) \doteq 5013$. So, the largest number of rabbits in the forest during the first 3 years is 5500.

e.



The graph is constantly decreasing. The y-intercept is at the point $(0, 5500)$. Rabbit populations normally grow exponentially, but this population is shrinking exponentially. Perhaps a large number of rabbit predators such as snakes recently began to appear in the forest. A large number of predators would quickly shrink the rabbit population.

7. The highest concentration of the drug can be found by optimizing the given function.

$C(t) = 10e^{-2t} - 10e^{-3t}$

$C'(t) = (10e^{-2t})(-2t)' - (10e^{-3t})(-3t)'$

$= (10e^{-2t})(-2) - (10e^{-3t})(-3)$

$= -20e^{-2t} + 30e^{-3t}$

Set the derivative of the function equal to zero and find the critical points.

$0 = -20e^{-2t} + 30e^{-3t}$

$20e^{-2t} = 30e^{-3t}$

$\frac{2}{3}e^{-2t} = e^{-3t}$

$\frac{2}{3} = \frac{e^{-3t}}{e^{-2t}}$

$\frac{2}{3} = (e^{-3t})(e^{2t})$

$\frac{2}{3} = e^{-3t+2t}$

$\frac{2}{3} = e^{-t}$

$\ln\frac{2}{3} = -t$

$-\left(\ln\frac{2}{3}\right) = t$

Therefore, $t = -\left(\ln\frac{2}{3}\right) \doteq 0.41$ is the critical value.
Now, use the algorithm for finding extreme values.

$C(0) = 10(e^0 - e^0) = 0$

$C\left(-\left(\ln\frac{2}{3}\right)\right) \doteq 1.48$ (using a calculator)

$C(5) = 0.0005$

So, the function has a maximum when $t = -\left(\ln\frac{2}{3}\right) \doteq 0.41$. Therefore, during the first five hours, the highest concentration occurs at about 0.41 hours.

8. $y = ce^{kx}$

$y' = cke^{kx}$

The original function is increasing when its derivative is positive and decreasing when its derivative is negative.

$e^{kx} > 0$ for all $k, x \in \mathbf{R}$.

So, the original function represents growth when $ck > 0$, meaning that c and k must have the same sign. The original function represents decay when c and k have opposite signs.

9. a. $A(t) = 5000e^{0.02t}$

$= 5000e^{0.02(0)}$

$= 5000$

The initial population is 5000.

b. at $t = 7$

$A(7) = 5000e^{0.02(7)} = 5751$

After a week, the population is 5751.

c. at $t = 30$

$A(30) = 5000e^{0.02(30)} = 9111$

After 30 days, the population is 9111.

10. a. $P(5) = 760e^{-0.125(5)}$

$\doteq 406.80$ mm Hg

b. $P(7) = 760e^{-0.125(7)}$

$\doteq 316.82$ mm Hg

c. $P(9) = 760e^{-0.125(9)}$

$\doteq 246.74$ mm Hg

11. $A = 100e^{-0.3x}$

$A' = 100e^{-0.3x}(-0.3)$

$= -30e^{-0.3x}$

When 50% of the substance is gone, $y = 50$

$50 = 100e^{-0.3x}$

$0.5 = e^{-0.3x}$

$\ln(0.5) = \ln e^{-0.3x}$

$\ln(0.5) = -0.3x \ln e$

$$\frac{\ln 0.5}{\ln e} = -0.3x$$
$$-\frac{\ln 0.5}{0.3 \ln e} = x$$
$$x = 2.31$$
$$A' = -30e^{-0.3x}$$
$$A'(2.31) = -30e^{-0.3(2.31)}$$
$$A' \doteq -15$$

When 50% of the substance is gone, the rate of decay is 15% per year.

12. $f(x) = xe^x$

$$f'(x) = xe^x + (1)e^x$$
$$= e^x(x + 1)$$

So $e^x > 0$

$x + 1 > 0$

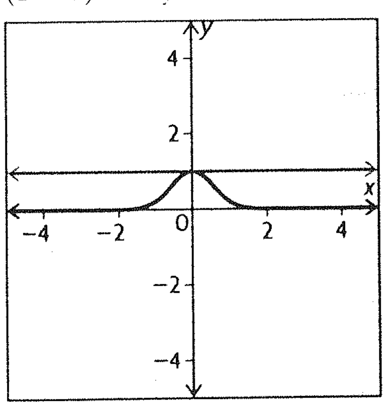
$x > -1$

This means that the function is increasing when $x > -1$.

13. $y = 5^{-x^2}$

When $x = 1$,

$$y = \frac{1}{5}$$
$$y' = 5^{-x^2}(-2x)\ln 5$$
$$y' = -\frac{2}{5}\ln 5$$
$$5y - \frac{1}{5} = -\frac{2}{5}\ln 5(x - 1)$$
$$5y - 1 = -2\ln 5(x - 1)$$
$$5y - 1 = (-2\ln 5)x + 2\ln 5$$
$$(2\ln 5)x + 5y = 2\ln 5 + 1$$



14. a. $A = P(1 + i)^t$

$$A(t) = 1000(1 + 0.06)^t$$
$$= 1000(1.06)^t$$

b. $A'(t) = 1000(1.06)^t(1)\ln(1.06)$

$$= 1000(1.06)^t \ln 1.06$$

c. $A'(2) = 1000(1.06)^2 \ln 1.06$

$$= \$65.47$$
$$A'(5) = 1000(1.06)^5 \ln 1.06$$
$$= \$77.98$$
$$A'(10) = 1000(1.06)^{10} \ln 1.06$$
$$= \$104.35$$

d. No, the rate is not constant.

e. $\frac{A'(2)}{A(2)} = \ln 1.06$

$$\frac{A'(5)}{A(5)} = \ln 1.06$$
$$\frac{A'(10)}{A(10)} = \ln 1.06$$

f. All the ratios are equivalent (they equal ln 1.06, which is about 0.058 27), which means that $\frac{A'(t)}{A(t)}$ is constant.

15. $y = ce^x$

$$y' = c(e^x) + (0)e^x$$
$$= ce^x$$
$$y = y' = ce^x$$

5.4 The Derivatives of $y = \sin x$ and $y = \cos x$, pp. 256–257

1. a. $\frac{dy}{dx} = (\cos 2x) \cdot \frac{d(2x)}{dx}$

$$= 2\cos 2x$$

b. $\frac{dy}{dx} = -2(\sin 3x) \cdot \frac{d(3x)}{dx}$

$$= -6\sin 3x$$

c. $\frac{dy}{dx} = (\cos(x^3 - 2x + 4)) \cdot \frac{d(x^3 - 2x + 4)}{dx}$

$$= (3x^2 - 2)(\cos(x^3 - 2x + 4))$$

d. $\frac{dy}{dx} = -2\sin(-4x) \cdot \frac{d(-4x)}{dx}$

$$= 8\sin(-4x)$$

e. $\frac{dy}{dx} = \cos(3x) \cdot \frac{d(3x)}{dx} + \sin(4x) \cdot \frac{d(4x)}{dx}$

$$= 3\cos(3x) + 4\sin(4x)$$

f. $\frac{dy}{dx} = 2^x(\ln 2) + 2\cos x + 2\sin x$

g. $\frac{dy}{dx} = \cos(e^x) \cdot \frac{d(e^x)}{dx}$

$$= e^x\cos(e^x)$$

h. $\frac{dy}{dx} = 3\cos(3x + 2\pi) \cdot \frac{d(3x + 2\pi)}{dx}$

$$= 9\cos(3x + 2\pi)$$

i. $\frac{dy}{dx} = 2x - \sin x + 0$
$= 2x - \sin x$

j. $\frac{dy}{dx} = \cos\left(\frac{1}{x}\right) \cdot \frac{d\left(\frac{1}{x}\right)}{dx}$
$= -\frac{1}{x^2}\cos\left(\frac{1}{x}\right)$

2. a. $\frac{dy}{dx} = (2\sin x)(-\sin x) + (\cos x)(2\cos x)$
$= -2\sin^2 x + 2\cos^2 x$
$= 2(\cos^2 x - \sin^2 x)$
$= 2\cos(2x)$

b. $y = (x^{-1})(\cos 2x)$
$\frac{dy}{dx} = (x^{-1})(-2\sin 2x) + (\cos 2x)(-x^{-2})$
$= -\frac{2\sin 2x}{x} - \frac{\cos 2x}{x^2}$

c. $\frac{dy}{dx} = -\sin(\sin 2x) \cdot \frac{d(\sin 2x)}{dx}$
$= -\sin(\sin 2x) \cdot 2\cos 2x$

d. $y = (\sin x)(1 + \cos x)^{-1}$
$\frac{dy}{dx} = (\sin x)(-(1 + \cos x)^{-2} \cdot (-\sin x)$
$+ (1 + \cos x)^{-1}(\cos x)$
$= \frac{-\sin^2 x}{-(1+\cos x)^2} + \frac{\cos x}{1 + \cos x}$
$= \frac{\sin^2 x}{(1+\cos x)^2} + \frac{\cos x(1 + \cos x)}{(1+\cos x)^2}$
$= \frac{\sin^2 x + \cos^2 x + \cos x}{(1+\cos x)^2}$
$= \frac{1 + \cos x}{(1 + \cos x)^2}$
$= \frac{1}{1 + \cos x}$

e. $\frac{dy}{dx} = (e^x)(-\sin x + \cos x) + (\cos x + \sin x)(e^x)$
$= e^x(-\sin x + \cos x + \cos x + \sin x)$
$= e^x(2\cos x)$

f. $\frac{dy}{dx} = (2x^3)(\cos x) + (\sin x)(6x^2)$
$-[(3x)(-\sin x) + (\cos x)(3)]$
$= 2x^3\cos x + 6x^2\sin x + 3x\sin x - 3\cos x$

3. a. When $x = \frac{\pi}{3}$, $f(x) = f\left(\frac{\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$.
$f'(x) = \cos x$
$f'\left(\frac{\pi}{3}\right) = \cos\frac{\pi}{3}$
$= \frac{1}{2}$

So an equation for the tangent at the point $x = \frac{\pi}{3}$ is
$y - \frac{\sqrt{3}}{2} = \frac{1}{2}\left(x - \frac{\pi}{3}\right)$
$2y - \sqrt{3} = x - \frac{\pi}{3}$
$-x + 2y + \left(\frac{\pi}{3} - \sqrt{3}\right) = 0$

b. When $x = 0$, $f(x) = f(0) = 0 + \sin(0) = 0$.
$f'(x) = 1 + \cos x$
$f'(0) = 1 + \cos(0)$
$= 1 + 1$
$= 2$

So an equation for the tangent at the point $x = 0$ is
$y - 0 = 2(x - 0)$
$y = 2x$
$-2x + y = 0$

c. When $x = \frac{\pi}{4}$, $f(x) = f\left(\frac{\pi}{4}\right) = \cos\left(4 \cdot \frac{\pi}{4}\right)$
$= \cos(\pi)$
$= -1$
$f'(x) = -\sin(4x) \cdot \frac{d(4x)}{dx}$
$= -4\sin(4x)$
$f'\left(\frac{\pi}{4}\right) = -4\sin\left(4 \cdot \frac{\pi}{4}\right)$
$= -4\sin(\pi)$
$= 0$

So an equation for the tangent at the point $x = \frac{\pi}{4}$ is
$y - (-1) = 0\left(x - \frac{\pi}{4}\right)$
$y + 1 = 0$
$y = -1$

d. $f(x) = \sin 2x + \cos x$, $x = \frac{\pi}{2}$

The point of contact is $\left(\frac{\pi}{2}, 0\right)$. The slope of the tangent line at any point is $f'(x) = 2\cos 2x - \sin x$. At $\left(\frac{\pi}{2}, 0\right)$, the slope of the tangent line is

$2\cos\pi - \sin\frac{\pi}{2} = -3.$

An equation of the tangent line is $y = -3\left(x - \frac{\pi}{2}\right)$.

e. $f(x) = \cos\left(2x + \frac{\pi}{3}\right), x = \frac{\pi}{4}$

The point of tangency is $\left(\frac{\pi}{4}, -\frac{\sqrt{3}}{2}\right)$. The slope of the tangent line at any point is $f'(x) = -2\sin\left(2x + \frac{\pi}{3}\right)$.

At $\left(\frac{\pi}{4}, -\frac{\sqrt{3}}{2}\right)$, the slope of the tangent line is

$-2\sin\left(\frac{5\pi}{6}\right) = -1.$

An equation of the tangent line is

$y + \frac{\sqrt{3}}{2} = -\left(x - \frac{\pi}{4}\right).$

f. When $x = \frac{\pi}{2}$, $f(x) = f\left(\frac{\pi}{2}\right) = 2\sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right)$

$$= 2(1)(0) = 0$$

$$\begin{aligned} f'(x) &= (2\sin x)(-\sin x) + (\cos x)(2\cos x) \\ &= -2\sin^2 x + 2\cos^2 x \\ &= 2(\cos^2 x - \sin^2 x) \\ &= 2\cos(2x) \end{aligned}$$

$$\begin{aligned} f'\left(\frac{\pi}{2}\right) &= 2\cos\left(2 \cdot \frac{\pi}{2}\right) \\ &= 2\cos\pi \\ &= -2 \end{aligned}$$

So an equation for the tangent when $x = \frac{\pi}{2}$ is

$$y - 0 = -2\left(x - \frac{\pi}{2}\right)$$

$$y = -2x + \pi$$

$$2x + y - \pi = 0$$

4. a. One could easily find $f'(x)$ and $g'(x)$ to see that they both equal $2(\sin x)(\cos x)$. However, it is easier to notice a fundamental trigonometric identity. It is known that $\sin^2 x + \cos^2 x = 1$. So, $\sin^2 x = 1 - \cos^2 x$.

Therefore, one can notice that $f(x)$ is in fact equal to $g(x)$. So, because $f(x) = g(x)$, $f'(x) = g'(x)$.

b. $f'(x)$ and $g'(x)$ are negatives of each other. That is, $f'(x) = 2(\sin x)(\cos x)$ while $g'(x) = -2(\sin x)(\cos x)$.

5. a. $v(t) = (\sin(\sqrt{t}))^2$

$$\begin{aligned} v'(t) &= 2\sin(\sqrt{t}) \cdot \frac{d(\sin(\sqrt{t}))}{dt} \\ &= 2\sin(\sqrt{t}) \cdot \cos(\sqrt{t}) \cdot \frac{d(\sqrt{t})}{dt} \\ &= 2\sin(\sqrt{t}) \cdot \cos(\sqrt{t}) \cdot \frac{1}{2}t^{-\frac{1}{2}} \\ &= 2\sin(\sqrt{t}) \cdot \cos(\sqrt{t}) \cdot \frac{1}{2\sqrt{t}} \\ &= \frac{\sin(\sqrt{t})\cos(\sqrt{t})}{\sqrt{t}} \end{aligned}$$

b. $v(t) = (1 + \cos t + \sin^2 t)^{\frac{1}{2}}$

$$\begin{aligned} v'(t) &= \frac{1}{2}(1 + \cos t + \sin^2 t)^{-\frac{1}{2}} \\ &\quad \times \frac{d(1 + \cos t + (\sin t)^2)}{dt} \\ &= \frac{-\sin t + 2(\sin t) \cdot \frac{d(\sin t)}{dt}}{2\sqrt{1 + \cos t + \sin^2 t}} \\ &= \frac{-\sin t + 2(\sin t)(\cos t)}{2\sqrt{1 + \cos t + \sin^2 t}} \end{aligned}$$

c. $h(x) = \sin x \sin 2x \sin 3x$

So, treat $\sin x \sin 2x$ as one function, say $f(x)$ and treat $\sin 3x$ as another function, say $g(x)$.

Then, the product rule may be used with the chain rule:

$$\begin{aligned} h'(x) &= f(x)g'(x) + g(x)f'(x) \\ &= (\sin x \sin 2x)(3\cos 3x) \\ &\quad + (\sin 3x)[(\sin x)(2\cos 2x) \\ &\quad + (\sin 2x)(\cos x)] \\ &= 3\sin x \sin 2x \cos 3x \\ &\quad + 2\sin x \sin 3x \cos 2x \\ &\quad + \sin 2x \sin 3x \cos x \end{aligned}$$

d.
$$\begin{aligned} m'(x) &= 3(x^2 + \cos^2 x)^2 \cdot \frac{d(x^2 + (\cos x)^2)}{dx} \\ &= 3(x^2 + \cos^2 x)^2 \cdot (2x + 2(\cos x)(-\sin x)) \\ &= 3(x^2 + \cos^2 x)^2 \cdot (2x - 2\sin x \cos x) \end{aligned}$$

6. By the algorithm for finding extreme values, the maximum and minimum values occur at points on the graph where $f'(x) = 0$, or at an endpoint of the interval.

a. $\frac{dy}{dx} = -\sin x + \cos x$

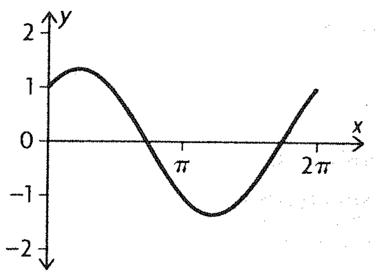
Set $\frac{dy}{dx} = 0$ and solve for x to find any critical points.

$$\begin{aligned} \cos x - \sin x &= 0 \\ \cos x &= \sin x \\ 1 &= \frac{\sin x}{\cos x} \\ 1 &= \tan x \\ x &= \frac{\pi}{4}, \frac{5\pi}{4} \end{aligned}$$

Evaluate $f(x)$ at the critical numbers, including the endpoints of the interval.

x	0	$\frac{\pi}{4}$	$\frac{5\pi}{4}$	2π
$f(x) = \cos x + \sin x$	1	$\sqrt{2}$	$-\sqrt{2}$	1

So, the absolute maximum value on the interval is $\sqrt{2}$ when $x = \frac{\pi}{4}$ and the absolute minimum value on the interval is $-\sqrt{2}$ when $x = \frac{5\pi}{4}$.



b. $\frac{dy}{dx} = 1 - 2\sin x$

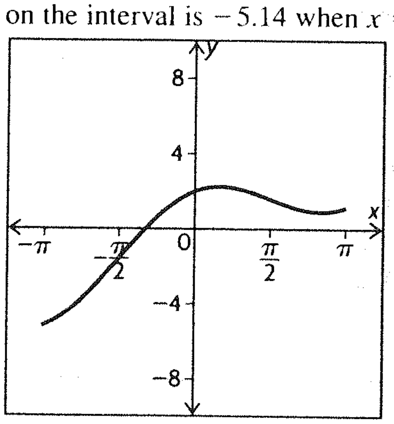
Set $\frac{dy}{dx} = 0$ and solve for x to find any critical points.

$$1 - 2\sin x = 0$$
$$1 = 2\sin x$$
$$\frac{1}{2} = \sin x$$
$$x = \frac{\pi}{6}, \frac{5\pi}{6}$$

Evaluate $f(x)$ at the critical numbers, including the endpoints of the interval.

x	$-\pi$	$-\frac{\pi}{6}$	$\frac{\pi}{6}$	π
$f(x) = x + 2\cos x$	$-\pi - 2$ $\doteq -5.14$	$-\frac{\pi}{6} + \sqrt{3}$ $\doteq 1.21$	$\frac{\pi}{6} + \sqrt{3}$ $\doteq 2.26$	$\pi - 2$ $\doteq 1.14$

So, the absolute maximum value on the interval is 2.26 when $x = \frac{\pi}{6}$ and the absolute minimum value on the interval is -5.14 when $x = -\pi$.



c. $\frac{dy}{dx} = \cos x + \sin x$

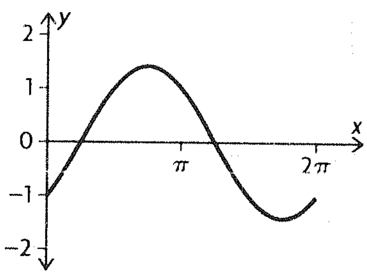
Set $\frac{dy}{dx} = 0$ and solve for x to find any critical points.

$$\cos x + \sin x = 0$$
$$\sin x = -\cos x$$
$$\frac{\sin x}{\cos x} = -1$$
$$\tan x = -1$$
$$x = \frac{3\pi}{4}, \frac{7\pi}{4}$$

Evaluate $f(x)$ at the critical numbers, including the endpoints of the interval.

x	0	$\frac{3\pi}{4}$	$\frac{7\pi}{4}$	2
$f(x) = \sin x - \cos x$	-1	$\sqrt{2}$	$-\sqrt{2}$	-1

So, the absolute maximum value on the interval is $\sqrt{2}$ when $x = \frac{3\pi}{4}$ and the absolute minimum value on the interval is $-\sqrt{2}$ when $x = \frac{7\pi}{4}$.



d. $\frac{dy}{dx} = 3\cos x - 4\sin x$

Set $\frac{dy}{dx} = 0$ and solve for x to find any critical points.

$$3\cos x - 4\sin x = 0$$
$$3\cos x = 4\sin x$$
$$\frac{3}{4} = \frac{\sin x}{\cos x}$$
$$\frac{3}{4} = \tan x$$
$$\tan^{-1}\left(\frac{3}{4}\right) = \tan^{-1}(\tan x)$$

Using a calculator, $x \doteq 0.6435$.

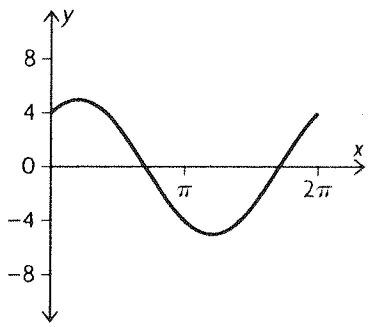
This is a critical value, but there is also one more in the interval $0 \le x \le 2\pi$. The period of $\tan x$ is π, so adding π to the one solution will give another solution in the interval.

$x = 0.6435 + \pi \doteq 3.7851$

Evaluate $f(x)$ at the critical numbers, including the endpoints of the interval.

x	0	0.64	3.79	2π
$f(x) = 3\sin x + 4\cos x$	4	5	−5	4

So, the absolute maximum value on the interval is 5 when $x \doteq 0.64$ and the absolute minimum value on the interval is -5 when $x \doteq 3.79$.



7. a. The particle will change direction when the velocity, $s'(t)$, changes from positive to negative.

$s'(t) = 16\cos 2t$

Set $s'(t) = 0$ and solve for t to find any critical points.

$0 = 16\cos 2t$

$0 = \cos 2t$

$\frac{\pi}{2}, \frac{3\pi}{2} = 2t$

$\frac{\pi}{4}, \frac{3\pi}{4} = t$

Also, there is no given interval so it will be beneficial to locate all solutions.

Therefore, $t = \frac{\pi}{4} + \pi k, \frac{3\pi}{4} + \pi k$ for some positive integer k constitutes all solutions.

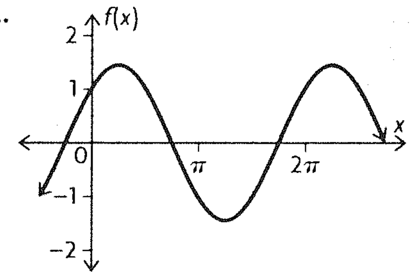
One can create a table and notice that on each side of any value of t, the function is increasing on one side and decreasing on the other. So, each t value is either a maximum or a minimum.

t	$\frac{\pi}{4}$	$\frac{3\pi}{4}$	$\frac{5\pi}{4}$	$\frac{7\pi}{4}$
$s(t) = 8\sin 2t$	8	−8	8	−8

The table continues in this pattern for all critical values t. So, the particle changes direction at all critical values. That is, it changes direction for $t = \frac{\pi}{4} + \pi k, \frac{3\pi}{4} + \pi k$ for positive integers k.

b. From the table or a graph, one can see that the particle's maximum velocity is 8 at the time $t = \frac{\pi}{4} + \pi k$.

8. a.



b. The tangent to the curve $f(x)$ is horizontal at the point(s) where $f'(x)$ is zero.

$f'(x) = -\sin x + \cos x$

Set $f'(x) = 0$ and solve for x to find any critical points.

$\cos x - \sin x = 0$

$\cos x = \sin x$

$1 = \frac{\sin x}{\cos x}$

$1 = \tan x$

$x = \frac{\pi}{4}$ (*Note:* The solution $x = \frac{5\pi}{4}$ is not in the interval $0 \le x \le \pi$ so it is not included.) When $x = \frac{\pi}{4}, f(x) = f\left(\frac{\pi}{4}\right) = \sqrt{2}$.

So, the coordinates of the point where the tangent to the curve of $f(x)$ is horizontal is $\left(\frac{\pi}{4}, \sqrt{2}\right)$.

9. $\csc x = \frac{1}{\sin x} = (\sin x)^{-1}$

$\sec x = \frac{1}{\cos x} = (\cos x)^{-1}$

Now, the power rule can be used to compute the derivates of $\csc x$ and $\sec x$.

$$((\sin x)^{-1})' = -(\sin x)^{-2} \cdot \frac{d(\sin x)}{dx}$$
$$= -(\sin x)^{-2} \cdot \cos x$$
$$= -\frac{\cos x}{(\sin x)^2}$$

$$((\sin x)^{-1})' = -(\sin x)^{-2} \cdot \frac{d(\sin x)}{dx}$$
$$= -\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x}$$
$$= -\csc x \cot x$$

$$((\cos x)^{-1})' = -(\cos x)^{-2} \cdot \frac{d(\cos x)}{dx}$$
$$= -(\cos x)^{-2} \cdot (-\sin x)$$
$$= \frac{\sin x}{(\cos x)^2}$$
$$= \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}$$
$$= \sec x \tan x$$

10. $\frac{dy}{dx} = -2\sin 2x$

At the point $\left(\frac{\pi}{6}, \frac{1}{2}\right)$,

$$\begin{aligned}\frac{dy}{dx} &= -2\sin\left(2\cdot\frac{\pi}{6}\right)\\ &= -2\sin\left(\frac{\pi}{3}\right)\\ &= -2\left(\frac{\sqrt{3}}{2}\right)\\ &= -\sqrt{3}.\end{aligned}$$

Therefore, at the point $\left(\frac{\pi}{6}, \frac{1}{2}\right)$, the slope of the tangent to the curve $y = \cos 2x$ is $-\sqrt{3}$.

11. a. The particle will change direction when the velocity, $s'(t)$ changes from positive to negative.

$s'(t) = 16\cos 4t$

Set $s'(t) = 0$ and solve for t to find any critical points.

$$\begin{aligned}0 &= 16\cos 4t\\ 0 &= \cos 4t\end{aligned}$$

$\frac{\pi}{2}, \frac{3\pi}{2} = 4t$

$\frac{\pi}{8}, \frac{3\pi}{8} = t$

Also, there is no given interval so it will be beneficial to locate all solutions.

Therefore, $t = \frac{\pi}{8} + \pi k, \frac{3\pi}{8} + \pi k$ for some positive integer k constitutes all solutions.

One can create a table and notice that on each side of any value of t, the function is increasing on one side and decreasing on the other. So, each t value is either a maximum or a minimum.

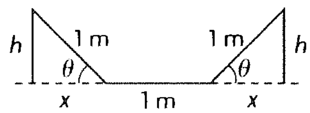
t	$\frac{\pi}{8}$	$\frac{3\pi}{8}$	$\frac{5\pi}{8}$	$\frac{7\pi}{8}$
$s(t) = 4\sin 4t$	4	−4	4	−4

The table continues in this pattern for all critical values t. So, the particle changes direction at all critical values. That is, it changes direction for $t = \frac{\pi}{4} + \pi k, \frac{3\pi}{4} + \pi k$ for positive integers k.

b. From the table or a graph, one can see that the particle's maximum velocity is 4 at the time $t = \frac{\pi}{4} + \pi k$.

c. At $t = 0$, $s = 0$, so the minimum distance from the origin is 0. The maximum value of the sine function is 1, so the maximum distance from the origin is 4(1) or 4.

12.



Label the base of a triangle x and the height h. So $\cos\theta = \frac{x}{1} = x$ and $\sin\theta = \frac{h}{1} = h$.

Therefore, $x = \cos\theta$ and $h = \sin\theta$.

The irrigation channel forms a trapezoid and the area of a trapezoid is $\frac{(b_1 + b_2)h}{2}$ where b_1 and b_2 are the bottom and top bases of the trapezoid and h is the height.

$b_1 = 1$

$b_2 = x + 1 + x = \cos\theta + 1 + \cos\theta = 2\cos\theta + 1$

$h = \sin\theta$

Therefore, the area equation is given by

$$\begin{aligned}A &= \frac{(2\cos\theta + 1 + 1)\sin\theta}{2}\\ &= \frac{(2\cos\theta + 2)\sin\theta}{2}\\ &= \frac{2\cos\theta\sin\theta + 2\sin\theta}{2}\\ &= \sin\theta\cos\theta + \sin\theta\end{aligned}$$

To maximize the cross-sectional area, differentiate:

$$\begin{aligned}A' &= (\sin\theta)(-\sin\theta) + (\cos\theta)(\cos\theta) + \cos\theta\\ &= -\sin^2\theta + \cos^2\theta + \cos\theta\end{aligned}$$

Using the trig identity $\sin^2\theta + \cos^2\theta = 1$, use the fact that $\sin^2\theta = 1 - \cos^2\theta$.

$$\begin{aligned}A' &= -(1 - \cos^2\theta) + \cos^2\theta + \cos\theta\\ &= -1 + \cos^2\theta + \cos^2\theta + \cos\theta\\ &= 2\cos^2\theta + \cos\theta - 1\end{aligned}$$

Set $A' = 0$ to find the critical points.

$0 = 2\cos^2\theta + \cos\theta - 1$

$0 = (2\cos\theta - 1)(\cos\theta + 1)$

Solve the two expressions for θ.

$$\begin{aligned}2\cos\theta &= 1\\ \cos\theta &= \frac{1}{2}\\ \theta &= \frac{\pi}{3}\end{aligned}$$

Also, $\cos\theta = -1$

$\theta = \pi$

(*Note:* The question only seeks an answer around $0 \le \theta \le \frac{\pi}{2}$. So, there is no need to find all solutions by adding $k\pi$ for all integer values of k.)

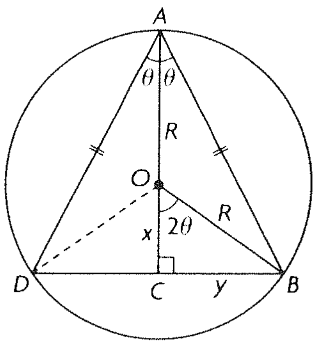
The area, A, when $\theta = \pi$ is 0 so that answer is disregarded for this problem.

When $\theta = \frac{\pi}{3}$,

$$A = \sin\frac{\pi}{3}\cos\frac{\pi}{3} + \sin\frac{\pi}{3}$$
$$= \left(\frac{\sqrt{3}}{2}\cdot\frac{1}{2}\right) + \frac{\sqrt{3}}{2}$$
$$= \frac{\sqrt{3}}{4} + \frac{2\sqrt{3}}{4}$$
$$= \frac{3\sqrt{3}}{4}$$

The area is maximized by the angle $\theta = \frac{\pi}{3}$.

13. Let O be the centre of the circle with line segments drawn and labeled, as shown.



In $\triangle OCB$, $\angle COB = 2\theta$.

Thus, $\frac{y}{R} = \sin 2\theta$ and $\frac{x}{R} = \cos 2\theta$,

so $y = R\sin 2\theta$ and $x = R\cos 2\theta$.

The area A of $\triangle ABD$ is

$$A = \frac{1}{2}|DB||AC|$$
$$= y(R + x)$$
$$= R\sin 2\theta(R + R\cos 2\theta)$$
$$= R^2(\sin 2\theta + \sin 2\theta\cos 2\theta), \text{ where } 0 < 2\theta < \pi$$

$$\frac{dA}{d\theta} = R^2(2\cos 2\theta + 2\cos 2\theta\cos 2\theta + \sin 2\theta(-2\sin 2\theta)).$$

We solve $\frac{dA}{d\theta} = 0$:

$$2\cos^2 2\theta - 2\sin^2 2\theta + 2\cos 2\theta = 0$$
$$2\cos^2 2\theta + \cos 2\theta - 1 = 0$$
$$(2\cos 2\theta - 1)(\cos 2\theta + 1) = 0$$
$$\cos 2\theta = \frac{1}{2} \text{ or } \cos 2\theta = -1$$
$$2\theta = \frac{\pi}{3} \text{ or } 2\theta = \pi \text{ (not in domain).}$$

As $2\theta \to 0$, $A \to 0$ and as $2\theta \to \pi$, $A \to 0$. The maximum area of the triangle is $\frac{3\sqrt{3}}{4}R^2$ when $2\theta = \frac{\pi}{3}$, i.e., $\theta = \frac{\pi}{6}$.

14. First find y''.

$$y = A\cos kt + B\sin kt$$
$$y' = -kA\sin kt + kB\cos kt$$
$$y'' = -k^2A\cos kt - k^2B\sin kt$$

So, $y'' + k^2y$

$$= -k^2A\cos kt - k^2B\sin kt + k^2(A\cos kt + B\sin kt)$$
$$= -k^2A\cos kt - k^2B\sin kt + k^2A\cos kt + k^2B\sin kt$$
$$= 0$$

Therefore, $y'' + k^2y = 0$.

5.5 The Derivative of $y = \tan x$, p. 260

1. a. $\frac{dy}{dx} = \sec^2 3x\left(\frac{d}{dx}3x\right)$
$$= 3\sec^2 3x$$

b. $\frac{dy}{dx} = 2\sec^2 x - \sec 2x\left(\frac{d}{dx}2x\right)$
$$= 2\sec^2 x - 2\sec 2x$$

c. $\frac{dy}{dx} = 2\tan(x^3)\left(\frac{d}{dx}\tan(x^3)\right)$

$$\frac{dy}{dx} = 2\tan(x^3)\left(\frac{d}{dx}\tan(x^3)\right)$$
$$= 2\tan(x^3)\sec^2(x^3)\left(\frac{d}{dx}x^3\right)$$
$$= 6x^2\tan(x^3)\sec^2(x^3)$$

d. $\frac{dy}{dx} = \frac{2x\tan\pi x - x^2\sec^2\pi x\left(\frac{d}{dx}\pi x\right)}{\tan^2\pi x}$
$$= \frac{2x\tan\pi x - \pi x^2\sec^2\pi x}{\tan^2\pi x}$$
$$= \frac{x(2\tan\pi x - \pi x\sec^2\pi x)}{\tan^2\pi x}$$

e. $\frac{dy}{dx} = \sec^2(x^2)\left(\frac{d}{dx}x^2\right) - 2\tan x\left(\frac{d}{dx}\right)(\tan x)$
$$= 2x\sec^2(x^2) - 2\tan x\sec^2 x$$

f. $\frac{dy}{dx} = \tan 5x(3\cos 5x)\left(\frac{d}{dx}5x\right) + 3\sin 5x\sec^2 5x\left(\frac{d}{dx}5x\right)$
$$= 15\tan 5x\cos 5x + 15\sin 5x\sec^2 5x$$
$$= 15(\tan 5x\cos 5x + \sin 5x\sec^2 5x)$$

2. a. The general equation for the line tangent to the function $f(x)$ at the point (a, b) is
$y - b = f'(x)(x - a)$.

$$f(x) = \tan x$$
$$f'(x) = \sec^2 x$$

$f\left(\frac{\pi}{4}\right) = 0$

$f'\left(\frac{\pi}{4}\right) = 2$

The equation for the line tangent to the function $f(x)$ at $x = \frac{\pi}{4}$ is $y = 2\left(x - \frac{\pi}{4}\right)$.

b. The general equation for the line tangent to the function $f(x)$ at the point (a, b) is $y - b = f'(x)(x - a)$.

$f(x) = 6 \tan x - \tan 2x$

$f'(x) = 6 \sec^2 x - \sec^2 2x\left(\frac{d}{dx}2x\right)$

$f'(x) = 6 \sec^2 x - 2 \sec^2 2x$

$f(0) = 0$

$f'(0) = -2$

The equation for the line tangent to the function $f(x)$ at $x = 0$ is $y = -2x$.

3. a. $\frac{dy}{dx} = \sec^2 x(\sin x)\left(\frac{d}{dx} \sin x\right)$

$= \cos x \sec^2 (\sin x)$

b. $\frac{dy}{dx} = -2\,[\tan (x^2 - 1)]^{-3}\left(\frac{d}{dx} \tan (x^2 - 1)\right)$

$= -2\,[\tan (x^2 - 1)]^{-3} \sec^2 (x^2 - 1) \times \left(\frac{d}{dx}(x^2 - 1)\right)$

$= -4x\,[\tan (x^2 - 1)]^{-3} \sec^2 (x^2 - 1)$

c. $\frac{dy}{dx} = 2 \tan (\cos x)\left(\frac{d}{dx} \tan (\cos x)\right)$

$= 2 \tan (\cos x) \sec^2 (\cos x)\left(\frac{d}{dx} \cos x\right)$

$= -2 \tan (\cos x) \sec^2 (\cos x) \sin x$

d. $\frac{dy}{dx} = 2\,(\tan x + \cos x)\left(\frac{d}{dx} \tan x + \cos x\right)$

$= 2\,(\tan x + \cos x)(\sec^2 x - \sin x)$

e. $\frac{dy}{dx} = \tan x\,(3 \sin^2 x)\left(\frac{d}{dx} \sin x\right) + \sin^3 x \sec^2 x$

$= 3 \tan x \sin^2 x \cos x + \sin^3 x \sec^2 x$

$= \sin^2 x\,(3 \tan x \cos x + \sin x \sec^2 x)$

f. $\frac{dy}{dx} = e^{\tan \sqrt{x}}\left(\frac{d}{dx} \tan \sqrt{x}\right)$

$= e^{\tan \sqrt{x}}\,(\sec^2 \sqrt{x})\left(\frac{d}{dx}\sqrt{x}\right)$

$= \frac{1}{2\sqrt{x}}\, e^{\tan \sqrt{x}} \sec^2 \sqrt{x}$

4. a. $\frac{dy}{dx} = \tan x \cos x + \sin x \sec^2 x$

$= \frac{\sin x}{\cos x} \cdot \cos x + \sin x \cdot \frac{1}{\cos^2 x}$

$= \sin x + \frac{\sin x}{\cos^2 x}$

$\frac{d^2y}{dx^2} = \cos x + \frac{\cos^3 x}{\cos^4 x} - \frac{\sin x(2 \cos x)\left(\frac{d}{dx} \cos x\right)}{\cos^4 x}$

$= \cos x + \frac{\cos^3 x + 2 \sin^2 x \cos x}{\cos^4 x}$

$= \cos x + \frac{1}{\cos x} + \frac{2 \sin^2 x}{\cos^3 x}$

$= \cos x + \sec x + \frac{2 \sin^2 x}{\cos^3 x}$

b. $\frac{dy}{dx} = 2 \tan x\left(\frac{d}{dx} \tan x\right)$

$= 2 \tan x \sec^2 x$

$= \frac{2 \sin x}{\cos x} \cdot \frac{1}{\cos^2 x}$

$= \frac{2 \sin x}{\cos^3 x}$

$\frac{d^2y}{dx^2} = \frac{2 \cos^4 x - 6 \sin x \cos^2 x\left(\frac{d}{dx}\cos x\right)}{\cos^6 x}$

$= \frac{2 \cos^4 x + 6 \sin^2 x \cos^2 x}{\cos^6 x}$

$= \frac{2}{\cos^2 x} + \frac{6 \sin^2 x}{\cos^2 x} \cdot \frac{1}{\cos^2 x}$

$= 2 \sec^2 x + 6 \tan^2 x \sec^2 x$

$= 2 \sec^2 x(1 + 3 \tan^2 x)$

5. The slope of $f(x) = \sin x \tan x$ equals zero when the derivative equals zero.

$f(x) = \sin x \tan x$

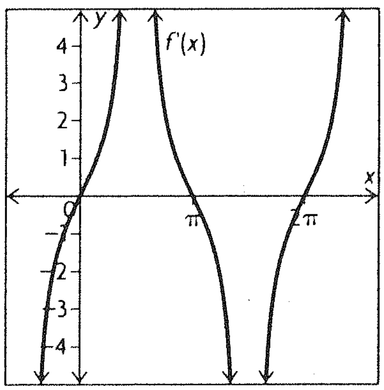
$f'(x) = \sin x(\sec^2 x) + \tan x(\cos x)$

$= \sin x(\sec^2 x) + \frac{\sin x}{\cos x}(\cos x)$

$= \sin x(\sec^2 x) + \sin x$

$= \sin x(\sec^2 x + 1)$

$\sec^2 x + 1$ is always positive, so the derivative is 0 only when $\sin x = 0$. So, $f'(x)$ equals 0 when $x = 0$, $x = \pi$, and $x = 2\pi$. The solutions can be verified by examining the graph of the derivative function shown below.



6. The local maximum point occurs when the derivative equals zero.

$$\frac{dy}{dx} = 2 - \sec^2 x$$

$$2 - \sec^2 x = 0$$
$$\sec^2 x = 2$$
$$\sec x = \pm\sqrt{2}$$
$$x = \pm\frac{\pi}{4}$$

$\frac{dy}{dx} = 0$ when $x = \pm\frac{\pi}{4}$, so the local maximum point occurs when $x = \pm\frac{\pi}{4}$. Solve for y when $x = \frac{\pi}{4}$.

$$y = 2\left(\frac{\pi}{4}\right) - \tan\left(\frac{\pi}{4}\right)$$
$$y = \frac{\pi}{2} - 1$$
$$y = 0.57$$

Solve for y when $x = -\frac{\pi}{4}$.

$$y = 2\left(-\frac{\pi}{4}\right) - \tan\left(-\frac{\pi}{4}\right)$$
$$y = -\frac{\pi}{2} + 1$$
$$y = -0.57$$

The local maximum occurs at the point $\left(\frac{\pi}{4}, 0.57\right)$.

7. $y = \sec x + \tan x$

$$= \frac{1}{\cos x} + \frac{\sin x}{\cos x}$$
$$= \frac{1 + \sin x}{\cos x}$$
$$\frac{dy}{dx} = \frac{\cos^2 x - (1 + \sin x)(-\sin x)}{\cos^2 x}$$
$$= \frac{\cos^2 x - (-\sin x - \sin^2 x)}{\cos^2 x}$$
$$= \frac{\cos^2 x + \sin x + \sin^2 x}{\cos^2 x}$$
$$= \frac{1 + \sin x}{\cos^2 x}$$

The denominator is never negative. $1 + \sin x > 0$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$, since $\sin x$ reaches its minimum of -1 at $x = \frac{\pi}{2}$. Since the derivative of the original function is always positive in the specified interval, the function is always increasing in that interval.

8. When $x = \frac{\pi}{4}$, $y = 2\tan\left(\frac{\pi}{4}\right)$

$$= 2$$
$$y' = 2\sec^2 x$$

When $x = \frac{\pi}{4}$, $y' = 2\left(\sec\frac{\pi}{4}\right)^2$

$$= 2(\sqrt{2})^2$$
$$= 4$$

So an equation for the tangent at the point $x = \frac{\pi}{4}$ is

$$y - 2 = 4\left(x - \frac{\pi}{4}\right)$$
$$y - 2 = 4x - \pi$$
$$-4x + y - (2 - \pi) = 0$$

9. Write $\tan x = \frac{\sin x}{\cos x}$ and use the quotient rule to derive the derivative of the tangent function.

10. $y = \cot x$

$$y = \frac{1}{\tan x}$$
$$\frac{dy}{dx} = \frac{\tan x(0) - (1)\sec^2 x}{\tan^2 x}$$
$$= \frac{-\sec^2 x}{\tan^2 x}$$
$$= \frac{\frac{-1}{\cos^2 x}}{\frac{\sin^2 x}{\cos^2 x}}$$
$$= \frac{-1}{\sin^2 x}$$
$$= -\csc^2 x$$

11. Using the fact from question 10 that the derivative of $\cot x$ is $-\csc^2 x$,

$$f'(x) = -4\csc^2 x$$
$$= -4(\csc x)^2$$
$$f''(x) = -8(\csc x) \cdot \frac{d(\csc x)}{dx}$$
$$= -8(\csc x) \cdot (-\csc x \cot x)$$
$$= 8\csc^2 x \cot x$$

Review Exercise, pp. 263–265

1. a. $y' = 0 - e^x$
$= -e^x$

b. $y' = 2 + 3e^x$

c. $y' = e^{2x+3} \cdot \dfrac{d(2x+3)}{dx}$
$= 2e^{2x+3}$

d. $y' = e^{-3x^2+5x} \cdot \dfrac{d(-3x^2+5x)}{dx}$
$= (-6x+5)e^{-3x^2+5x}$

e. $y' = (x)(e^x) + (e^x)(1)$
$= e^x(x+1)$

f. $s' = \dfrac{(e^t+1)(e^t) - (e^t-1)(e^t)}{(e^t+1)^2}$
$= \dfrac{e^{2t} + e^t - (e^{2t} - e^t)}{(e^t+1)^2}$
$= \dfrac{2e^t}{(e^t+1)^2}$

2. a. $\dfrac{dy}{dx} = 10^x \ln 10$

b. $\dfrac{dy}{dx} = 4^{3x^2} \cdot \ln 4 \cdot \dfrac{d(3x^2)}{dx}$
$= 6x(4^{3x^2})\ln 4$

c. $\dfrac{dy}{dx} = (5x)(5^x \ln 5) + (5^x)(5)$
$= 5 \cdot 5^x(x \ln 5 + 1)$

d. $\dfrac{dy}{dx} = (x^4)(2^x \ln 2) + (2^x)(4x^3)$
$= x^3 \cdot 2^x(x \ln 2 + 4)$

e. $y = (4x)(4^{-x})$
$\dfrac{dy}{dx} = (4x)(-4^{-x}\ln 4) + (4^{-x})(4)$
$= 4 \cdot 4^{-x}(-x \ln 4 + 1)$
$= \dfrac{4 - 4x \ln 4}{4^x}$

f. $y = (5^{\sqrt{x}})(x^{-1})$
$\dfrac{dy}{dx} = (5^{\sqrt{x}})(-x^{-2}) + (x^{-1})\left(5^{\sqrt{x}} \cdot \ln 5 \cdot \dfrac{d(\sqrt{x})}{dx}\right)$
$= (5^{\sqrt{x}})\left(-\dfrac{1}{x^2}\right) + (x^{-1})\left(5^{\sqrt{x}} \cdot \ln 5 \cdot \dfrac{1}{2\sqrt{x}}\right)$
$= 5^{\sqrt{x}}\left(-\dfrac{1}{x^2} + \dfrac{\ln 5}{2x\sqrt{x}}\right)$

3. a. $\dfrac{dy}{dx} = 3\cos(2x) \cdot \dfrac{d(2x)}{dx} + 4\sin(2x) \cdot \dfrac{d(2x)}{dx}$
$= 6\cos(2x) + 8\sin(2x)$

b. $\dfrac{dy}{dx} = \sec^2(3x) \cdot \dfrac{d(3x)}{dx}$
$= 3\sec^2(3x)$

c. $y = (2 - \cos x)^{-1}$
$\dfrac{dy}{dx} = -(2-\cos x)^{-2} \cdot \dfrac{d(2-\cos x)}{dx}$
$= -\dfrac{\sin x}{(2-\cos x)^2}$

d. $\dfrac{dy}{dx} = (x)\left(\sec^2(2x) \cdot \dfrac{d(2x)}{dx}\right) + (\tan(2x))(1)$
$= 2x\sec^2(2x) + \tan 2x$

e. $\dfrac{dy}{dx} = (\sin 2x)\left(e^{3x} \cdot \dfrac{d(3x)}{dx}\right)$
$+ (e^{3x})\left(\cos 2x \cdot \dfrac{d(2x)}{dx}\right)$
$= 3e^{3x}\sin 2x + 2e^{3x}\cos 2x$
$= e^{3x}(3\sin 2x + 2\cos 2x)$

f. $y = (\cos(2x))^2$
$\dfrac{dy}{dx} = 2(\cos(2x)) \cdot \dfrac{d(\cos(2x))}{dx}$
$= 2(\cos(2x)) \cdot -\sin(2x) \cdot \dfrac{d(2x)}{dx}$
$= -4\cos(2x)\sin(2x)$

4. a. $f(x) = e^x \cdot x^{-1}$
$f'(x) = (e^x)(-x^{-2}) + (x^{-1})(e^x)$
$= e^x\left(-\dfrac{1}{x^2} + \dfrac{1}{x}\right)$
$= e^x\left(\dfrac{-x + x^2}{x^3}\right)$

Now, set $f'(x) = 0$ and solve for x.
$0 = e^x\left(\dfrac{-x + x^2}{x^3}\right)$

Solve $e^x = 0$ and $\dfrac{x^2 - x}{x^3} = 0$.
e^x is never zero.
$\dfrac{x^2 - x}{x^3} = 0$
$x^2 - x = 0$
$x(x-1) = 0$
So $x = 0$ or $x = 1$.
(Note, however, that x cannot be zero because this would cause division by zero in the original function.)
So $x = 1$.

b. The function has a horizontal tangent at $(1, e)$.

5. a. $f'(x) = (x)\left(e^{-2x} \cdot \frac{d(-2x)}{dx}\right) + (e^{-2x})(1)$
$= -2xe^{-2x} + e^{-2x}$
$= e^{-2x}(-2x + 1)$
$f'\left(\frac{1}{2}\right) = e^{-2\frac{1}{2}}\left(-2 \cdot \frac{1}{2} + 1\right)$
$= e^{-1}(-1 + 1)$
$= 0$

b. This means that the slope of the tangent to $f(x)$ at the point with x-coordinate $\frac{1}{2}$ is 0.

6. a. $y' = (x)(e^x) + (e^x)(1) - e^x$
$= xe^x$
$y'' = (x)(e^x) + (e^x)(1)$
$= xe^x + e^x$
$= e^x(x + 1)$

b. $y' = (x)(10e^{10x}) + (e^{10x})(1)$
$= 10xe^{10x} + e^{10x}$
$y'' = (10x)(10e^{10x}) + (e^{10x})(10) + 10e^{10x}$
$= 100xe^{10x} + 10e^{10x} + 10e^{10x}$
$= 100xe^{10x} + 20e^{10x}$
$= 20e^{10x}(5x + 1)$

7. $y = \frac{e^{2x} - 1}{e^{2x} + 1}$
$\frac{dy}{dx} = \frac{2e^{2x}(e^{2x} + 1) - (e^{2x} - 1)(2e^{2x})}{(e^{2x} + 1)^2}$
$= \frac{2e^{4x} + 2e^{2x} - 2e^{4x} + 2e^{2x}}{(e^{2x} + 1)^2}$
$= \frac{4e^{2x}}{(e^{2x} + 1)^2}$

Now, $1 - y^2 = 1 - \frac{e^{4x} - 2e^{2x} + 1}{(e^{2x} + 1)^2}$
$= \frac{e^{4x} + 2e^{2x} + 1 - e^{4x} + 2e^{2x} - 1}{(e^{2x} + 1)^2}$
$= \frac{4e^{2x}}{(3^{2x} + 1)^2} = \frac{dy}{dx}$

8. The slope of the required tangent line is 3. The slope at any point on the curve is given by $\frac{dy}{dx} = 1 + e^{-x}$.
To find the point(s) on the curve where the tangent has slope 3, we solve:
$1 + e^{-x} = 3$
$e^{-x} = 2$
$-x = \ln 2$
$x = -\ln 2.$
The point of contact of the tangent is $(-\ln 2, -\ln 2 - 2)$.
The equation of the tangent line is
$y + \ln 2 + 2 = 3(x + \ln 2)$ or
$3x - y + 2\ln 2 - 2 = 0.$

9. When $x = \frac{\pi}{2}$,
$y = f(x) = f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\sin\left(\frac{\pi}{2}\right) = \frac{\pi}{2}(1) = \frac{\pi}{2}$
$y' = f'(x) = (x)(\cos x) + (\sin x)(1)$
$= x\cos x + \sin x$
$f'\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2}$
$= \frac{\pi}{2}(0) + 1$
$= 1$
So an equation for the tangent at the point $x = \frac{\pi}{2}$ is
$y - \frac{\pi}{2} = 1\left(x - \frac{\pi}{2}\right)$
$y - \frac{\pi}{2} = x - \frac{\pi}{2}$
$y = x$
$-x + y = 0$

10. If $s(t) = \frac{\sin t}{3 + \cos 2t}$ is the function describing an object's position at time t, then $v(t) = s'(t)$ is the function describing the object's velocity at time t. So
$v(t) = s'(t)$
$= \frac{(3 + \cos 2t)(\cos t) - (\sin t)(-2\sin 2t)}{(3 + \cos 2t)^2}$
$s'\left(\frac{\pi}{4}\right) = \frac{\left(3 + \cos 2 \cdot \frac{\pi}{4}\right)\left(\cos\frac{\pi}{4}\right)}{\left(3 + \cos 2 \cdot \frac{\pi}{4}\right)^2} - \frac{\left(\sin\frac{\pi}{4}\right)\left(-2\sin 2 \cdot \frac{\pi}{4}\right)}{\left(3 + \cos 2 \cdot \frac{\pi}{4}\right)^2}$
$= \frac{\left(3 + \cos\frac{\pi}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(-2\sin\frac{\pi}{2}\right)}{\left(3 + \cos\frac{\pi}{2}\right)^2}$
$= \frac{(3 + 0)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)(-2 \cdot 1)}{(3 + 0)^2}$
$= \frac{\frac{3\sqrt{2}}{2} + \sqrt{2}}{9}$
$= \frac{3\sqrt{2} + 2\sqrt{2}}{2} \cdot \frac{1}{9}$
$= \frac{5\sqrt{2}}{18}$

So, the object's velocity at time $t = \frac{\pi}{4}$ is

$\frac{5\sqrt{2}}{18} \doteq 0.3928$ metres per unit of time.

11. a. The question asks for the time t when $N'(t) = 0$.

$$N(t) = 60\,000 + 2000te^{-\frac{t}{20}}$$
$$N'(t) = 0 + (2000t)\left(-\frac{1}{20}e^{-\frac{t}{20}}\right) + (e^{-\frac{t}{20}})(2000)$$
$$= -100te^{-\frac{t}{20}} + 2000e^{-\frac{t}{20}}$$
$$= 100e^{-\frac{t}{20}}(-t + 20)$$

Set $N'(t) = 0$ and solve for t.

$$0 = 100e^{-\frac{t}{20}}(-t + 20)$$

$100e^{-\frac{t}{20}}$ is never equal to zero.

$$-t + 20 = 0$$
$$20 = t$$

Therefore, the rate of change of the number of bacteria is equal to zero when time $t = 20$.

b. The question asks for $\frac{dM}{dt} = M'(t)$ when $t = 10$. That is, it asks for $M'(10)$.

$$M(t) = (N + 1000)^{\frac{1}{3}}$$
$$M'(t) = \frac{1}{3}(N + 1000)^{-\frac{2}{3}} \cdot \frac{d(N + 1000)}{dt}$$
$$= \frac{1}{3(N + 1000)^{\frac{2}{3}}} \cdot \frac{dN}{dt}$$

From part a., $\frac{dN}{dt} = N'(t) = 100e^{-\frac{t}{20}}(-t + 20)$ and

$$N(t) = 60000 + 2000te^{-\frac{t}{20}}$$

So $M'(t) = \dfrac{100e^{-\frac{t}{20}}(-t + 20)}{3(N + 1000)^{\frac{2}{3}}}$

First calculate $N(10)$.

$$N(10) = 60000 + 2000(10)e^{-\frac{10}{20}}$$
$$= 60000 + 20000e^{-\frac{1}{2}}$$
$$\doteq 72131$$

So $M'(10) = \dfrac{100e^{-\frac{10}{20}}(-10 + 20)}{3(N(10) + 1000)^{\frac{2}{3}}}$

$$= \frac{100e^{-\frac{1}{2}}(10)}{3(72131 + 1000)^{\frac{2}{3}}}$$
$$\doteq \frac{606.53}{5246.33}$$
$$\doteq 0.1156$$

So, after 10 days, about 0.1156 mice are infected per day. Essentially, almost 0 mice are infected per day when $t = 10$.

12. a. $c_1(t) = te^{-t}; c_1(0) = 0$

$$c_1'(t) = e^{-t} - te^{-t}$$
$$= e^{-t}(1 - t)$$

Since $e^{-t} > 0$ for all t, $c_1'(t) = 0$ when $t = 1$.
Since $c_1'(t) > 0$ for $0 \leq t < 1$, and $c_1'(t) < 0$ for all $t > 1$, $c_1(t)$ has a maximum value of $\frac{1}{e} \doteq 0.368$ at $t = 1$ h.

$$c_2(t) = t^2e^{-t}; c_2(0) = 0$$
$$c_2'(t) = 2te^{-t} - t^2e^{-t}$$
$$= te^{-t}(2 - t)$$

$c_2'(t) = 0$ when $t = 0$ or $t = 2$.
Since $c_2'(t) > 0$ for $0 < t < 2$ and $c_2'(t) < 0$ for all $t > 2$, $c_2(t)$ has a maximum value of $\frac{4}{e^2} \doteq 0.541$ at $t = 2$ h. The larger concentration occurs for medicine c_2.

b. $c_1(0.5) = 0.303$

$c_2(0.5) = 0.152$

In the first half-hour, the concentration of c_1 increases from 0 to 0.303, and that of c_2 increases from 0 to 0.152. Thus, c_1 has the larger concentration over this interval.

13. a. $y = (2 + 3e^{-x})^3$

$$y' = 3(2 + 3e^{-x})^2[0 + 3e^{-x}(-1)]$$
$$= 3(2 + 3e^{-x})^2(-3e^{-x})$$
$$= -9e^{-x}(2 + 3e^{-x})^2$$

b. $y = x^e$

$$y' = ex^{e-1}$$

c. $y = e^{e^x}$

$$y' = e^{e^x}(e^x)(1)$$
$$= e^{x+e^x}$$

d. $y = (1 - e^{5x})^5$

$$y' = 5(1 - e^{5x})^4[0 - e^{5x}(5)]$$
$$= -25e^{5x}(1 - e^{5x})^4$$

14. a. $y = 5^x$

$$y' = 5^x \ln 5$$

b. $y = (0.47)^x$

$$y' = (0.47)^x \ln(0.47)$$

c. $y = (52)^{2x}$

$$y' = (52)^{2x}(2)\ln 52$$
$$= 2(52)^{2x}\ln 52$$

d. $y = 5(2)^x$

$$y' = 5(2)^x \ln 2$$

e. $y = 4e^x$

$$y' = 4e^x(1)\ln e$$
$$= 4e^x$$

f. $y = -2(10)^{3x}$

$$y' = -2(3)10^{3x}\ln 10$$
$$= -6(10)^{3x}\ln 10$$

15. a. $y' = \cos 2^x \cdot \dfrac{d(2^x)}{dx}$

$$= 2^x \ln 2 \cos 2^x$$

b. $y' = (x^2)(\cos x) + (\sin x)(2x)$
$= x^2 \cos x + 2x \sin x$

c. $y' = \cos\left(\frac{\pi}{2} - x\right) \cdot \frac{d(\frac{\pi}{2} - x)}{dx}$
$= -\cos\left(\frac{\pi}{2} - x\right)$

d. $y' = (\cos x)(\cos x) + (\sin x)(-\sin x)$
$= \cos^2 x - \sin^2 x$

e. $y = (\cos x)^2$
$y' = 2(\cos x) \cdot \frac{d(\cos x)}{dx}$
$= -2 \cos x \sin x$

f. $y = \cos x\,(\sin x)^2$
$y' = (\cos x)(2\,(\sin x)(\cos x)) + (\sin x)^2(-\sin x)$
$= 2 \sin x \cos^2 x - \sin^3 x$

16. Compute $\frac{dy}{dx}$ when $x = \frac{\pi}{2}$ to find the slope of the line at the given point.
$y' = -\sin x$
So, at the point $x = \frac{\pi}{2}$, $y' = f'(x)$ is

$$f'\left(\frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2}\right) = -1.$$

Therefore, an equation of the line tangent to the curve at the given point is

$$y - 0 = -1\left(x - \frac{\pi}{2}\right)$$

$$y = -x + \frac{\pi}{2}$$

$$x + y - \frac{\pi}{2} = 0$$

17. The velocity of the object at any time t is $v = \frac{ds}{dt}$.
Thus, $v = 8\,(\cos(10\pi t))(10\pi)$
$= 80\pi \cos(10\pi t)$.
The acceleration at any time t is $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$.
Hence, $a = 80\pi(-\sin(10\pi t))(10\pi) = -800\pi^2 \sin(10\pi t)$.
Now, $\frac{d^2s}{dt^2} + 100\pi^2 s$
$= -800\pi^2 \sin(10\pi t)$
$+ 100\pi^2(8 \sin(10\pi t)) = 0$.

18. Since $s = 5 \cos\left(2t + \frac{\pi}{4}\right)$,

$$v = \frac{ds}{dt} = 5\left(-\sin\left(2t + \frac{\pi}{4}\right)\right)$$

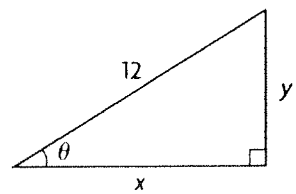
$$= -10 \sin\left(2t + \frac{\pi}{4}\right)$$

and $a = \frac{dv}{dt} = -10\left(\cos\left(2t + \frac{\pi}{4}\right)\right)$

$$= -20 \cos\left(2t + \frac{\pi}{4}\right).$$

The maximum values of the displacement, velocity, and acceleration are 5, 10, and 20, respectively.

19. Let the base angle be θ, $0 < \theta < \frac{\pi}{2}$, and let the sides of the triangle have lengths x and y, as shown. Let the perimeter of the triangle be P cm.



Now, $\frac{x}{12} = \cos\theta$ and $\frac{y}{12} = \sin\theta$
so $x = 12 \cos\theta$ and $y = 12 \sin\theta$.
Therefore, $P = 12 + 12\cos\theta + 12\sin\theta$ and
$\frac{dP}{d\theta} = -12\sin\theta + 12\cos\theta$.
For critical values, $-12\sin\theta + 12\cos\theta = 0$
$\sin\theta = \cos\theta$
$\tan\theta = 1$

$\theta = \frac{\pi}{4}$, since $0 < \theta < \frac{\pi}{2}$.

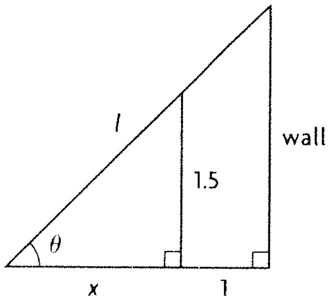
When $\theta = \frac{\pi}{4}$, $P = 12 + \frac{12}{\sqrt{2}} + \frac{12}{\sqrt{2}}$
$= 12 + \frac{24}{\sqrt{2}}$
$= 12 + 12\sqrt{2}$.

As $\theta \to 0^+$, $\cos\theta \to 1$, $\sin\theta \to 0$, and $P \to 12 + 12 + 0 = 24$.
As $\theta \to \frac{\pi}{2}$, $\cos\theta \to 0$, $\sin\theta \to 1$ and $P \to 12 + 0 + 12 = 24$.
Therefore, the maximum value of the perimeter is $12 + 12\sqrt{2}$ cm, and occurs when the other two angles are each $\frac{\pi}{4}$ rad, or 45°.

20. Let l be the length of the ladder, θ be the angle between the foot of the ladder and the ground, and x be the distance of the foot of the ladder from the fence, as shown.

Thus, $\frac{x + 1}{l} = \cos\theta$ and $\frac{1.5}{x} = \tan\theta$

$$x + 1 = l\cos\theta \text{ where } x = \frac{1.5}{\tan\theta}.$$



Replacing x, $\frac{1.5}{\tan\theta} + 1 = l\cos\theta$

$$l = \frac{1.5}{\sin\theta} + \frac{1}{\cos\theta},\ 0 < \theta < \frac{\pi}{2}$$

$$\frac{dl}{d\theta} = -\frac{1.5\cos\theta}{\sin^2\theta} + \frac{\sin\theta}{\cos^2\theta}$$

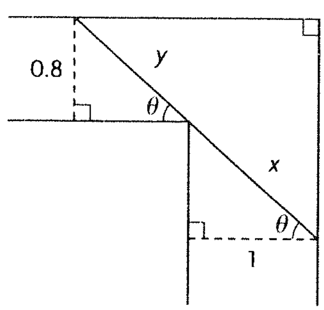
$$= \frac{-1.5\cos^3\theta + \sin^3\theta}{\sin^2\theta\cos^2\theta}.$$

Solving $\frac{dl}{d\theta} = 0$ yields:

$$\sin^3\theta - 1.5\cos^3\theta = 0$$
$$\tan^3\theta = 1.5$$
$$\tan\theta = \sqrt[3]{1.5}$$
$$\theta \doteq 0.46365.$$

The length of the ladder corresponding to this value of θ is $l \doteq 4.5$ m. As $\theta \to 0^+$ and $\frac{\pi^-}{2}$, l increases without bound. Therefore, the shortest ladder that goes over the fence and reaches the wall has a length of 4.5 m.

21. The longest pole that can fit around the corner is determined by the minimum value of $x + y$. Thus, we need to find the minimum value of $l = x + y$.



From the diagram, $\frac{0.8}{y} = \sin\theta$ and $\frac{1}{x} = \cos\theta$.

Thus, $l = \frac{1}{\cos\theta} + \frac{0.8}{\sin\theta},\ 0 \le \theta \le \frac{\pi}{2}$:

$$\frac{dl}{d\theta} = \frac{1\sin\theta}{\cos^2\theta} - \frac{0.8\cos\theta}{\sin^2\theta}$$

$$= \frac{0.8\sin^3\theta - \cos^3\theta}{\cos^2\theta\sin^2\theta}.$$

Solving $\frac{dl}{d\theta} = 0$ yields:

$$0.8\sin^3\theta - \cos^3\theta = 0$$
$$\tan^3\theta = 1.25$$
$$\tan\theta = \sqrt[3]{1.25}$$
$$\tan\theta \doteq 1.077$$
$$\theta \doteq 0.822.$$

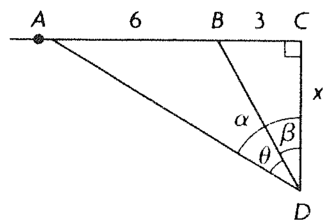
Now, $l = \frac{0.8}{\cos(0.822)} + \frac{1}{\sin(0.822)} \doteq 2.5.$

When $\theta = 0$, the longest possible pole would have a length of 0.8 m. When $\theta = \frac{\pi}{2}$, the longest possible pole would have a length of 1 m. Therefore, the longest pole that can be carried horizontally around the corner is one of length 2.5 m.

22. We want to find the value of x that maximizes θ. Let $\angle ADC = \alpha$ and $\angle BDC = \beta$.

Thus, $\theta = \alpha - \beta$:

$$\tan\theta = \tan(\alpha - \beta)$$
$$= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}.$$



From the diagram, $\tan\alpha = \frac{9}{x}$ and $\tan\beta = \frac{3}{x}$.

Hence, $\tan\theta = \frac{\frac{9}{x} - \frac{3}{x}}{1 + \frac{27}{x^2}}$

$$= \frac{9x - 3x}{x^2 + 27}$$
$$= \frac{6x}{x^2 + 27}.$$

We differentiate implicitly with respect to x:

$$\sec^2\theta\frac{d\theta}{dx} = \frac{6(x^2 + 27) - 6x(2x)}{(x^2 + 27)^2}$$

$$\frac{d\theta}{dx} = \frac{162 - 6x^2}{\sec^2\theta(x^2 + 27)^2}$$

Solving $\frac{d\theta}{dx} = 0$ yields:

$$162 - 6x^2 = 0$$
$$x^2 = 27$$
$$x = 3\sqrt{3}.$$

23. a. $f(x) = 4(\sin(x-2))^2$
$$f'(x) = 8\sin(x-2)\cos(x-2)$$
$$f''(x) = (8\sin(x-2))(-\sin(x-2)) + (\cos(x-2))(8\cos(x-2))$$
$$= -8\sin^2(x-2) + 8\cos^2(x-2)$$

b. $f(x) = (2\cos x)(\sec x)^2$
$$f'(x) = (2\cos x)(2\sec x \cdot \sec x \tan x) + (\sec x)^2(-2\sin x)$$
$$= (4\cos x)(\sec^2 x\tan x) - 2\sin x(\sec x)^2$$

Using the product rule multiple times,
$$f''(x) = (4\cos x)\left[\sec^2 x \cdot \sec^2 x + \tan x(2\sec x \cdot \sec x\tan x)\right] + (\sec^2 x\tan x)(-4\sin x) + (-2\sin x)(2\sec x \cdot \sec x\tan x) + (\sec x)^2(-2\cos x)$$
$$= 4\cos x\sec^4 x + 8\cos x\tan^2 x\sec^2 x - 4\sin x\tan x\sec^2 x - 4\sin x\tan x\sec^2 x - 2\cos x\sec^2 x$$
$$= 4\cos x\sec^4 x + 8\cos x\tan^2 x\sec^2 x - 8\sin x\tan x\sec^2 x - 2\cos x\sec^2 x$$

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1. a. $y = e^{-2x^2}$
$$\frac{dy}{dx} = -4xe^{-2x^2}$$

b. $y = 3^{x^2+3x}$
$$\frac{dy}{dx} = 3^{x^2+3x} \cdot \ln 3 \cdot (2x+3)$$

c. $y = \frac{e^{3x} + e^{-3x}}{2}$
$$\frac{dy}{dx} = \frac{1}{2}[3e^{3x} - 3e^{-3x}]$$
$$= \frac{3}{2}[e^{3x} - e^{-3x}]$$

d. $y = 2\sin x - 3\cos 5x$
$$\frac{dy}{dx} = 2\cos x - 3(-\sin 5x)(5)$$
$$= 2\cos x + 15\sin 5x$$

e. $y = \sin^3(x^2)$
$$\frac{dy}{dx} = 3\sin^2(x^2)(\cos(x^2)(2x))$$
$$= 6x\sin^2(x^2)\cos(x^2)$$

f. $y = \tan\sqrt{1-x}$
$$\frac{dy}{dx} = \sec^2\sqrt{1-x}\left(\frac{1}{2} \times \frac{1}{\sqrt{1-x}}\right)(-1)$$
$$= -\frac{\sec^2\sqrt{1-x}}{2\sqrt{1-x}}$$

2. The given line is $-6x + y = 2$ or $y = 6x + 2$, so the slope is 6.
$$y = 2e^{3x}$$
$$\frac{dy}{dx} = 2e^{3x}(3)$$
$$= 6e^{3x}$$
In order for the tangent line to be parallel to the given line, the derivative has to equal 6 at the tangent point.
$$6e^{3x} = 6$$
$$e^{3x} = 1$$
$$x = 0$$
When $x = 0$, $y = 2$.
The equation of the tangent line is $y - 2 = 6(x - 0)$ or $-6x + y = 2$. The tangent line is the given line.

3. $y = e^x + \sin x$
$$\frac{dy}{dx} = e^x + \cos x$$
When $x = 0$, $\frac{dy}{dx} = 1 + 1$ or 2, so the slope of the tangent line at $(0, 1)$ is 2.
The equation of the tangent line at $(0, 1)$ is $y - 1 = 2(x - 0)$ or $-2x + y = 1$.

4. $v(t) = 10e^{-kt}$

a. $a(t) = v'(t) = -10ke^{-kt}$
$$= -k(10e^{-kt})$$
$$= -kv(t)$$
Thus, the acceleration is a constant multiple of the velocity. As the velocity of the particle decreases, the acceleration increases by a factor of k.

b. At time $t = 0$, $v = 10$ cm/s.

c. When $v = 5$, we have $10e^{-kt} = 5$
$$e^{-kt} = \frac{1}{2}$$
$$-kt = \ln\left(\frac{1}{2}\right) = -\ln 2$$
$$t = \frac{\ln 2}{k}.$$
After $\frac{\ln 2}{k}$ s have elapsed, the velocity of the particle is 5 cm/s. The acceleration of the particle is $-5k$ at this time.

5. a. $f(x) = (\cos x)^2$
$$f'(x) = 2(\cos x) \cdot \frac{d(\cos x)}{dx}$$
$$= 2(\cos x) \cdot (-\sin x)$$
$$= -2\sin x\cos x$$
$$f''(x) = (-2\sin x)(-\sin x) + (\cos x)(-2\cos x)$$
$$= 2\sin^2 x - 2\cos^2 x$$
$$= 2(\sin^2 x - \cos^2 x)$$

b. $f(x) = \cos x \cot x$

$f'(x) = (\cos x)(-\csc^2 x) + (\cot x)(-\sin x)$

$= -\cos x \cdot \frac{1}{\sin^2 x} - \frac{\cos x}{\sin x} \cdot \sin x$

$= -\frac{\cos x}{\sin x} \cdot \frac{1}{\sin x} - \cos x$

$= -\cot x \csc x - \cos x$

$f''(x) =$

$(-\cot x)(-\csc x \cot x) + (\csc x)(\csc^2 x) + \sin x$

$= \csc x \cot^2 x + \csc^3 x + \sin x$

6. $f(x) = (\sin x)^2$

To find the absolute extreme values, first find the derivative, set it equal to zero, and solve for x.

$f'(x) = 2 (\sin x) \cdot \frac{d(\sin x)}{dx}$

$= 2 \sin x \cos x$

$= \sin 2x$

Now set $f'(x) = 0$ and solve for x.

$0 = \sin 2x$

$2x = 0, \pi, 2\pi$

$x = 0, \frac{\pi}{2}, \pi$ in the interval $0 \le x \le \pi$.

Evaluate $f(x)$ at the critical numbers, including the endpoints of the interval.

x	0	$\frac{\pi}{2}$	π
$f(x) = (\sin^2 x)$	0	1	0

So, the absolute maximum value on the interval is 1 when $x = \frac{\pi}{2}$ and the absolute minimum value on the interval is 0 when $x = 0$ and $x = \pi$.

7. $y = f(x) = 5^x$

Find the derivative, $f'(x)$, and evaluate the derivative at $x = 2$ to find the slope of the tangent when $x = 2$.

$f'(x) = 5^x \ln 5$

$f'(2) = 5^2 \ln 5$

$= 25 \ln 5$

$\doteq 40.24$

8. $y = xe^x + 3e^x$

To find the maximum and minimum values, first find the derivative, set it equal to zero, and solve for x.

$y' = (x)(e^x) + (e^x)(1) + 3e^x$

$= xe^x + e^x + 3e^x$

$= xe^x + 4e^x$

$= e^x(x + 4)$

Now set $y' = 0$ and solve for x.

$0 = e^x(x + 4)$

e^x is never equal to zero.

$(x + 4) = 0$

So $x = -4$.

Therefore, the critical value is -4.

Interval	$e^x(x + 4)$
$x < -4$	$-$
$-4 < x$	$+$

So $f(x)$ is decreasing on the left of $x = -4$ and increasing on the right of $x = -4$. Therefore, the function has a minimum value at $\left(-4, -\frac{1}{e^4}\right)$. There is no maximum value.

9. $f(x) = 2 \cos x - \sin 2x$

So, $f(x) = 2 \cos x - 2 \sin x \cos x$.

a. $f'(x) = -2 \sin x - (2 \sin x)(-\sin x)$

$- (\cos x)(2 \cos x)$

$= -2 \sin x + 2 \sin^2 x - 2 \cos^2 x$

Set $f'(x) = 0$ to solve for the critical values.

$-2 \sin x + 2 \sin^2 x - 2 \cos^2 x = 0$

$-2 \sin x + 2 \sin^2 x - 2(1 - \sin^2 x) = 0$

$-2 \sin x + 2 \sin^2 x - 2 + 2 \sin^2 x = 0$

$4 \sin^2 x - 2 \sin x - 2 = 0$

$(2 \sin x + 1)(2 \sin x - 2) = 0$

$2 \sin x + 1 = 0$ and $2 \sin x - 2 = 0$

So, $\sin x = -\frac{1}{2}$.

In the given interval, this occurs when $x = -\frac{\pi}{6}, -\frac{5\pi}{6}$.

Also, $\sin x = 1$.

In the given interval, this occurs when $x = \frac{\pi}{2}$.

Therefore, on the given interval, the critical numbers for $f(x)$ are $x = -\frac{\pi}{6}, -\frac{5\pi}{6}, \frac{\pi}{2}$.

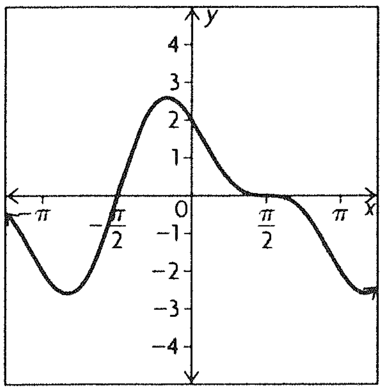
b. To determine the intervals where $f(x)$ is increasing and where $f(x)$ is decreasing, find the slope of $f(x)$ in the intervals between the endpoints and the critical numbers. To do this, it helps to make a table.

x	slope of $f(x)$
$-\pi \le x < -\frac{5\pi}{6}$	$-$
$-\frac{5\pi}{6} < x < -\frac{\pi}{6}$	$+$
$-\frac{\pi}{6} < x < \frac{\pi}{2}$	$-$
$\frac{\pi}{2} < x \le \pi$	$-$

So, $f(x)$ is increasing on the interval $-\frac{5\pi}{6} < x < -\frac{\pi}{6}$ and $f(x)$ is decreasing on the intervals $-\pi \le x < -\frac{5\pi}{6}$ and $-\frac{\pi}{6} < x < \pi$.

c. From the table in part b., it can be seen that there is a local maximum at the point where $x = -\frac{\pi}{6}$ and there is a local minimum at the point where $x = -\frac{5\pi}{6}$.

d.



Cumulative Review of Calculus

1. a. $f(x) = 3x^2 + 4x - 5$

$$m = \lim_{h\to 0} \frac{f(a + h) - f(a)}{h}$$
$$= \lim_{h\to 0} \frac{f(2 + h) - 15}{h}$$
$$= \lim_{h\to 0} \frac{3(2 + h)^2 + 4(2 + h) - 5 - 15}{h}$$
$$= \lim_{h\to 0} \frac{12 + 12h + 3h^2 + 8 + 4h - 20}{h}$$
$$= \lim_{h\to 0} \frac{3h^2 + 16h}{h}$$
$$= \lim_{h\to 0} 3h + 16$$
$$= 16$$

b. $f(x) = \frac{2}{x - 1}$

$$m = \lim_{h\to 0} \frac{f(a + h) - f(a)}{h}$$
$$= \lim_{h\to 0} \frac{f(2 + h) - 2}{h}$$
$$= \lim_{h\to 0} \frac{\frac{2}{2 + h - 1} - 2}{h}$$
$$= \lim_{h\to 0} \frac{\frac{2}{1 + h} - \frac{2(1 + h)}{1 + h}}{h}$$
$$= \lim_{h\to 0} \frac{2 - 2(1 + h)}{h(1 + h)}$$
$$= \lim_{h\to 0} \frac{-2h}{h(1 + h)}$$
$$= \lim_{h\to 0} \frac{-2}{1 + h}$$
$$= -2$$

c. $f(x) = \sqrt{x + 3}$

$$m = \lim_{h\to 0} \frac{f(a + h) - f(a)}{h}$$
$$= \lim_{h\to 0} \frac{f(6 + h) - 3}{h}$$
$$= \lim_{h\to 0} \frac{\sqrt{h + 9} - 3}{h}$$
$$= \lim_{h\to 0} \frac{(\sqrt{h + 9} - 3)(\sqrt{h + 9} + 3)}{h(\sqrt{h + 9} + 3)}$$
$$= \lim_{h\to 0} \frac{h + 9 - 9}{h(\sqrt{h + 9} + 3)}$$
$$= \lim_{h\to 0} \frac{h}{h(\sqrt{h + 9} + 3)}$$
$$= \lim_{h\to 0} \frac{1}{(\sqrt{h + 9} + 3)}$$
$$= \frac{1}{6}$$

d. $f(x) = 2^{5x}$

$$m = \lim_{h\to 0} \frac{f(a + h) - f(a)}{h}$$
$$= \lim_{h\to 0} \frac{2^{5(1 + h)} - 32}{h}$$
$$= \lim_{h\to 0} \frac{2^5 \cdot 2^{5h} - 32}{h}$$
$$= \lim_{h\to 0} \frac{32(2^{5h} - 1)}{h}$$
$$= 32 \lim_{h\to 0} \frac{5(2^{5h} - 1)}{5h}$$
$$= 160 \lim_{h\to 0} \frac{(2^{5h} - 1)}{5h}$$
$$= 160 \ln 2$$

2. a. average velocity $= \frac{\text{change in distance}}{\text{change in time}}$

$$= \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$
$$= \frac{[2(4)^2 + 3(4) + 1] - [(2(1)^2 + 3(1) + 1)]}{4 - 1}$$
$$= \frac{45 - 6}{3}$$
$$= 13 \text{ m/s}$$

b. instantaneous velocity = slope of the tangent

$$m = \lim_{h\to 0} \frac{s(a + h) - s(a)}{h}$$

$$= \lim_{h\to 0} \frac{s(3+h) - s(3)}{h}$$

$$= \lim_{h\to 0} \left[\frac{2(3+h)^2 + 3(3+h) + 1}{h} - \frac{(2(3)^2 + 3(3) + 1)}{h}\right]$$

$$= \lim_{h\to 0} \frac{18 + 12h + 2h^2 + 9 + 3h + 1 - 28}{h}$$

$$= \lim_{h\to 0} \frac{15h + 2h^2}{h}$$

$$= \lim_{h\to 0} (15 + 2h)$$

$$= 15 \text{ m/s}$$

3. $$m = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

$$\lim_{h\to 0} \frac{(4+h)^3 - 64}{h} = \lim_{h\to 0} \frac{f(4+h) - f(4)}{h}$$

$$(4+h)^3 - 64 = f(4+h) - f(4)$$

Therefore, $f(x) = x^3$.

4. a. Average rate of change in distance with respect to time is average velocity, so

$$\text{average velocity} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

$$= \frac{s(3) - s(1)}{3 - 1}$$

$$= \frac{4.9(3)^2 - 4.9(1)}{3 - 1}$$

$$= 19.6 \text{ m/s}$$

b. Instantaneous rate of change in distance with respect to time = slope of the tangent.

$$m = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h\to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h\to 0} \frac{4.9(2+h)^2 - 4.9(2)^2}{h}$$

$$= \lim_{h\to 0} \frac{19.6 + 19.6h + 4.9h^2 - 19.6}{h}$$

$$= \lim_{h\to 0} \frac{19.6h + 4.9h^2}{h}$$

$$= \lim_{h\to 0} 19.6 + 4.9h$$

$$= 19.6 \text{ m/s}$$

c. First, we need to determine t for the given distance:

$146.9 = 4.9t^2$

$29.98 = t^2$

$5.475 = t$

Now use the slope of the tangent to determine the instantaneous velocity for $t = 5.475$:

$$m = \lim_{h\to 0} \frac{f(5.475 + h) - f(5.475)}{h}$$

$$= \lim_{h\to 0} \frac{4.9(5.475 + h)^2 - 4.9(5.475)^2}{h}$$

$$= \lim_{h\to 0} \frac{146.9 + 53.655h + 4.9h^2 - 146.9}{h}$$

$$= \lim_{h\to 0} \frac{53.655h + 4.9h^2}{h}$$

$$= \lim_{h\to 0} [53.655 + 4.9h]$$

$$= 53.655 \text{ m/s}$$

5. a. Average rate of population change

$$= \frac{p(t_2) - p(t_1)}{t_2 - t_1}$$

$$= \frac{2(8)^2 + 3(8) + 1 - (2(0) + 3(0) + 1)}{8 - 0}$$

$$= \frac{128 + 24 + 1 - 1}{8 - 0}$$

$= 19$ thousand fish/year

b. Instantaneous rate of population change

$$= \lim_{h\to 0} \frac{p(t+h) - p(t)}{h}$$

$$= \lim_{h\to 0} \frac{p(5+h) - p(5)}{h}$$

$$= \lim_{h\to 0} \left[\frac{2(5+h)^2 + 3(5+h) + 1}{h} - \frac{(2(5)^2 + 3(5) + 1)}{h}\right]$$

$$= \lim_{h\to 0} \frac{50 + 20h + 2h^2 + 15 + 3h + 1 - 66}{h}$$

$$= \lim_{h\to 0} \frac{2h^2 + 23h}{h}$$

$$= \lim_{h\to 0} 2h + 23$$

$= 23$ thousand fish/year

6. a. i. $f(2) = 3$

ii. $\lim_{x\to 2^-} f(x) = 1$

iii. $\lim_{x\to 2^+} f(x) = 3$

iv. $\lim_{x\to 6} f(x) = 2$

b. No, $\lim_{x\to 4} f(x)$ does not exist. In order for the limit to exist, $\lim_{x\to 4^-} f(x)$ and $\lim_{x\to 4^+} f(x)$ must exist and they must be equal. In this case, $\lim_{x\to 4} f(x) = \infty$, but $\lim_{x\to 4^+} f(x) = -\infty$, so $\lim_{x\to 4} f(x)$ does not exist.

7. $f(x)$ is discontinuous at $x = 2$. $\lim_{x \to 2^+} f(x) = 5$, but $\lim_{x \to 2^-} f(x) = 3$.

8. a. $$\lim_{x\to 0} \frac{2x^2+1}{x-5} = \frac{2(0)^2+1}{0-5} = -\frac{1}{5}$$

b. $$\lim_{x\to 3} \frac{x-3}{\sqrt{x+6}-3}$$
$$= \lim_{x\to 3} \frac{(x-3)(\sqrt{x+6}+3)}{(\sqrt{x+6}-3)(\sqrt{x+6}+3)}$$
$$= \lim_{x\to 3} \frac{(x-3)(\sqrt{x+6}+3)}{x+6-9}$$
$$= \lim_{x\to 3} \frac{(x-3)(\sqrt{x+6}+3)}{x-3}$$
$$= \lim_{x\to 3} \sqrt{x+6}+3$$
$$= 6$$

c. $$\lim_{x\to -3} \frac{\frac{1}{x}+\frac{1}{3}}{x+3}$$
$$= \lim_{x\to -3} \frac{\frac{x+3}{3x}}{x+3}$$
$$= \lim_{x\to -3} \frac{x+3}{3x(x+3)}$$
$$= \lim_{x\to -3} \frac{1}{3x}$$
$$= -\frac{1}{9}$$

d. $$\lim_{x\to 2} \frac{x^2-4}{x^2-x-2}$$
$$= \lim_{x\to 2} \frac{(x+2)(x-2)}{(x+1)(x-2)}$$
$$= \lim_{x\to 2} \frac{x+2}{x+1}$$
$$= \frac{4}{3}$$

e. $$\lim_{x\to 2} \frac{x-2}{x^3-8}$$
$$= \lim_{x\to 2} \frac{x-2}{(x-2)(x^2+2x+4)}$$
$$= \lim_{x\to 2} \frac{1}{x^2+2x+4}$$
$$= \frac{1}{12}$$

f. $$\lim_{x\to 0} \frac{\sqrt{x+4}-\sqrt{4-x}}{x}$$
$$= \lim_{x\to 0} \frac{(\sqrt{x+4}-\sqrt{4-x})(\sqrt{x+4}+\sqrt{4-x})}{x(\sqrt{x+4}+\sqrt{4-x})}$$
$$= \lim_{x\to 0} \frac{x+4-(4-x)}{x(\sqrt{x+4}+\sqrt{4-x})}$$
$$= \lim_{x\to 0} \frac{2x}{x(\sqrt{x+4}+\sqrt{4-x})}$$
$$= \lim_{x\to 0} \frac{2}{(\sqrt{x+4}+\sqrt{4-x})}$$
$$= \frac{1}{2}$$

9. a. $f(x) = 3x^2 + x + 1$
$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0} \left[\frac{3(x+h)^2+(x+h)+1}{h} - \frac{(3x^2+x+1)}{h}\right]$$
$$= \lim_{h\to 0} \left[\frac{3x^2+6hx+6h^2+x+h}{h} + \frac{1-3x^2-x-1}{h}\right]$$
$$= \lim_{h\to 0} \frac{6hx+6h^2+h}{h}$$
$$= \lim_{h\to 0} 6x+6h+1$$
$$= 6x+1$$

b. $f(x) = \frac{1}{x}$
$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0} \frac{\frac{1}{x+h}-\frac{1}{x}}{h}$$
$$= \lim_{h\to 0} \frac{x-(x+h)}{h(x)(x+h)}$$
$$= \lim_{h\to 0} \frac{-h}{h(x)(x+h)}$$
$$= \lim_{h\to 0} \frac{-1}{x(x+h)}$$
$$= -\frac{1}{x^2}$$

10. a. To determine the derivative, use the power rule:
$y = x^3 - 4x^2 + 5x + 2$
$$\frac{dy}{dx} = 3x^2 - 8x + 5$$

b. To determine the derivative, use the chain rule:

$$y = \sqrt{2x^3 + 1}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{2x^3 + 1}}(6x^2)$$

$$= \frac{3x^2}{\sqrt{2x^3 + 1}}$$

c. To determine the derivative, use the quotient rule:

$$y = \frac{2x}{x + 3}$$

$$\frac{dy}{dx} = \frac{2(x + 3) - 2x}{(x + 3)^2}$$

$$= \frac{6}{(x + 3)^2}$$

d. To determine the derivative, use the product rule:

$$y = (x^2 + 3)^2(4x^5 + 5x + 1)$$

$$\frac{dy}{dx} = 2(x^2 + 3)(2x)(4x^5 + 5x + 1) + (x^2 + 3)^2(20x^4 + 5)$$

$$= 4x(x^2 + 3)(4x^5 + 5x + 1) + (x^2 + 3)^2(20x^4 + 5)$$

e. To determine the derivative, use the quotient rule:

$$y = \frac{(4x^2 + 1)^5}{(3x - 2)^3}$$

$$\frac{dy}{dx} = \frac{5(4x^2 + 1)^4(8x)(3x - 2)^3}{(3x - 2)^6} - \frac{3(3x - 2)^2(3)(4x^2 + 1)^5}{(3x - 2)^6}$$

$$= (4x^2 + 1)^4(3x - 2)^2 \times \frac{40x(3x - 2) - 9(4x^2 + 1)}{(3x - 2)^6}$$

$$= \frac{(4x^2 + 1)^4(120x^2 - 80x - 36x^2 - 9)}{(3x - 2)^4}$$

$$= \frac{(4x^2 + 1)^4(84x^2 - 80x - 9)}{(3x - 2)^4}$$

f. $y = [x^2 + (2x + 1)^3]^5$

Use the chain rule

$$\frac{dy}{dx} = 5[x^2 + (2x + 1)^3]^4[2x + 6(2x + 1)^2]$$

11. To determine the equation of the tangent line, we need to determine its slope at the point $(1, 2)$. To do this, determine the derivative of y and evaluate for $x = 1$:

$$y = \frac{18}{(x + 2)^2}$$

$$= 18(x + 2)^{-2}$$

$$\frac{dy}{dx} = -36(x + 2)^{-3}$$

$$= \frac{-36}{(x + 2)^3}$$

$$m = \frac{-36}{(x + 2)^3}$$

$$= \frac{-36}{27} = \frac{-4}{3}$$

Since we have a given point and we know the slope, use point-slope form to write the equation of the tangent line:

$$y - 2 = \frac{-4}{3}(x - 1)$$

$$3y - 6 = -4x + 4$$

$$4x + 3y - 10 = 0$$

12. The intersection point of the two curves occurs when

$$x^2 + 9x + 9 = 3x$$

$$x^2 + 6x + 9 = 0$$

$$(x + 3)^2 = 0$$

$$x = -3.$$

At a point x, the slope of the line tangent to the curve $y = x^2 + 9x + 9$ is given by

$$\frac{dy}{dx} = \frac{d}{dx}(x^2 + 9x + 9)$$

$$= 2x + 9.$$

At $x = -3$, this slope is $2(-3) + 9 = 3$.

13. a. $p'(t) = \frac{d}{dt}(2t^2 + 6t + 1100)$

$$= 4t + 6$$

b. 1990 is 10 years after 1980, so the rate of change of population in 1990 corresponds to the value

$$p'(10) = 4(10) + 6$$

$$= 46 \text{ people per year.}$$

c. The rate of change of the population will be 110 people per year when

$$4t + 6 = 110$$

$$t = 26.$$

This corresponds to 26 years after 1980, which is the year 2006.

14. a. $f'(x) = \frac{d}{dx}(x^5 - 5x^3 + x + 12)$

$$= 5x^4 - 15x^2 + 1$$

$$f''(x) = \frac{d}{dx}(5x^4 - 15x^2 + 1)$$

$$= 20x^3 - 30x$$

b. $f(x)$ can be rewritten as $f(x) = -2x^{-2}$

$$f'(x) = \frac{d}{dx}(-2x^{-2})$$
$$= 4x^{-3}$$
$$= \frac{4}{x^3}$$
$$f''(x) = \frac{d}{dx}(4x^{-3})$$
$$= -12x^{-4}$$
$$= -\frac{12}{x^4}$$

c. $f(x)$ can be rewritten as $f(x) = 4x^{-\frac{1}{2}}$

$$f'(x) = \frac{d}{dx}(4x^{-\frac{1}{2}})$$
$$= -2x^{-\frac{3}{2}}$$
$$= -\frac{2}{\sqrt{x^3}}$$
$$f''(x) = \frac{d}{dx}(-2x^{-\frac{3}{2}})$$
$$= 3x^{-\frac{5}{2}}$$
$$= \frac{3}{\sqrt{x^5}}$$

d. $f(x)$ can be rewritten as $f(x) = x^4 - x^{-4}$

$$f'(x) = \frac{d}{dx}(x^4 - x^{-4})$$
$$= 4x^3 + 4x^{-5}$$
$$= 4x^3 + \frac{4}{x^5}$$
$$f''(x) = \frac{d}{dx}(4x^3 + 4x^{-5})$$
$$= 12x^2 - 20x^{-6}$$
$$= 12x^2 - \frac{20}{x^6}$$

15. Extreme values of a function on an interval will only occur at the endpoints of the interval or at a critical point of the function.

a. $f'(x) = \frac{d}{dx}(1 + (x + 3)^2)$

$$= 2(x + 3)$$

The only place where $f'(x) = 0$ is at $x = -3$, but that point is outside of the interval in question. The extreme values therefore occur at the endpoints of the interval:

$f(-2) = 1 + (-2 + 3)^2 = 2$

$f(6) = 1 + (6 + 3)^2 = 82$

The maximum value is 82, and the minimum value is 6

b. $f(x)$ can be rewritten as $f(x) = x + x^{-\frac{1}{2}}$

$$f'(x) = \frac{d}{dx}(x + x^{-\frac{1}{2}})$$
$$= 1 + -\frac{1}{2}x^{-\frac{3}{2}}$$
$$= 1 - \frac{1}{2\sqrt{x^3}}$$

On this interval, $x \geq 1$, so the fraction on the right is always less than or equal to $\frac{1}{2}$. This means that $f'(x) > 0$ on this interval and so the extreme values occur at the endpoints.

$$f(1) = 1 + \frac{1}{\sqrt{1}} = 2$$
$$f(9) = 9 + \frac{1}{\sqrt{9}} = 9\frac{1}{3}$$

The maximum value is $9\frac{1}{3}$, and the minimum value is 2.

c. $f'(x) = \frac{d}{dx}\left(\frac{e^x}{1 + e^x}\right)$

$$= \frac{(1 + e^x)(e^x) - (e^x)(e^x)}{(1 + e^x)^2}$$
$$= \frac{e^x}{(1 + e^x)^2}$$

Since e^x is never equal to zero, $f'(x)$ is never zero, and so the extreme values occur at the endpoints of the interval.

$$f(0) = \frac{e^0}{1 + e^0} = \frac{1}{2}$$
$$f(4) = \frac{e^4}{1 + e^4}$$

The maximum value is $\frac{e^4}{1 + e^4}$, and the minimum value is $\frac{1}{2}$.

d. $f'(x) = \frac{d}{dx}(2 \sin(4x) + 3)$

$$= 8 \cos(4x)$$

Cosine is 0 when its argument is a multiple of $\frac{\pi}{2}$ or $\frac{3\pi}{2}$.

$$4x = \frac{\pi}{2} + 2k\pi \text{ or } 4x = \frac{3\pi}{2} + 2k\pi$$
$$x = \frac{\pi}{8} + \frac{\pi}{2}k \qquad x = \frac{3\pi}{8} + \frac{\pi}{2}k$$

Since $x \in [0, \pi]$, $x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$.

Also test the function at the endpoints of the interval.

$$f(0) = 2 \sin 0 + 3 = 3$$
$$f\left(\frac{\pi}{8}\right) = 2 \sin \frac{\pi}{2} + 3 = 5$$

$f\left(\frac{3\pi}{8}\right) = 2\sin\frac{3\pi}{2} + 3 = 1$

$f\left(\frac{5\pi}{8}\right) = 2\sin\frac{5\pi}{2} + 3 = 5$

$f\left(\frac{7\pi}{8}\right) = 2\sin\frac{7\pi}{2} + 3 = 1$

$f(\pi) = 2\sin(4\pi) + 3 = 3$

The maximum value is 5, and the minimum value is 1.

16. a. The velocity of the particle is given by

$v(t) = s'(t)$

$= \frac{d}{dt}(3t^3 - 40.5t^2 + 162t)$

$= 9t^2 - 81t + 162.$

The acceleration is

$a(t) = v'(t)$

$= \frac{d}{dt}(9t^2 - 81t + 162)$

$= 18t - 81$

b. The object is stationary when $v(t) = 0$:

$9t^2 - 81t + 162 = 0$

$9(t - 6)(t - 3) = 0$

$t = 6$ or $t = 3$

The object is advancing when $v(t) > 0$ and retreating when $v(t) < 0$. Since $v(t)$ is the product of two linear factors, its sign can be determined using the signs of the factors:

t-values	$t - 3$	$t - 6$	$v(t)$	Object
$0 < t < 3$	< 0	< 0	> 0	Advancing
$3 < t < 6$	> 0	< 0	< 0	Retreating
$6 < t < 8$	> 0	> 0	> 0	Advancing

c. The velocity of the object is unchanging when the acceleration is 0; that is, when

$a(t) = 18t - 81 = 0$

$t = 4.5$

d. The object is decelerating when $a(t) < 0$, which occurs when

$18t - 81 < 0$

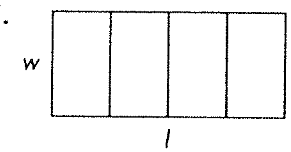
$0 \le t < 4.5$

e. The object is accelerating when $a(t) > 0$, which occurs when

$18t - 81 > 0$

$4.5 < t \le 8$

17.



Let the length and width of the field be l and w, as shown. The total amount of fencing used is then $2l + 5w$. Since there is 750 m of fencing available, this gives

$2l + 5w = 750$

$l = 375 - \frac{5}{2}w$

The total area of the pens is

$A = lw$

$= 375w - \frac{5}{2}w^2$

The maximum value of this area can be found by expressing A as a function of w and examining its derivative to determine critical points.

$A(w) = 375w - \frac{5}{2}w^2$, which is defined for $0 \le w$ and $0 \le l$. Since $l = 375 - \frac{5}{2}w$, $0 \le l$ gives the restriction $w \le 150$. The maximum area is therefore the maximum value of the function $A(w)$ on the interval $0 \le w \le 150$.

$A'(w) = \frac{d}{dw}\left(375w - \frac{5}{2}w^2\right)$

$= 375 - 5w$

Setting $A'(w) = 0$ shows that $w = 75$ is the only critical point of the function. The only values of interest are therefore:

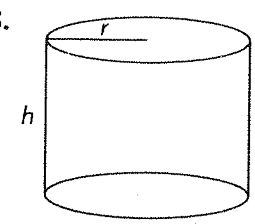
$A(0) = 375(0) - \frac{5}{2}(0)^2 = 0$

$A(75) = 375(75) - \frac{5}{2}(75)^2 = 14\,062.5$

$A(150) = 375(150) - \frac{5}{2}(150)^2 = 0$

The maximum area is 14 062.5 m^2

18.



Let the height and radius of the can be h and r, as shown. The total volume of the can is then $\pi r^2 h$. The volume of the can is also give at 500 mL, so

$\pi r^2 h = 500$

$h = \frac{500}{\pi r^2}$

The total surface area of the can is

$A = 2\pi rh + 2\pi r^2$

$= \frac{1000}{r} + 2\pi r^2$

The minimum value of this surface area can be found by expressing A as a function of r and examining its derivative to determine critical points. $A(r) = \frac{1000}{r} + 2\pi r^2$, which is defined for $0 < r$ and $0 < h$. Since $h = \frac{500}{\pi r^2}$, $0 < h$ gives no additional restriction on r. The maximum area is therefore the maximum value of the function $A(r)$ on the interval $0 < r$.

$$A'(r) = \frac{d}{dr}\left(\frac{1000}{r} + 2\pi r^2\right)$$
$$= -\frac{1000}{r^2} + 4\pi r$$

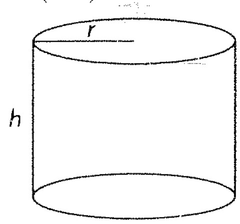
The critical points of $A(r)$ can be found by setting $A'(r) = 0$:

$$-\frac{1000}{r^2} + 4\pi r = 0$$
$$4\pi r^3 = 1000$$
$$r = \sqrt[3]{\frac{1000}{4\pi}} \doteq 4.3 \text{ cm}$$

So $r = 4.3$ cm is the only critical point of the function. This gives the value

$$h = \frac{500}{\pi(4.3)^2} \doteq 8.6 \text{ cm.}$$

19.



Let the radius be r and the height h.

Minimize the cost:

$C = 2\pi r^2(0.005) + 2\pi rh(0.0025)$

$V = \pi r^2 h = 4000$

$h = \frac{4000}{\pi r^2}$

$$C(r) = 2\pi r^2(0.005) + 2\pi r\left(\frac{4000}{\pi r^2}\right)(0.0025)$$
$$= 0.01\pi r^2 + \frac{20}{r}, 1 \le r \le 36$$
$$C'(r) = 0.02\pi r - \frac{20}{r^2}.$$

For a maximum or minimum value, let $C'(r) = 0$.

$$0.02\pi r^2 - \frac{20}{r^2} = 0$$
$$r^3 = \frac{20}{0.02\pi}$$
$$r \doteq 6.8$$

Using the max min algorithm:

$C(1) = 20.03$, $C(6.8) = 4.39$, $C(36) = 41.27$.

The dimensions for the cheapest container are a radius of 6.8 cm and a height of 27.5 cm.

20. a. Let the length, width, and depth be l, w, and d, respectively. Then, the given information is that $l = x$, $w = x$, and $l + w + d = 140$. Substituting gives

$$2x + d = 140$$
$$d = 140 - 2x$$

b. The volume of the box is $V = lwh$. Substituting in the values from part **a.** gives

$V = (x)(x)(140 - 2x)$

$= 140x^2 - 2x^3$

In order for the dimensions of the box to make sense, the inequalities $l \ge 0$, $w \ge 0$, and $h \ge 0$ must be satisfied. The first two give $x \ge 0$, the third requires $x \le 70$. The maximum volume is therefore the maximum value of $V(x) = 140x^2 - 2x^3$ on the interval $0 \le x \le 70$, which can be found by determining the critical points of the derivative $V'(x)$.

$$V'(x) = \frac{d}{dx}(140x^2 - 2x^3)$$
$$= 280x - 6x^2$$
$$= 2x(140 - 3x)$$

Setting $V'(x) = 0$ shows that $x = 0$ and $x = \frac{140}{3} \doteq 46.7$ are the critical points of the function. The maximum value therefore occurs at one of these points or at one of the endpoints of the interval:

$V(0) = 140(0)^2 - 2(0)^3 = 0$

$V(46.7) = 140(46.7)^2 - 2(46.7)^3 = 101\,629.5$

$V(0) = 140(70)^2 - 2(70)^3 = 0$

So the maximum volume is 101 629.5 cm^3, from a box with length and width 46.7 cm and depth $140 - 2(46.7) = 46.6$ cm.

21. The revenue function is

$R(x) = x(50 - x^2)$

$= 50x - x^3$. Its maximum for $x \ge 0$ can be found by examining its derivative to determine critical points.

$$R'(x) = \frac{d}{dx}(50x - x^3)$$
$$= 50 - 3x^2$$

The critical points can be found by setting $R'(x) = 0$:

$$50 - 3x^2 = 0$$
$$x = \pm\sqrt{\frac{50}{3}} \doteq \pm 4.1$$

Only the positive root is of interest since the number of MP3 players sold must be positive. The number must also be an integer, so both $x = 4$ and $x = 5$ must be tested to see which is larger.

$R(4) = 50(4) - 4^3 = 136$

$R(4) = 50(5) - 5^3 = 125$

So the maximum possible revenue is \$136, coming from a sale of 4 MP3 players.

22. Let x be the fare, and $p(x)$ be the number of passengers per year. The given information shows that p is a linear function of x such that an increase of 10 in x results in a decrease of 1000 in p. This means that the slope of the line described by $p(x)$ is $\frac{-1000}{10} = -100$. Using the initial point given,

$$p(x) = -100(x - 50) + 10\,000$$
$$= -100x + 15\,000$$

The revenue function can now be written:

$$R(x) = xp(x)$$
$$= x(-100x + 15\,000)$$
$$= 15\,000x - 100x^2$$

Its maximum for $x \geq 0$ can be found by examining its derivative to determine critical points.

$$R'(x) = \frac{d}{dx}(15\,000x - 100x^2)$$
$$= 15\,000 - 200x$$

Setting $R'(x) = 0$ shows that $x = 75$ is the only critical point of the function. The problem states that only \$10 increases in fare are possible, however, so the two nearest must be tried to determine the maximum possible revenue:

$R(70) = 15\,000(70) - 100(70)^2 = 560\,000$

$R(80) = 15\,000(80) - 100(80)^2 = 560\,000$

So the maximum possible revenue is \$560 000, which can be achieved by a fare of either \$70 or \$80.

23. Let the number of \$30 price reductions be n. The resulting number of tourists will be $80 + n$ where $0 \leq n \leq 70$. The price per tourist will be $5000 - 30n$ dollars. The revenue to the travel agency will be $(5000 - 30n)(80 + n)$ dollars. The cost to the agency will be $250\,000 + 300(80 + n)$ dollars.

Profit = Revenue − Cost

$$P(n) = (5000 - 30n)(80 + n) - 250\,000 - 300(80 + n), 0 \leq n \leq 70$$

$$P'(n) = -30(80 + n) + (5000 - 30n)(1) - 300$$
$$= 2300 - 60n$$

$P'(n) = 0$ when $n = 38\frac{1}{3}$

Since n must be an integer, we now evaluate $P(n)$ for $n = 0, 38, 39,$ and 70. (Since $P(n)$ is a quadratic function whose graph opens downward with vertex at $38\frac{1}{3}$, we know $P(38) > P(39)$.)

$$P(0) = 126\,000$$
$$P(38) = (3860)(118) - 250\,000 - 300(118)$$
$$= 170\,080$$
$$P(39) = (3830)(119) - 250\,000 - 300(119)$$
$$= 170\,070$$
$$P(70) = (2900)(150) - 250\,000 - 300(150)$$
$$= 140\,000$$

The price per person should be lowered by \$1140 (38 decrements of \$30) to realize a maximum profit of \$170 080.

24. a. $$\frac{dy}{dx} = \frac{d}{dx}(-5x^2 + 20x + 2)$$
$$= -10x + 20$$

Setting $\frac{dy}{dx} = 0$ shows that $x = 2$ is the only critical number of the function.

x	$x < 2$	$x = 2$	$x > 2$
y'	+	0	−
Graph	Inc.	Local Max	Dec.

b. $$\frac{dy}{dx} = \frac{d}{dx}(6x^2 + 16x - 40)$$
$$= 12x + 16$$

Setting $\frac{dy}{dx} = 0$ shows that $x = -\frac{4}{3}$ is the only critical number of the function.

x	$x < -\frac{4}{3}$	$x = -\frac{4}{3}$	$x > -\frac{4}{3}$
y'	−	0	+
Graph	Dec.	Local Min	Inc.

c. $$\frac{dy}{dx} = \frac{d}{dx}(2x^3 - 24x)$$
$$= 6x^2 - 24$$

The critical numbers are found by setting $\frac{dy}{dx} = 0$:

$$6x^2 - 24 = 0$$
$$6x^2 = 24$$
$$x = \pm 2$$

x	$x < -2$	$x = -2$	$-2 < x < 2$	$x = 2$	$x > 2$
y'	+	0	−	0	+
Graph	Inc.	Local Max	Dec.	Local Min	Inc.

d. $$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{x - 2}\right)$$
$$= \frac{(x - 2)(1) - x(1)}{(x - 2)^2}$$
$$= \frac{-2}{(x - 2)^2}$$

This derivative is never equal to zero, so the function has no critical numbers. Since the numerator is always negative and the denominator is never negative, the derivative is always negative. This means that the function is decreasing everywhere it is defined, that is, $x \neq 2$.

25. a. This function is discontinuous when

$x^2 - 9 = 0$

$x = \pm 3$. The numerator is non-zero at these points, so these are the equations of the vertical asymptotes.

To check for a horizontal asymptote:

$$\lim_{x\to\infty} \frac{8}{x^2 - 9} = \lim_{x\to\infty} \frac{8}{x^2\left(1 - \frac{9}{x^2}\right)}$$

$$= \frac{\lim_{x\to\infty}(8)}{\lim_{x\to\infty} x^2\left(1 - \frac{9}{x^2}\right)}$$

$$= \frac{\lim_{x\to\infty}(8)}{\lim_{x\to\infty}(x)^2 \times \lim_{x\to\infty}\left(1 - \frac{9}{x^2}\right)}$$

$$= \lim_{x\to\infty} \frac{1}{x^2} \times \frac{8}{1 - 0}$$

$$= 0$$

Similarly, $\lim_{x\to-\infty} \frac{8}{x^2 - 9} = 0$, so $y = 0$ is a horizontal asymptote of the function.

There is no oblique asymptote because the degree of the numerator does not exceed the degree of the denominator by 1.

Local extrema can be found by examining the derivative to determine critical points:

$$y' = \frac{(x^2 - 9)(0) - (8)(2x)}{(x^2 - 9)^2}$$

$$= \frac{-16x}{(x^2 - 9)^2}$$

Setting $y' = 0$ shows that $x = 0$ is the only critical point of the function.

x	$x < 0$	$x = 0$	$x > 0$
y'	+	0	+
Graph	Inc.	Local Max	Dec.

So $(0, -\frac{8}{9})$ is a local maximum.

b. This function is discontinuous when

$x^2 - 1 = 0$

$x = \pm 1$. The numerator is non-zero at these points, so these are the equations of the vertical asymptotes.

To check for a horizontal asymptote:

$$\lim_{x\to\infty} \frac{4x^3}{x^2 - 1} = \lim_{x\to\infty} \frac{x^3(4)}{x^2\left(1 - \frac{1}{x^2}\right)}$$

$$= \lim_{x\to\infty} \frac{x(4)}{1 - \frac{1}{x^2}}$$

$$= \frac{\lim_{x\to\infty}(x(4))}{\lim_{x\to\infty}\left(1 - \frac{1}{x^2}\right)}$$

$$= \frac{\lim_{x\to\infty}(x) \times \lim_{x\to\infty}(4)}{\lim_{x\to\infty}\left(1 - \frac{1}{x^2}\right)}$$

$$= \lim_{x\to\infty}(x) \times \frac{4}{1 - 0}$$

$$= \infty$$

Similarly, $\lim_{x\to-\infty} \frac{4x^3}{x^2 - 1} = \lim_{x\to-\infty}(x) = -\infty$, so this function has no horizontal asymptote.

To check for an oblique asymptote:

$$\begin{array}{r} 4x \\ x^2 - 1\overline{)4x^3 + 0x^2 + 0x + 0} \\ \underline{4x^3 + 0x^2 - 4x} \\ 0 \quad + 0 \quad + 4x + 0 \end{array}$$

So y can be written in the form

$y = 4x + \frac{4x}{x^2 - 1}$. Since

$$\lim_{x\to\infty} \frac{4x}{x^2 - 1} = \lim_{x\to\infty} \frac{x(4)}{x^2\left(1 - \frac{1}{x^2}\right)}$$

$$= \lim_{x\to\infty} \frac{4}{x\left(1 - \frac{1}{x^2}\right)}$$

$$= \frac{\lim_{x\to\infty}(4)}{\lim_{x\to\infty}\left(x\left(1 - \frac{1}{x^2}\right)\right)}$$

$$= \frac{\lim_{x\to\infty}(4)}{\lim_{x\to\infty}(x) \times \lim_{x\to\infty}\left(1 - \frac{1}{x^2}\right)}$$

$$= \lim\left(\frac{1}{x}\right) \times \frac{4}{1 - 0}$$

$$= 0,$$

and similarly $\lim_{x\to-\infty} \frac{4x}{x^2 - 1} = 0$, the line $y = 4x$ is an asymptote to the function y.

Local extrema can be found by examining the derivative to determine critical points:

$$y' = \frac{(x^2 - 1)(12x^2) - (4x^3)(2x)}{(x^2 - 1)^2}$$
$$= \frac{12x^4 - 12x^2 - 8x^4}{(x^2 - 1)^2}$$
$$= \frac{4x^4 - 12x^2}{(x^2 - 1)^2}$$

Setting $y' = 0$:
$4x^4 - 12x^2 = 0$
$x^2(x^2 - 3) = 0$
so $x = 0$, $x = \pm\sqrt{3}$ are the critical points of the function
$(-\sqrt{3}, -6\sqrt{3})$ is a local maximum, $(\sqrt{3}, 6\sqrt{3})$ is a local minimum, and $(0, 0)$ is neither.

x	$x < -\sqrt{3}$	$x = -\sqrt{3}$	$-\sqrt{3} < x < 0$	$x = 0$
y'	+	0	−	0
Graph	Inc.	Local Max	Dec.	Horiz.

x	$0 < x < \sqrt{3}$	$x = \sqrt{3}$	$x > \sqrt{3}$
y'	−	0	−
Graph	Dec.	Local Min	Inc.

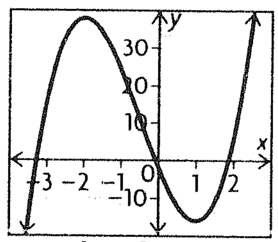
26. a. This function is continuous everywhere, so it has no vertical asymptotes. To check for a horizontal asymptote:
$$\lim_{x\to\infty} (4x^3 + 6x^2 - 24x - 2)$$
$$= \lim_{x\to\infty} x^3\left(4 + \frac{6}{x} - \frac{24}{x^2} - \frac{2}{x^3}\right)$$
$$= \lim_{x\to\infty} (x^3) \times \lim_{x\to\infty}\left(4 + \frac{6}{x} - \frac{24}{x^2} - \frac{2}{x^3}\right)$$
$$= \lim_{x\to\infty} (x^3) \times (4 + 0 - 0 - 0)$$
$$= \infty$$
Similarly,
$\lim_{x\to-\infty} (4x^3 + 6x^2 - 24x - 2) = \lim_{x\to-\infty} (x^3) = -\infty$,
so this function has no horizontal asymptote.

The y-intercept can be found by letting $x = 0$, which gives $y = 4(0)^3 + 6(0)^2 - 24(0) - 2$
$= -2$

The derivative is of the function is
$$y' = \frac{d}{dx}(4x^3 + 6x^2 - 24x - 2)$$
$$= 12x^2 + 12x - 24$$
$= 12(x + 2)(x - 1)$, and the second derivative is
$$y'' = \frac{d}{dx}(12x^2 + 12x - 24)$$
$$= 24x + 12$$
Letting $f'(x) = 0$ shows that $x = -2$ and $x = 1$ are critical points of the function. Letting $y'' = 0$ shows that $x = -\frac{1}{2}$ is an inflection point of the function.

x	$x < -2$	$x = -2$	$-2 < x$	$x = -\frac{1}{2}$
y'	+	0	−	−
Graph	Inc.	Local Max	Dec.	Dec.
y''	−	−	−	0
Concavity	Down	Down	Down	Infl.

x	$-\frac{1}{2} < x < 1$	$x = 1$	$x > 1$
y'	−	0	+
Graph	Dec.	Local Min	Inc.
y''	+	+	+
Concavity	Up	Up	Up



$y = 4x^3 + 6x^2 - 24x - 2$

b. This function is discontinuous when
$$x^2 - 4 = 0$$
$$(x + 2)(x - 2) = 0$$
$x = 2$ or $x = -2$. The numerator is non-zero at these points, so the function has vertical asymptotes at both of them. The behaviour of the function near these asymptotes is:

x-values	$3x$	$x + 2$	$x - 2$	y	lim y
$x \to -2^-$	< 0	< 0	< 0	< 0	$-\infty$
$x \to -2^+$	< 0	> 0	< 0	> 0	$+\infty$
$x \to 2^-$	> 0	> 0	< 0	< 0	$-\infty$
$x \to 2^+$	> 0	> 0	> 0	> 0	$+\infty$

To check for a horizontal asymptote:
$$\lim_{x\to\infty} \frac{3x}{x^2 - 4} = \lim_{x\to\infty} \frac{x(3)}{x^2\left(1 - \frac{4}{x^2}\right)}$$
$$= \lim_{x\to\infty} \frac{3}{x\left(1 - \frac{4}{x^2}\right)}$$

$$= \frac{\lim_{x\to\infty}(3)}{\lim_{x\to\infty}\left(x\left(1 - \frac{4}{x^2}\right)\right)}$$

$$= \frac{\lim_{x\to\infty}(3)}{\lim_{x\to\infty}(x) \times \lim_{x\to\infty}\left(1 - \frac{4}{x^2}\right)}$$

$$= \lim_{x\to\infty}\frac{1}{x} \times \frac{3}{1 - 0}$$

$$= 0$$

Similarly, $\lim_{x\to -\infty}\frac{3x}{x^2 - 4} = 0$, so $y = 0$ is a horizontal asymptote of the function.

This function has $y = 0$ when $x = 0$, so the origin is both the x- and y-intercept.

The derivative is

$$y' = \frac{(x^2 - 4)(3) - (3x)(2x)}{(x^2 - 4)^2}$$

$$= \frac{-3x^2 - 12}{(x^2 - 4)^2}, \text{ and the second derivative is}$$

$$y'' = \frac{(x^2 - 4)^2(-6x)}{(x^2 - 4)^4} - \frac{(-3x^2 - 12)(2(x^2 - 4)(2x))}{(x^2 - 4)^4}$$

$$= \frac{-6x^3 + 24x + 12x^3 + 48x}{(x^2 - 4)^3}$$

$$= \frac{6x^3 + 72x}{(x^2 - 4)^3}$$

The critical points of the function can be found by letting $y' = 0$, so

$-3x^2 - 12 = 0$

$x^2 + 4 = 0$. This has no real solutions, so the function y has no critical points.

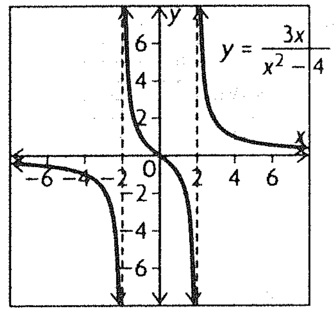
The inflection points can be found by letting $y'' = 0$, so

$6x^3 + 72x = 0$

$6x(x^2 + 12) = 0$

The only real solution to this equation is $x = 0$, so that is the only possible inflection point.

x	$x < -2$	$-2 < x < 0$	$x = 0$	$0 < x < 2$	$x > 2$
y'	−	−	−	−	−
Graph	Dec.	Dec.	Dec.	Dec.	Dec.
y''	−	+	0	−	+
Concavity	Down	Up	Infl.	Down	Up



27. a. $f'(x) = \frac{d}{dx}((-4)e^{5x+1})$

$= (-4)e^{5x+1} \times \frac{d}{dx}(5x + 1)$

$= (-20)e^{5x+1}$

b. $f'(x) = \frac{d}{dx}(xe^{3x})$

$= xe^{3x} \times \frac{d}{dx}(3x) + (1)e^{3x}$

$= e^{3x}(3x + 1)$

c. $y' = \frac{d}{dx}(6^{3x-8})$

$= (\ln 6)6^{3x-8} \times \frac{d}{dx}(3x - 8)$

$= (3\ln 6)6^{3x-8}$

d. $y' = \frac{d}{dx}(e^{\sin x})$

$= e^{\sin x} \times \frac{d}{dx}(\sin x)$

$= (\cos x)e^{\sin x}$

28. The slope of the tangent line at $x = 1$ can be found by evaluating the derivative $\frac{dy}{dx}$ for $x = 1$:

$\frac{dy}{dx} = \frac{d}{dx}(e^{2x-1})$

$= e^{2x-1} \times \frac{d}{dx}(2x - 1)$

$= 2e^{2x-1}$

Substituting $x = 1$ shows that the slope is $2e$. The value of the original function at $x = 1$ is e, so the equation of the tangent line at $x = 1$ is

$y = 2e(x - 1) + e$.

29. a. The maximum of the function modelling the number of bacteria infected can be found by examining its derivative.

$N'(t) = \frac{d}{dt}((15t)e^{-\frac{t}{5}})$

$= 15te^{-\frac{t}{5}} \times \frac{d}{dt}\left(-\frac{t}{5}\right) + (15)e^{-\frac{t}{5}}$

$= e^{-\frac{t}{5}}(15 - 3t)$

Setting $N'(t) = 0$ shows that $t = 5$ is the only critical point of the function (since the exponential function is never zero). The maximum number of infected bacteria therefore occurs after 5 days.

b. $N(5) = (15(5))e^{-\frac{5}{5}}$
$= 27$ bacteria

30. a. $\frac{dy}{dx} = \frac{d}{dx}(2\sin x - 3\cos 5x)$

$= 2\cos x - 3(-\sin 5x) \times \frac{d}{dx}(5x)$

$= 2\cos x + 15\sin 5x$

b. $\frac{dy}{dx} = \frac{d}{dx}(\sin 2x + 1)^4$

$= 4(\sin 2x + 1)^3 \times \frac{d}{dx}(\sin 2x + 1)$

$= 4(\sin 2x + 1)^3 \times (\cos 2x) \times \frac{d}{dx}(2x)$

$= 8\cos 2x(\sin 2x + 1)^3$

c. y can be rewritten as $y = (x^2 + \sin 3x)^{\frac{1}{2}}$. Then,

$\frac{dy}{dx} = \frac{d}{dx}(x^2 + \sin 3x)^{\frac{1}{2}}$

$= \frac{1}{2}(x^2 + \sin 3x)^{-\frac{1}{2}} \times \frac{d}{dx}(x^2 + \sin 3x)$

$= \frac{1}{2}(x^2 + \sin 3x)^{-\frac{1}{2}}$

$\times \left(2x + \cos 3x \times \frac{d}{dx}(3x)\right)$

$= \frac{2x + 3\cos 3x}{2\sqrt{x^2 + \sin 3x}}$

d. $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\sin x}{\cos x + 2}\right)$

$= \frac{(\cos x + 2)(\cos x) - (\sin x)(-\sin x)}{(\cos x + 2)^2}$

$= \frac{\cos^2 x + \sin^2 x + 2\cos x}{(\cos x + 2)^2}$

$= \frac{1 + 2\cos x}{(\cos x + 2)^2}$

e. $\frac{dy}{dx} = \frac{d}{dx}(\tan x^2 - \tan^2 x)$

$= \frac{d}{dx}\sec^2 x^2 \times \frac{d}{dx}(x^2)$

$- 2\tan x \times \frac{d}{dx}(\tan x)$

$= 2x\sec^2 x^2 - 2\tan x \sec^2 x$

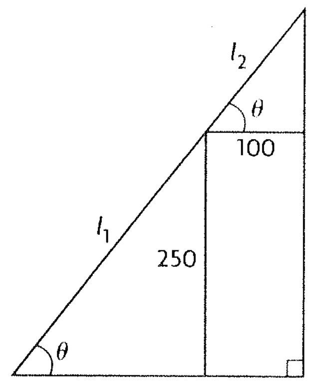
f. $\frac{dy}{dx} = \frac{d}{dx}(\sin(\cos x^2))$

$= \cos(\cos x^2) \times \frac{d}{dx}(\cos x^2)$

$= \cos(\cos x^2) \times (-\sin x^2) \times \frac{d}{dx}(x^2)$

$= -2x\sin x^2 \cos(\cos x^2)$

31.



As shown in the diagram, let θ be the angle between the ladder and the ground, and let the total length of the ladder be $l = l_1 + l_2$, where l_1 is the length from the ground to the top corner of the shed and $l_2$is the length from the corner of the shed to the wall.

$\sin\theta = \frac{250}{l_1}$ $\qquad \cos\theta = \frac{100}{l_2}$

$l_1 = 250\csc\theta$ $\qquad l_2 = 100\sec\theta$

$l = 250\csc\theta + 100\sec\theta$

$\frac{dl}{d\theta} = -250\csc\theta\cot\theta + 100\sec\theta\tan\theta$

$= -\frac{250\cos\theta}{\sin^2\theta} + \frac{100\sin\theta}{\cos^2\theta}$

To determine the minimum, solve $\frac{dl}{d\theta} = 0$.

$\frac{250\cos\theta}{\sin^2\theta} = \frac{100\sin\theta}{\cos^2\theta}$

$250\cos^3\theta = 100\sin^3\theta$

$2.5 = \tan^3\theta$

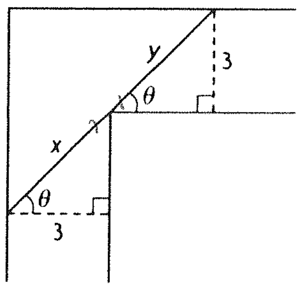
$\tan\theta = \sqrt[3]{2.5}$

$\theta \doteq 0.94$

At $\theta = 0.94$, $l = 250\csc 0.94 + 100\sec 0.94$
$\doteq 479$ cm

The shortest ladder is about 4.8 m long.

32. The longest rod that can fit around the corner is determined by the minimum value of $x + y$. So, determine the minimum value of $l = x + y$.



From the diagram, $\sin\theta = \frac{3}{y}$ and $\cos\theta = \frac{3}{x}$. So,

$l = \frac{3}{\cos\theta} + \frac{3}{\sin\theta}$, for $0 \le \theta \le \frac{\pi}{2}$.

$$\frac{dl}{d\theta} = \frac{3\sin\theta}{\cos^2\theta} - \frac{3\cos\theta}{\sin^2\theta}$$
$$= \frac{3\sin^3\theta - 3\cos^3\theta}{\cos^2\theta\sin^2\theta}$$

Solving $\frac{dl}{d\theta} = 0$ yields:

$$3\sin^3\theta - 3\cos^3\theta = 0$$
$$\tan^3\theta = 1$$
$$\tan\theta = 1$$
$$\theta = \frac{\pi}{4}$$

So $l = \frac{3}{\cos\frac{\pi}{4}} + \frac{3}{\sin\frac{\pi}{4}}$
$= 3\sqrt{2} + 3\sqrt{2}$
$= 6\sqrt{2}$

When $\theta = 0$ or $\theta = \frac{\pi}{2}$, the longest possible rod would have a length of 3 m. Therefore the longest rod that can be carried horizontally around the corner is one of length $6\sqrt{2}$, or about 8.5 m.

CHAPTER 6
Introduction to Vectors

Review of Prerequisite Skills, p. 273

1. a. $\frac{\sqrt{3}}{2}$ **c.** $\frac{1}{2}$ **e.** $\frac{\sqrt{2}}{2}$

b. $-\sqrt{3}$ **d.** $\frac{\sqrt{3}}{2}$ **f.** 1

2. Find BC using the Pythagorean theorem, $AC^2 = AB^2 + BC^2$.

$$BC^2 = AC^2 - AB^2 = 10^2 - 6^2 = 64$$
$$BC = 8$$

Next, use the ratio $\tan A = \frac{\text{opposite}}{\text{adjacent}}$.

$$\tan A = \frac{BC}{AB} = \frac{8}{6} = \frac{4}{3}$$

3. a. To solve ΔABC, find measures of the sides and angles whose values are not given: AB, $\angle B$, and $\angle C$. Find AB using the Pythagorean theorem, $BC^2 = AB^2 + AC^2$.

$$AB^2 = BC^2 - AC^2 = (37.0)^2 - (22.0)^2 = 885$$
$$AB = \sqrt{885} \doteq 29.7$$

Find $\angle B$ using the ratio $\sin B = \frac{\text{opposite}}{\text{hypotenuse}}$.

$$\sin B = \frac{AC}{BC} = \frac{22.0}{37.0}$$
$$\angle B \doteq 36.5°$$
$$\angle C = 90° - \angle B$$
$$\angle C = 90° - 36.5°$$
$$\angle C \doteq 53.5°$$

b. Find measures of the angles whose values are not given. Find $\angle A$ using the cosine law,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$
$$= \frac{5^2 + 8^2 - 10^2}{2(5)(8)} = \frac{-11}{80}$$
$$\angle A \doteq 97.9°$$

Find $\angle B$ using the sine law.

$$\frac{\sin B}{b} = \frac{\sin A}{a}$$
$$\frac{\sin B}{5} = \frac{\sin(97.9°)}{10}$$
$$\sin B \doteq 0.5$$
$$\angle B \doteq 29.7°$$

Find $\angle C$ using the sine law.

$$\frac{\sin C}{c} = \frac{\sin A}{a}$$
$$\frac{\sin C}{8} = \frac{\sin(97.9°)}{10}$$
$$\sin C \doteq 0.8$$
$$\angle C \doteq 52.4°$$

4. Since the sum of the internal angles of a triangle equals 180°, determine the measure of $\angle Z$ using $\angle X = 60°$ and $\angle Y = 70°$.

$$\angle Z = 180° - (\angle X + \angle Y) = 180° - (60° + 70°) = 50°$$

Find XY and YZ using the sine law.

$$\frac{XY}{\sin Y} = \frac{XY}{\sin Z}$$
$$\frac{XY}{\sin 70°} = \frac{6}{\sin 50°}$$
$$XZ \doteq 7.36$$
$$\frac{YZ}{\sin X} = \frac{XY}{\sin Z}$$
$$\frac{YZ}{\sin 60°} = \frac{6}{\sin 50°}$$
$$YZ \doteq 6.78$$

5. Find each angle using the cosine law.

$$\cos R = \frac{RS^2 + RT^2 - ST^2}{2(RS)(RT)} = \frac{4^2 + 7^2 - 5^2}{2(4)(7)}$$

$$= \frac{5}{7}$$

$$\angle R \doteq 44°$$

$$\cos S = \frac{RS^2 + ST^2 - RT^2}{2(RS)(ST)}$$

$$= \frac{4^2 + 5^2 - 7^2}{2(4)(5)}$$

$$= -\frac{1}{5}$$

$$\angle S \doteq 102°$$

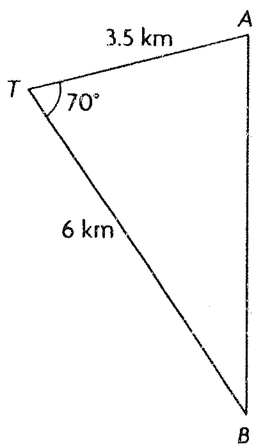
$$\cos T = \frac{RT^2 + ST^2 - RS^2}{2(RT)(ST)}$$

$$= \frac{7^2 + 5^2 - 4^2}{2(7)(5)}$$

$$= \frac{29}{35}$$

$$\angle T \doteq 34°$$

6.



Find AB (the distance between the airplanes) using the cosine law.

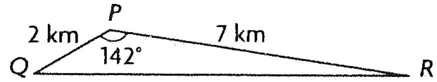
$$AB^2 = AT^2 + BT^2 - 2(AT)(BT)\cos T$$

$$= (3.5 \text{ km})^2 + (6 \text{ km})^2 - 2(3.5 \text{ km})(6 \text{ km}) \cos 70°$$

$$\doteq 33.89 \text{ km}^2$$

$$AB \doteq 5.82 \text{ km}$$

7.



Find QR using the cosine law.

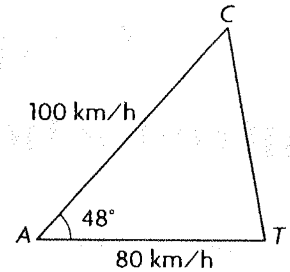
$$QR^2 = PQ^2 + PR^2 - 2(PQ)(PR)\cos P$$

$$= (2 \text{ km})^2 + (7 \text{ km})^2 - 2(2 \text{ km})(7 \text{ km}) \cos 142°$$

$$\doteq 75.06 \text{ km}^2$$

$$QR \doteq 8.66 \text{ km}$$

8.



Find AC and AT using the speed of each vehicle and the elapsed time (in hours) until it was located, distance = speed × time.

$$AC = 100 \text{ km/h} \times \frac{1}{4} \text{ h}$$

$$= 25 \text{ km}$$

$$AT = 80 \text{ km/h} \times \frac{1}{3} \text{ h}$$

$$= 26\frac{2}{3} \text{ km}$$

Find CT using the cosine law.

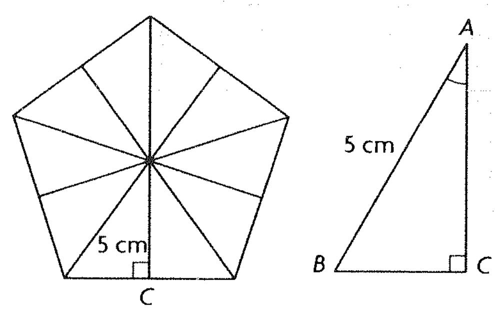
$$CT^2 = AC^2 + AT^2 - 2(AC)(AT)\cos A$$

$$= (25 \text{ km})^2 + \left(26\frac{2}{3} \text{ km}\right)^2 - 2(25 \text{ km})\left(26\frac{2}{3} \text{ km}\right)\cos 48°$$

$$\doteq 443.94 \text{ km}^2$$

$$CT \doteq 21.1 \text{ km}$$

9.



The pentagon can be divided into 10 congruent right triangles with height AC and base BC.

$$10 \times \angle A = 360°$$

$$\angle A = 36°$$

Find AC and BC using trigonometric ratios.

$$AC = AB \times \cos A$$

$$= 5 \cos 36°$$

$$\doteq 4.0 \text{ cm}$$

$$BC = AB \times \sin A$$

$$= 5 \sin 36°$$

$$\doteq 2.9 \text{ cm}$$

The area of the pentagon is the sum of the areas of the 10 right triangles. Use the area of $\triangle ABC$ to determine the area of the pentagon.

$$\text{Area}_{\text{pentagon}} = 10 \times \frac{1}{2}(BC)(AC)$$
$$= 10 \times \frac{1}{2}(2.9 \text{ cm})(4.0 \text{ cm})$$
$$= 59.4 \text{ cm}^2$$

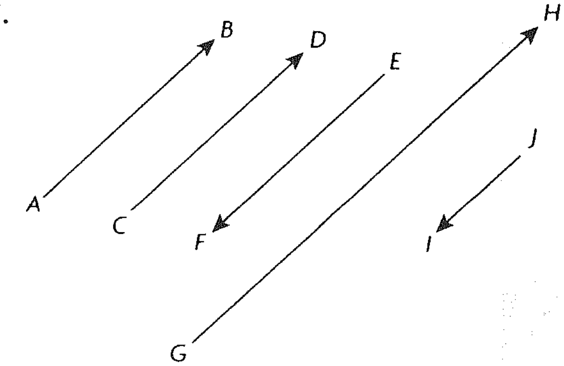
6.1 An Introduction to Vectors, pp. 279–281

1. a. False. Two vectors with the same magnitude can have different directions, so they are not equal.
b. True. Equal vectors have the same direction and the same magnitude.
c. False. Equal or opposite vectors must be parallel and have the same magnitude. If two parallel vectors have different magnitude, they cannot be equal or opposite.
d. False. Equal or opposite vectors must be parallel and have the same magnitude. Two vectors with the same magnitude can have directions that are not parallel, so they are not equal or opposite.
2. Vectors must have a magnitude and direction. For some scalars, it is clear what is meant by just the number. Other scalars are related to the magnitude of a vector.
- Height is a scalar. Height is the distance (see below) from one end to the other end. No direction is given.
- Temperature is a scalar. Negative temperatures are below freezing, but this is not a direction.
- Weight is a vector. It is the force (see below) of gravity acting on your mass.
- Mass is a scalar. There is no direction given.
- Area is a scalar. It is the amount space inside a two-dimensional object. It does not have direction.
- Volume is a scalar. It is the amount of space inside a three-dimensional object. No direction is given.
- Distance is a scalar. The distance between two points does not have direction.
- Displacement is a vector. Its magnitude is related to the scalar distance, but it gives a direction.
- Speed is a scalar. It is the rate of change of distance (a scalar) with respect to time, but does not give a direction.
- Force is a vector. It is a push or pull in a certain direction.
- Velocity is a vector. It is the rate of change of displacement (a vector) with respect to time. Its magnitude is related to the scalar speed.

3. Answers may vary. For example: Friction resists the motion between two surfaces in contact by acting in the opposite direction of motion.
- A rolling ball stops due to friction which resists the direction of motion.
- A swinging pendulum stops due to friction resisting the swinging pendulum.

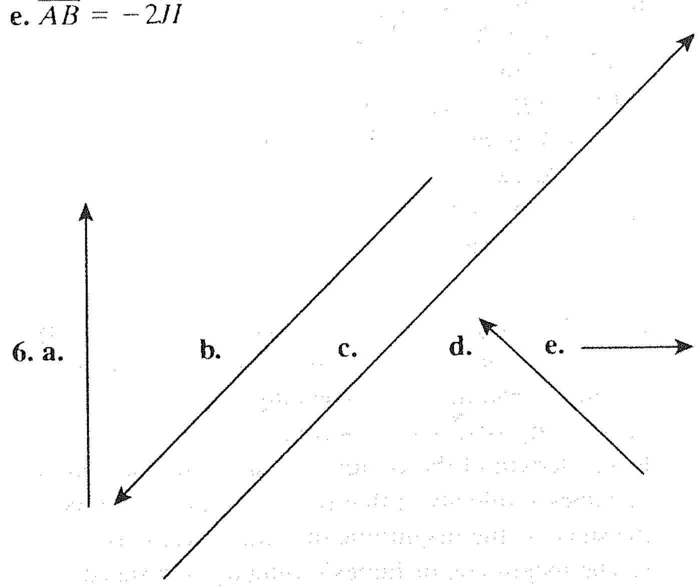
4. Answers may vary. For example:
a. $\overrightarrow{AD} = \overrightarrow{BC}$; $\overrightarrow{AB} = \overrightarrow{DC}$; $\overrightarrow{AE} = \overrightarrow{EC}$; $\overrightarrow{DE} = \overrightarrow{EB}$
b. $\overrightarrow{AD} = -\overrightarrow{CB}$; $\overrightarrow{AB} = -\overrightarrow{CD}$; $\overrightarrow{AE} = -\overrightarrow{CE}$; $\overrightarrow{ED} = -\overrightarrow{EB}$; $\overrightarrow{DA} = -\overrightarrow{BC}$
c. $\overrightarrow{AC}$ & $\overrightarrow{DB}$; $\overrightarrow{AE}$ & $\overrightarrow{EB}$; $\overrightarrow{EC}$ & $\overrightarrow{DE}$; $\overrightarrow{AB}$ & $\overrightarrow{CB}$

5.



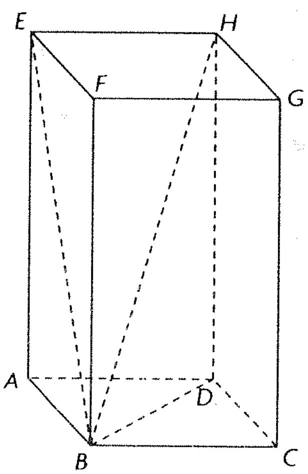
a. $\overrightarrow{AB} = \overrightarrow{CD}$
b. $\overrightarrow{AB} = -\overrightarrow{EF}$
c. $|\overrightarrow{AB}| = |\overrightarrow{EF}|$ but $\overrightarrow{AB} \neq \overrightarrow{EF}$
d. $\overrightarrow{GH} = 2\overrightarrow{AB}$
e. $\overrightarrow{AB} = -2\overrightarrow{JI}$

6.



7. a. 100 km/h, south
b. 50 km/h, west
c. 100 km/h, northeast
d. 25 km/h, northwest
e. 60 km/h, east

8. a. 400 km/h, due south
b. 70 km/h, southwesterly
c. 30 km/h southeasterly
d. 25 km/h, due east
9. a. i. False. They have equal magnitude, but opposite direction.
ii. True. They have equal magnitude.
iii. True. The base has sides of equal length, so the vectors have equal magnitude.
iv. True. They have equal magnitude and direction.
b.



To calculate $|\overrightarrow{BD}|$, $|\overrightarrow{BE}|$ and $|\overrightarrow{BH}|$, find the lengths of their corresponding line segments BD, BE and BH using the Pythagorean theorem.

$$BD^2 = AB^2 + AD^2$$
$$= 3^2 + 3^2$$
$$BD = \sqrt{18}$$
$$BE^2 = AB^2 + AE^2$$
$$= 3^2 + 8^2$$
$$BE = \sqrt{73}$$
$$BH^2 = BD^2 + DH^2$$
$$= (\sqrt{18})^2 + 8^2$$
$$BH = \sqrt{82}$$

10. a. The tangent vector describes James's velocity at that moment. At point A his speed is 15 km/h and he is heading north. The tangent vector shows his velocity is 15 km/h, north.
b. The length of the vector represents the magnitude of James's velocity at that point. James's speed is the same as the magnitude of James's velocity.
c. The magnitude of James's velocity (his speed) is constant, but the direction of his velocity changes at every point.
d. Point C
e. This point is halfway between D and A, which is $\frac{7}{8}$ of the way around the circle. Since he is running 15 km/h and the track is 1 km in circumference, he can run around the track 15 times in one hour. That means each lap takes him 4 minutes. $\frac{7}{8}$ of 4 minutes is 3.5 minutes.
f. When he has travelled $\frac{3}{8}$ of a lap, James will be halfway between B and C and will be heading southwest.
11. a. Find the magnitude of $\overrightarrow{AB}$ using the distance formula.

$$|\overrightarrow{AB}| = \sqrt{(x_A - x_B)^2 + (y_B - y_A)^2}$$
$$= \sqrt{(-4 + 1)^2 + (3 - 2)^2}$$
$$= \sqrt{10} \text{ or } 3.16$$

b. $\overrightarrow{CD} = \overrightarrow{AB}$. $\overrightarrow{AB}$ moves from $A(-4, 2)$ to $B(-1, 3)$ or $(x_B, y_B) = (x_A + 3, y_A + 1)$. Use this to find point D.

$$(x_D, y_D) = (x_C + 3, y_C + 1)$$
$$= (-6 + 3, 0 + 1)$$
$$= (-3, 1)$$

c. $\overrightarrow{EF} = \overrightarrow{AB}$. Find point E using

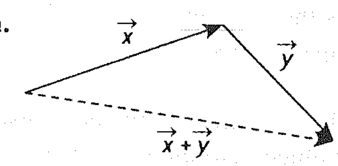
$$(x_A, y_A) = (x_B - 3, y_B - 1).$$
$$(x_E, y_E) = (x_F - 3, y_F - 1)$$
$$= (3 - 3, -2 - 1)$$
$$= (0, -3)$$

d. $\overrightarrow{GH} = -\overrightarrow{AB}$, and moves in the opposite direction as $\overrightarrow{AB}$.

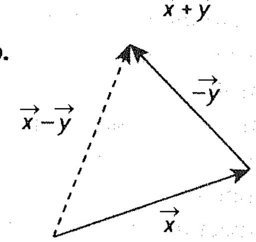
$$(x_H, y_H) = (x_G - 3, y_G - 1).$$
$$(x_H, y_H) = (x_G - 3, y_G - 1)$$
$$= (3 - 3, 1 - 1)$$
$$= (0, 0)$$

6.2 Vector Addition, pp. 290–292

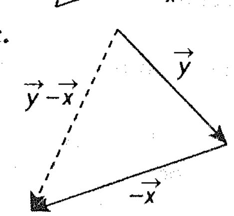
1. a.



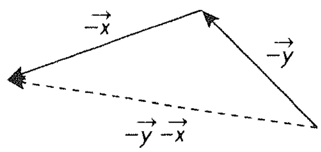
b.



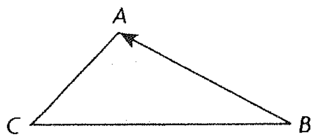
c.



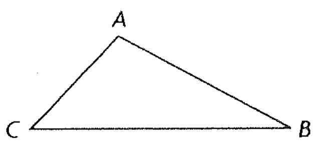
d.



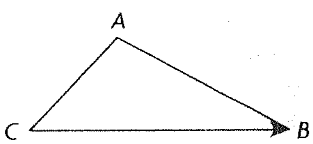
2. a. $\overline{BA}$



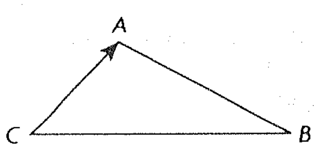
b. $\vec{0}$



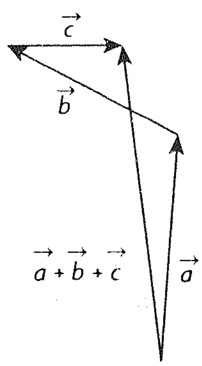
c. $\overline{CB}$



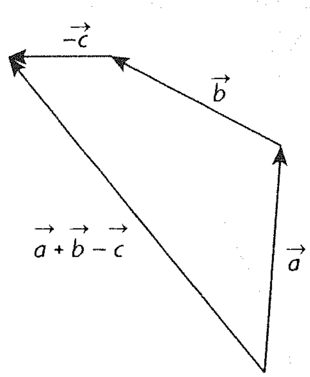
d. $\overline{CA}$



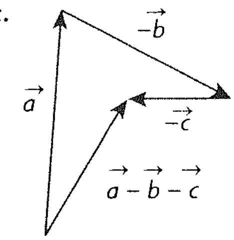
3. a.



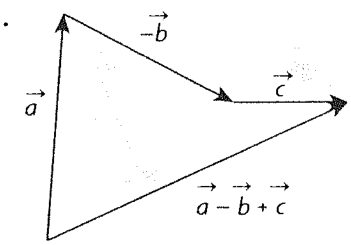
b.



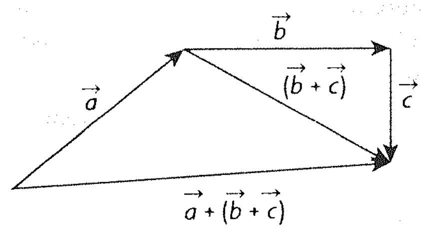
c.



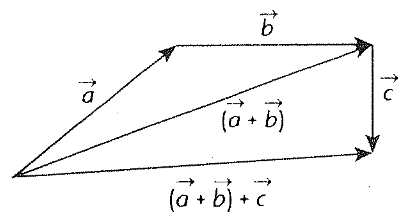
d.



4. a.



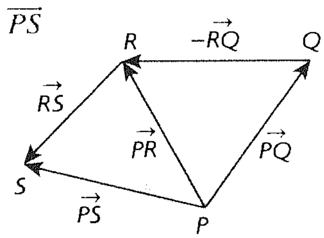
b.



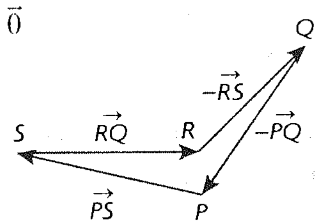
c. The resultant vectors are the same. The order in which you add vectors does not matter.

$(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$

5. a. $\overline{PS}$



b. $\vec{0}$



6. $\vec{x}+\vec{y}=\overrightarrow{MR}+\overrightarrow{RS}$

$=\overrightarrow{MS}$

$\vec{z}+\vec{t}=\overrightarrow{ST}+\overrightarrow{TQ}$

$=\overrightarrow{SQ}$

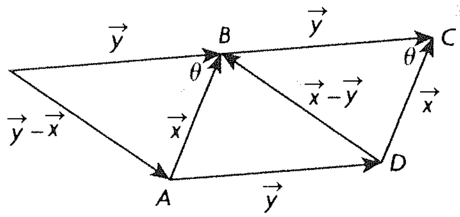
so

$(\vec{x}+\vec{y})+(\vec{z}+\vec{t})=\overrightarrow{MS}+\overrightarrow{SQ}$

$=\overrightarrow{MQ}$

7. a. $-\vec{x}$
b. $\vec{y}$
c. $\vec{x} + \vec{y}$
d. $-\vec{x} + \vec{y}$
e. $\vec{x} + \vec{y} + \vec{z}$
f. $-\vec{x} - \vec{y}$
g. $-\vec{x} + \vec{y} + \vec{z}$
h. $-\vec{x} - \vec{z}$

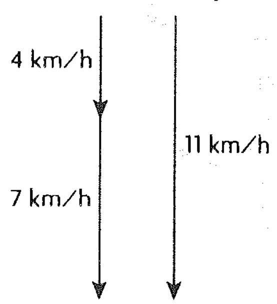
8. a.



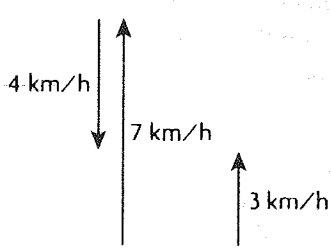
b. See the figure in part a. for the drawn vectors.
$|\vec{y} - \vec{x}|^2 = |\vec{y}|^2 + |\vec{x}|^2 - 2|\vec{y}||-\vec{x}| \cos(\theta)$ and $|-\vec{x}| = |\vec{x}|$, so $|\vec{y} - \vec{x}|^2 = |\vec{x} - \vec{y}|^2$

9. a. Maria's velocity is 11 km/h downstream.

b.

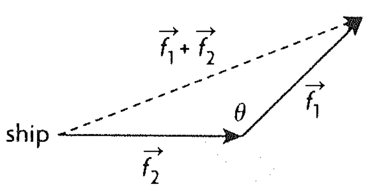


c.



Maria's speed is 3 km/h.

10. a.

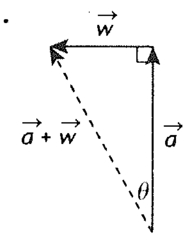


b. The vectors form a triangle with side lengths $|\vec{f}_1|$, $|\vec{f}_2|$ and $|\vec{f}_1 + \vec{f}_2|$. Find $|\vec{f}_1 + \vec{f}_2|$ using the cosine law.

$$|\vec{f}_1 + \vec{f}_2|^2 = |\vec{f}_1|^2 + |\vec{f}_2|^2 - 2|\vec{f}_1||\vec{f}_2| \cos(\theta)$$

$$|\vec{f}_1 + \vec{f}_2| = \sqrt{|\vec{f}_1|^2 + |\vec{f}_2|^2 - 2|\vec{f}_1||\vec{f}_2|\cos(\theta)}$$

11.



Find $|\vec{a} + \vec{w}|$ using the Pythagorean theorem.

$$|\vec{a} + \vec{w}|^2 = |\vec{a}|^2 + |\vec{w}|^2$$
$$= (150\text{ km/h})^2 + (80\text{ km/h})^2$$
$$= 28\,900$$
$$|\vec{a} + \vec{w}|^2 = 170$$

Find the direction of $\vec{a} + \vec{w}$ using the ratio

$$\tan(\theta) = \frac{|\vec{w}|}{|\vec{a}|}$$
$$\theta = \tan^{-1}\frac{80\text{ km/h}}{150\text{ km/h}}$$
$$\doteq \text{N } 28.1^\circ \text{ W}$$

$\vec{a} + \vec{w} = 170$ km/h, N 28.1° W

12. $\vec{x}$, $\vec{y}$, and $\vec{x} + \vec{y}$ form a right triangle. Find $|\vec{x} + \vec{y}|$ using the Pythagorean theorem.

$$|\vec{x} + \vec{y}|^2 = |\vec{x}|^2 + |\vec{y}|^2$$
$$= 7^2 + 24^2$$
$$= 625$$
$$|\vec{x} + \vec{y}| = 25$$

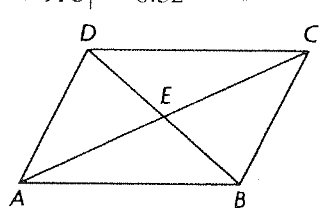
Find the angle between $\vec{x}$ and $\vec{x} + \vec{y}$ using the ratio

$$\tan(\theta) = \frac{|\vec{y}|}{|\vec{x}|}$$
$$\theta = \tan^{-1}\frac{24}{7}$$
$$\doteq 73.7^\circ$$

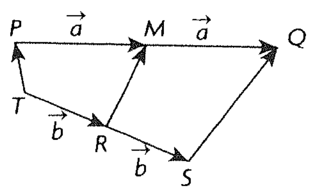
13. Find $|\overrightarrow{AB} + \overrightarrow{AC}|$ using the cosine law and the supplement to the angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$$|\overrightarrow{AB} + \overrightarrow{AC}|^2$$
$$= |\overrightarrow{AB}|^2 + |\overrightarrow{AC}|^2 - 2|\overrightarrow{AB}||\overrightarrow{AC}| \cos(30^\circ)$$
$$= 1^2 + 1^2 - 2(1)(1)\frac{\sqrt{3}}{2}$$
$$= 2 - \sqrt{3}$$
$$|\overrightarrow{AB} + \overrightarrow{AC}| \doteq 0.52$$

14.



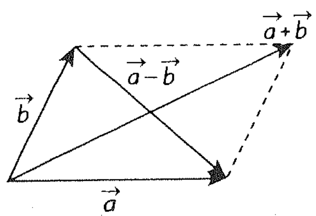
The diagonals of a parallelogram bisect each other.
So $\overrightarrow{EA} = -\overrightarrow{EC}$ and $\overrightarrow{ED} = -\overrightarrow{EB}$.
Therefore, $\overrightarrow{EA} + \overrightarrow{EB} + \overrightarrow{EC} + \overrightarrow{ED} = \vec{0}$.

15. 

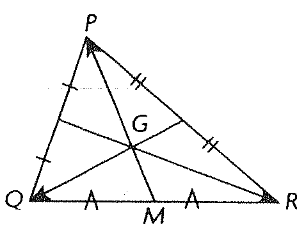
Multiple applications of the Triangle Law for adding vectors show that
$\overrightarrow{RM} + \vec{b} = \vec{a} + \overrightarrow{TP}$ (since both are equal to the undrawn vector $\overrightarrow{TM}$), and that
$\overrightarrow{RM} + \vec{a} = \vec{b} + \overrightarrow{SQ}$ (since both are equal to the undrawn vector $\overrightarrow{RQ}$)
Adding these two equations gives

$$2\,\overrightarrow{RM} + \vec{a} + \vec{b} = \vec{a} + \vec{b} + \overrightarrow{TP} + \overrightarrow{SQ}$$
$$2\,\overrightarrow{RM} = \overrightarrow{TP} + \overrightarrow{SQ}$$

16. $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ represent the diagonals of a parallelogram with sides $\vec{a}$ and $\vec{b}$.



Since $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$ and the only parallelogram with equal diagonals is a rectangle, the parallelogram must also be a rectangle.

17. 

Let point M be defined as shown. Two applications of the Triangle Law for adding vectors show that

$$\overrightarrow{GQ} + \overrightarrow{QM} + \overrightarrow{MG} = \vec{0}$$
$$\overrightarrow{GR} + \overrightarrow{RM} + \overrightarrow{MG} = \vec{0}$$

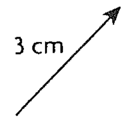
Adding these two equations gives

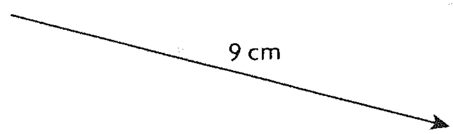
$$\overrightarrow{GQ} + \overrightarrow{QM} + 2\,\overrightarrow{MG} + \overrightarrow{GR} + \overrightarrow{RM} = \vec{0}$$

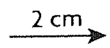
From the given information,
$2\,\overrightarrow{MG} = \overrightarrow{GP}$ and
$\overrightarrow{QM} + \overrightarrow{RM} = \vec{0}$ (since they are opposing vectors of equal length), so
$\overrightarrow{GQ} + \overrightarrow{GP} + \overrightarrow{GR} = \vec{0}$, as desired.

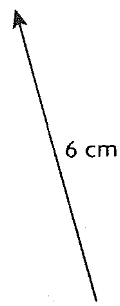
6.3 Multiplication of a Vector by a Scalar, pp. 298–301

1. A vector cannot equal a scalar.

2. a. 

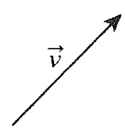
b. 

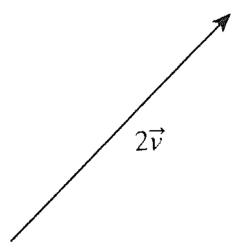
c. 

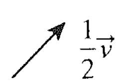
d. 

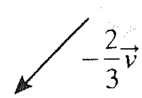
3. E25°N describes a direction that is 25° toward the north of due east (90° east of north), in other words $90° - 25° = 65°$ toward the east of due north. N65°E and "a bearing of 65°" both describe a direction that is 65° toward the east of due north. So, each is describing the same direction in a different way.

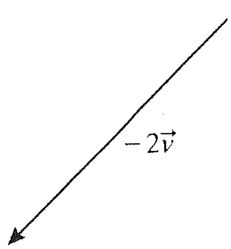
4. Answers may vary. For example:

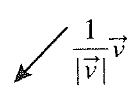


a. 

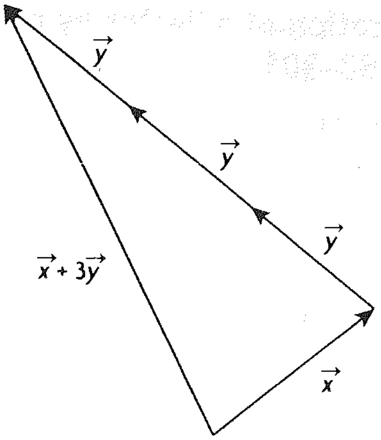
b. 

c. 

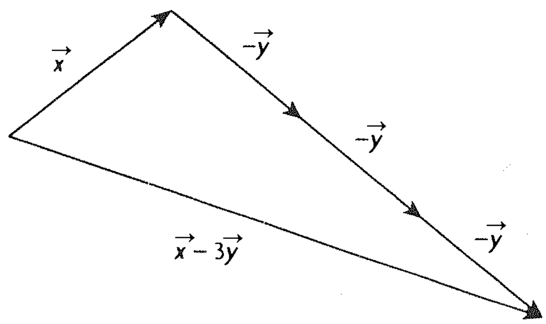
d. 

e. 

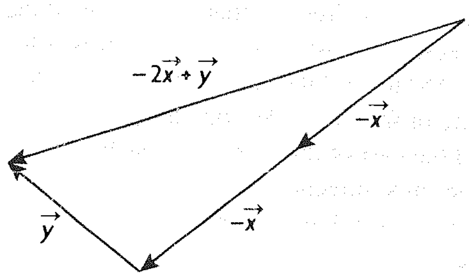
5. a.



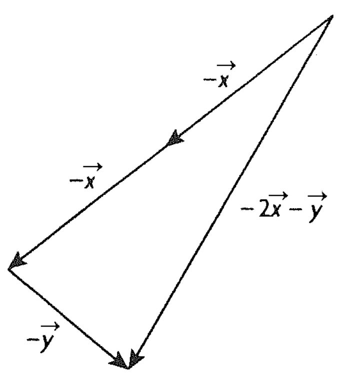
b.



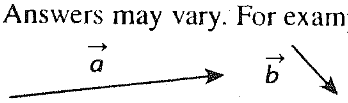
c.



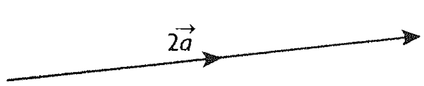
d.



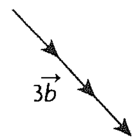
6. Answers may vary. For example:



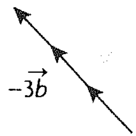
a.



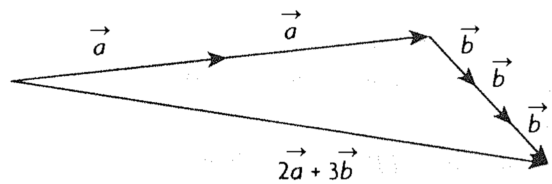
b.



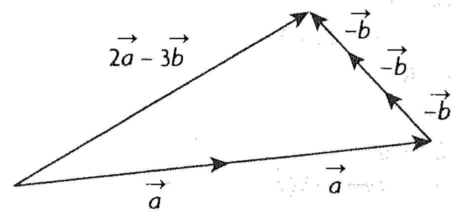
c.



d.



e.



7. a. $\vec{c} = 2\vec{a}, \vec{b} = \frac{3}{2}\vec{a}$

$$m\vec{c} + n\vec{b} = \vec{0}$$

$$m(2\vec{a}) + n\left(\frac{3}{2}\vec{a}\right) = \vec{0}$$

$$m(4\vec{a}) + n(3\vec{a}) = \vec{0}$$

$m = 3$ and $n = -4$ satisfy the equation, as does any multiple of the pair $(3, -4)$. There are infinitely many values possible.

b. $\vec{c} = 2\vec{a}, \vec{b} = \frac{3}{2}\vec{a}$

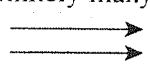
$$d\vec{a} + e\vec{b} + f\vec{c} = \vec{0}$$

$$d\vec{a} + e\left(\frac{3}{2}\vec{a}\right) + f(2\vec{a}) = \vec{0}$$

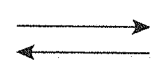
$$2d\vec{a} + 3e\vec{a} + 4f\vec{a} = \vec{0}$$

$d = 2$, $e = 0$, and $f = -1$ satisfy the equation, as does any multiple of the triple $(2, 0, -1)$. There are infinitely many values possible.

8.

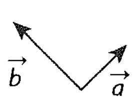


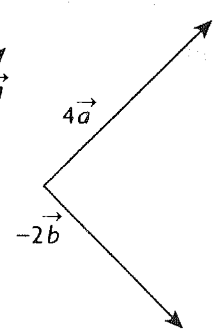
or



$\vec{a}$ and $\vec{b}$ are collinear, so $\vec{a} = k\vec{b}$, where k is a nonzero scalar. Since $|\vec{a}| = |\vec{b}|$, k can only be -1 or 1.

9.





Yes

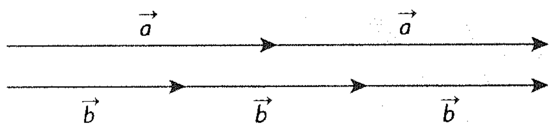
10. Two vectors are collinear if and only if they can be related by a scalar multiple. In this case $\vec{a} \neq k\vec{b}$

a. collinear
b. not collinear
c. not collinear
d. collinear

11. a. $\frac{1}{|\vec{x}|}\vec{x}$ is a vector with length 1 unit in the same direction as $\vec{x}$.

b. $-\frac{1}{|\vec{x}|}\vec{x}$ is a vector with length 1 unit in the opposite direction of $\vec{x}$.

12.



$m = \frac{2}{3}$

13. a. $-\frac{2}{3}\vec{a}$

b. $\frac{1}{3}\vec{a}$

c. $\frac{1}{3}|\vec{a}|$

d. $\frac{2}{3}|\vec{a}|$

e. $\frac{4}{3}\vec{a}$

14. $\vec{x}$ and $\vec{y}$ make an angle of 90°, so you may find $|2\vec{x} + \vec{y}|$ using the Pythagorean theorem.

$|2\vec{x} + \vec{y}|^2 = |2\vec{x}|^2 + |\vec{y}|^2$
$= 2^2 + 1^2$
$|2\vec{x} + \vec{y}| = \sqrt{5}$ or 2.24

Find the direction of $2\vec{x} + \vec{y}$ using the ratio

$\tan(\theta) = \frac{|\vec{y}|}{|2\vec{x}|}$

$\theta = \tan^{-1}\frac{1}{2}$
$\doteq 26.6°$ from $\vec{x}$ towards $2\vec{x} + \vec{y}$

15. Find $|2\vec{x} + \vec{y}|$ using the cosine law, and the supplement to the angle between $\vec{x}$ and $\vec{y}$.

$|2\vec{x} + \vec{y}|^2 = |2\vec{x}|^2 + |\vec{y}|^2 - 2|2\vec{x}||\vec{y}|\cos(150°)$
$= 2^2 + 1^2 - 2(2)(1)\frac{-\sqrt{3}}{2}$

$|2\vec{x} + \vec{y}| \doteq 2.91$

Find the direction of $2\vec{x} + \vec{y}$ using the sine law.

$\frac{\sin\theta}{|\vec{y}|} = \frac{\sin(150°)}{|2\vec{x} + \vec{y}|}$

$\sin\theta \doteq (1)\frac{\frac{1}{2}}{2.91}$

$\theta \doteq 9.9°$ from $\vec{x}$ towards $\vec{y}$

16. $\vec{b} = \frac{1}{|\vec{a}|}\vec{a}$

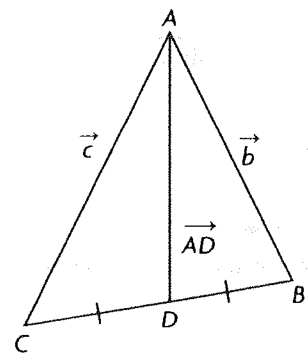
$|\vec{b}| = \left|\frac{1}{|\vec{a}|}\vec{a}\right|$

$|\vec{b}| = \frac{1}{|\vec{a}|}|\vec{a}|$

$|\vec{b}| = 1$

$\vec{b}$ is a positive multiple of $\vec{a}$, so it points in the same direction as $\vec{a}$ and has magnitude 1. It is a unit vector in the same direction as $\vec{a}$.

17.



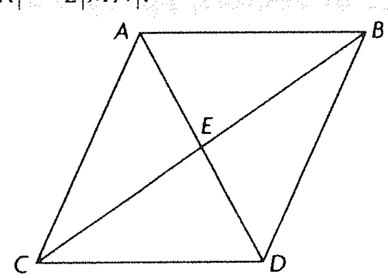
$\overrightarrow{AD} = \vec{c} + \overrightarrow{CD}$
$\overrightarrow{AD} = \vec{b} + \overrightarrow{BD}$
$2\overrightarrow{AD} = \vec{c} + \vec{b} + \overrightarrow{CD} + \overrightarrow{BD}$
But $\overrightarrow{CD} + \overrightarrow{BD} = 0$.
So $2\overrightarrow{AD} = \vec{c} + \vec{b}$, or $\overrightarrow{AD} = \frac{1}{2}\vec{c} + \frac{1}{2}\vec{b}$.

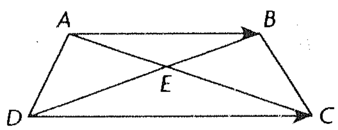
18. $\overrightarrow{PM} = \vec{a}$ and $\overrightarrow{PN} = \vec{b}$
so $\overrightarrow{MN} = \overrightarrow{PN} - \overrightarrow{PM}$
$= \vec{b} - \vec{a}$
$\overrightarrow{PQ} = 2\vec{a}$ and $\overrightarrow{PR} = 2\vec{b}$
so $\overrightarrow{QR} = \overrightarrow{PR} - \overrightarrow{PQ}$
$= 2\vec{b} - 2\vec{a}$

Notice that
$2\overrightarrow{MN} = 2\vec{b} - 2\vec{a}$
$= \overrightarrow{QR}$

We can conclude that $\overrightarrow{QR}$ is parallel to $\overrightarrow{MN}$ and $|\overrightarrow{QR}| = 2|\overrightarrow{MN}|$.

19.

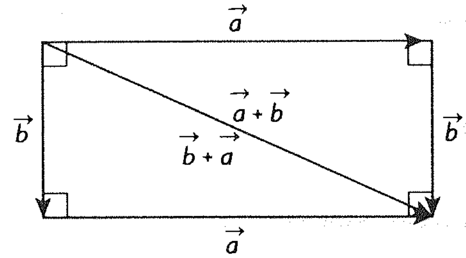


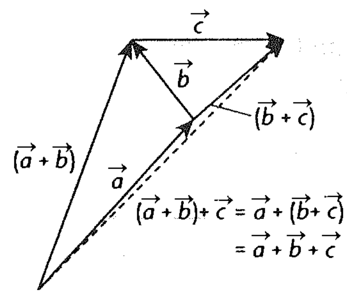
Answers may vary. For example:
a. $\vec{u} = \overrightarrow{AB}$ and $\vec{v} = \overrightarrow{CD}$
b. $\vec{u} = \overrightarrow{AD}$ and $\vec{v} = \overrightarrow{AE}$
c. $\vec{u} = \overrightarrow{AC}$ and $\vec{v} = \overrightarrow{DB}$
d. $\vec{u} = \overrightarrow{ED}$ and $\vec{v} = \overrightarrow{AD}$
20. a. Since the magnitude of $\vec{x}$ is three times the magnitude of $\vec{y}$ and because the given sum is 0, $m\vec{x}$ must be in the opposite direction of $n\vec{y}$ and $|n| = 3|m|$.
b. Whether $\vec{x}$ and $\vec{y}$ are collinear or not, $m = 0$ and $n = 0$ will make the given equation true.
21. a. $\overrightarrow{CD} = \vec{b} - \vec{a}$
b. $\overrightarrow{BE} = 2\vec{b} - 2\vec{a}$
$= 2(\vec{b} - \vec{a})$
$= 2\overrightarrow{CD}$
The two are therefore parallel (collinear) and $|\overrightarrow{BE}| = 2|\overrightarrow{CD}|$
22.


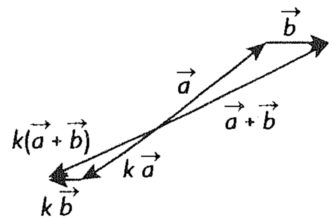
Applying the triangle law for adding vectors shows that
$\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC}$
The given information states that
$$\overrightarrow{AB} = \frac{2}{3}\overrightarrow{DC}$$
$$\frac{3}{2}\overrightarrow{AB} = \overrightarrow{DC}$$
By the properties of trapezoids, this gives
$\frac{3}{2}\overrightarrow{AE} = \overrightarrow{EC}$, and since
$\overrightarrow{AC} = \overrightarrow{AE} + \overrightarrow{EC}$, the original equation gives
$$\overrightarrow{AE} + \frac{3}{2}\overrightarrow{AE} = \overrightarrow{AD} + \frac{3}{2}\overrightarrow{AB}$$
$$\frac{5}{2}\overrightarrow{AE} = \overrightarrow{AD} + \frac{3}{2}\overrightarrow{AB}$$
$$\overrightarrow{AE} = \frac{2}{5}\overrightarrow{AD} + \frac{3}{5}\overrightarrow{AB}$$

6.4 Properties of Vectors, pp. 306–307

1. a. 0
b. 1
c. $\vec{0}$
d. 1

2. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$


3.


4. Answers may vary. For example:


5. $\overrightarrow{PQ} = \overrightarrow{RQ} + \overrightarrow{SR} + \overrightarrow{TS} + \overrightarrow{PT}$
$= (\overrightarrow{RQ} + \overrightarrow{SR}) + (\overrightarrow{TS} + \overrightarrow{PT})$
$= (\overrightarrow{SR} + \overrightarrow{RQ}) + (\overrightarrow{PT} + \overrightarrow{TS})$
$= \overrightarrow{SQ} + \overrightarrow{PS}$
$= \overrightarrow{PS} + \overrightarrow{SQ}$
$= \overrightarrow{PQ}$
6. a. $\overrightarrow{EC}$
b. $\vec{0}$
c. Yes, the diagonals of a rectangular prism are of equal length
7. $= 3\vec{a} - 6\vec{b} - 15\vec{c} - 6\vec{a} + 12\vec{b} - 6\vec{c} - \vec{a} + 3\vec{b} - 3\vec{c}$
$= -4\vec{a} + 9\vec{b} - 24\vec{c}$
8. a. $= 6\vec{i} - 8\vec{j} + 2\vec{k} + 6\vec{i} - 9\vec{j} + 3\vec{k}$
$= 12\vec{i} - 17\vec{j} + 5\vec{k}$
b. $= 3\vec{i} - 4\vec{j} + \vec{k} - 10\vec{i} + 15\vec{j} - 5\vec{k}$
$= -7\vec{i} + 11\vec{j} - 4\vec{k}$
c. $= 2(3\vec{i} - 4\vec{j} + \vec{k} + 6\vec{i} - 9\vec{j} + 3\vec{k})$
$-3(-6\vec{i} + 8\vec{j} - 2\vec{k} + 14\vec{i} - 21\vec{j} + 7\vec{k})$
$= -6\vec{i} + 13\vec{j} - 7\vec{k}$
9. Solve the first equation for $\vec{x}$.
$$\vec{x} = \frac{1}{2}\vec{a} - \frac{3}{2}\vec{y}$$

Substitute into the second equation.

$$6\vec{b} = -\left(\frac{1}{2}\vec{a} - \frac{3}{2}\vec{y}\right) + 5\vec{y}$$

$$\vec{y} = \frac{1}{13}\vec{a} + \frac{12}{13}\vec{b}$$

Lastly, find $\vec{x}$ in terms of $\vec{a}$ and $\vec{b}$.

$$\vec{x} = \frac{1}{2}\vec{a} - \frac{3}{2}\left(\frac{1}{13}\vec{a} + \frac{12}{13}\vec{b}\right)$$

$$= \frac{5}{13}\vec{a} - \frac{18}{13}\vec{b}$$

10. $\vec{a} = \vec{x} - \vec{y}$

$$= \frac{2}{3}\vec{y} + \frac{1}{3}\vec{z} - (\vec{b} + \vec{z})$$

$$= \frac{2}{3}\vec{y} - \frac{2}{3}\vec{z} - \vec{b}$$

$$= \frac{2}{3}(\vec{y} - \vec{z}) - \vec{b}$$

$$= \frac{2}{3}\vec{b} - \vec{b}$$

$$= -\frac{1}{3}\vec{b}$$

11. a. $\vec{AG} = \vec{a} + \vec{b} + \vec{c}$

$\vec{BH} = -\vec{a} + \vec{b} + \vec{c}$

$\vec{CE} = -\vec{a} - \vec{b} + \vec{c}$

$\vec{DF} = \vec{a} - \vec{b} + \vec{c}$

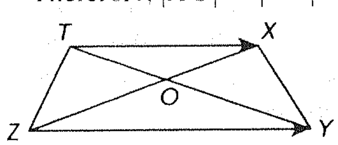
b. $|\vec{AG}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2$

$= |-\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2$

$= |\vec{BH}|^2$

Therefore, $|\vec{AG}| = |\vec{BH}|$

12.



Applying the triangle law for adding vectors shows that

$\vec{TY} = \vec{TZ} + \vec{ZY}$

The given information states that

$\vec{TX} = 2\,\vec{ZY}$

$\frac{1}{2}\vec{TX} = \vec{ZY}$

By the properties of trapezoids, this gives $\frac{1}{2}\vec{TO} = \vec{OY}$, and since $\vec{TY} = \vec{TO} + \vec{OY}$, the original equation gives

$$\vec{TO} + \frac{1}{2}\vec{TO} = \vec{TZ} + \frac{1}{2}\vec{TX}$$

$$\frac{3}{2}\vec{TO} = \vec{TZ} + \frac{1}{2}\vec{TX}$$

$$\vec{TO} = \frac{2}{3}\vec{TZ} + \frac{1}{3}\vec{TX}$$

Mid-Chapter Review, pp. 308–309

1. a. $\vec{AB} = \vec{DC}$

$\vec{BA} = \vec{CD}$

$\vec{AD} = \vec{BC}$

$\vec{CB} = \vec{DA}$

There is not enough information to determine if there is a vector equal to $\vec{AP}$.

b. $|\vec{PD}| = |\vec{DA}|$

$= |\vec{BC}|$ (parallelogram)

2. a. $\vec{RV}$

b. $\vec{RV}$

c. $\vec{PS}$

d. $\vec{RU}$

e. $\vec{PS}$

f. $\vec{PQ}$

3. a. Find $|\vec{a} + \vec{b}|$ using the cosine law, and the supplement to the angle between the vectors.

$$|\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}| \cos 60°$$

$$= 3^2 + 4^2 - 2(3)(4)\frac{1}{2}$$

$$= 3$$

$$|\vec{a} + \vec{b}| = \sqrt{3}$$

b. Find θ using the ratio

$$\tan \theta = \frac{|\vec{b}|}{|\vec{a}|}$$

$$= \frac{4}{3}$$

$$\theta = \tan^{-1}\frac{4}{3}$$

$$\doteq 53°$$

4. $t = 4$ or $t = -4$

5. In quadrilateral $PQRS$, look at $\triangle PQR$. Joining the midpoints B and C creates a vector $\vec{BC}$ that is parallel to $\vec{PR}$ and half the length of $\vec{PR}$. Look at $\triangle SPR$. Joining the midpoints A and D creates a vector $\vec{AD}$ that is parallel to $\vec{PR}$ and half the length of $\vec{PR}$. $\vec{BC}$ is parallel to $\vec{AD}$ and equal in length to $\vec{AD}$. Therefore, $ABCD$ is a parallelogram.

6. a. Find $|\vec{u} - \vec{v}|$ using the cosine law. Note $|-\vec{v}| = |\vec{v}|$ and the angle between $\vec{u}$ and $-\vec{v}$ is 120°.

$$|\vec{u} - \vec{v}|^2 = |\vec{u}|^2 + |-\vec{v}|^2 - 2|\vec{u}||-\vec{v}| \cos 60°$$

$$= 8^2 + 10^2 - 2(8)(10)\left(\frac{1}{2}\right)$$

$$|\vec{u} - \vec{v}| = 2\sqrt{21}$$

b. Find the direction of $\vec{u} - \vec{v}$ using the sine law.

$$\frac{\sin\theta}{|-\vec{v}|} = \frac{\sin 60°}{|\vec{u} - \vec{v}|}$$

$$\sin\theta = \frac{5}{\sqrt{21}}\sin 60°$$

$$\theta = \sin^{-1}\frac{5}{\sqrt{28}}$$

$$\doteq 71°$$

c. $\frac{1}{|\vec{u} + \vec{v}|}(\vec{u} + \vec{v}) = \frac{1}{2\sqrt{21}}(\vec{u} + \vec{v})$

d. Find $|5\vec{u} + 2\vec{v}|$ using the cosine law.

$$|5\vec{u} + 2\vec{v}|^2 = |5\vec{u}|^2 + |2\vec{v}|^2 - 2|5\vec{u}||2\vec{v}|\cos 120°$$

$$= 40^2 + 20^2 - 2(40)(20)\left(-\frac{1}{2}\right)$$

$$|5\vec{u} + 2\vec{v}| = 20\sqrt{7}$$

7. Find $|2\vec{p} - \vec{q}|$ using the cosine law.

$$|2\vec{p} - \vec{q}|^2 = |2\vec{p}|^2 + |-\vec{q}|^2 - 2|2\vec{p}||-\vec{q}|\cos 60°$$

$$= 2^2 + 1^2 - 2(2)(1)\left(\frac{1}{2}\right) = 3$$

8. $|\vec{m} + \vec{n}| = |\vec{m}| - |\vec{n}|$

9. $\overrightarrow{BC} = -\vec{y}$

$\overrightarrow{DC} = \vec{x}$

$\overrightarrow{BD} = -\vec{x} - \vec{y}$

$\overrightarrow{AC} = \vec{x} - \vec{y}$

10. Construct a parallelogram with sides $\overrightarrow{OA}$ and $\overrightarrow{OC}$. Since the diagonals bisect each other, $2\overrightarrow{OB}$ is the diagonal equal to $\overrightarrow{OA} + \overrightarrow{OC}$. Or $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$ and $\overrightarrow{AB} = \frac{1}{2}\overrightarrow{AC}$. So, $\overrightarrow{OB} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AC}$. And $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$. Now $\overrightarrow{OB} = \overrightarrow{OA} + \frac{1}{2}(\overrightarrow{OC} - \overrightarrow{OA})$ Multiplying by 2 gives $2\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{OC}$.

11. $\overrightarrow{AC} + \overrightarrow{CD} = \overrightarrow{AD}$

$3\vec{x} - \vec{y} + 2\vec{y} = \overrightarrow{AD}$

$3\vec{x} + \vec{y} = \overrightarrow{AD}$

$\overrightarrow{AB} + \overrightarrow{BD} = \overrightarrow{AD}$

$\vec{x} + \overrightarrow{BD} = 3\vec{x} + \vec{y}$

$\overrightarrow{BD} = 2\vec{x} + \vec{y}$

$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

$\vec{x} + \overrightarrow{BC} = 3\vec{x} - \vec{y}$

$\overrightarrow{BC} = 2\vec{x} - \vec{y}$

12. The air velocity of the airplane ($\vec{V}_{air}$) and the wind velocity ($\vec{W}$) have opposite directions.

$$\vec{V}_{ground} = \vec{V}_{air} - \vec{W}$$

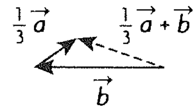
$$= 460 \text{ km/h due south}$$

13. a. $\overline{PT}$

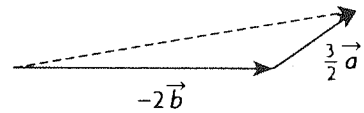
b. $\overline{PT}$

c. $\overline{SR}$

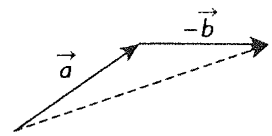
14. a.



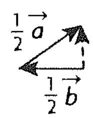
b.



c.



d.



15. $\overrightarrow{PS} = \overrightarrow{PQ} + \overrightarrow{QS}$

$= 3\vec{b} - \vec{a}$

$\overrightarrow{RS} = \overrightarrow{QS} - \overrightarrow{QR}$

$= -3\vec{a}$

6.5 Vectors in R^2 and R^3, pp. 316–318

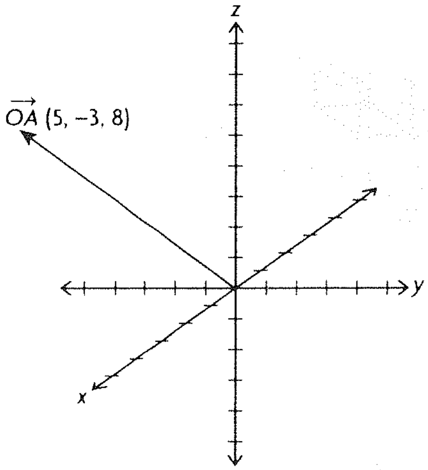
1. No, as the y-coordinate is not a real number.

2. a. We first arrange the x-, y-, and z-axes (each a copy of the real line) in a way so that each pair of axes are perpendicular to each other (i.e., the x- and y-axes are arranged in their usual way to form the xy-plane, and the z-axis passes through the origin of the xy-plane and is perpendicular to this plane). This is easiest viewed as a "right-handed system," where, from the viewer's perspective, the positive z-axis points upward, the positive x-axis points out of the page, and the positive y-axis points rightward in the plane of the page. Then, given point $P(a, b, c)$, we locate this point's unique position by moving a units along the x-axis, then from there b units parallel to the y-axis, and finally c units parallel to the z-axis. It's associated unique position vector is determined by drawing a vector with tail at the origin $O(0, 0, 0)$ and head at P.

b. Since this position vector is unique, its coordinates are unique. Therefore $a = -4$, $b = -3$, and $c = -8$.

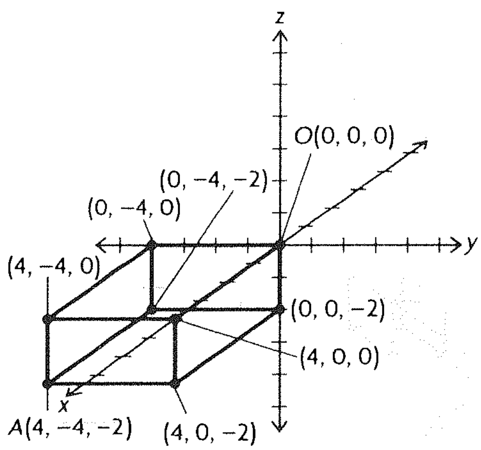
3. a. Since A and B are really the same point, we can equate their coordinates. Therefore $a = 5$, $b = -3$, and $c = 8$.

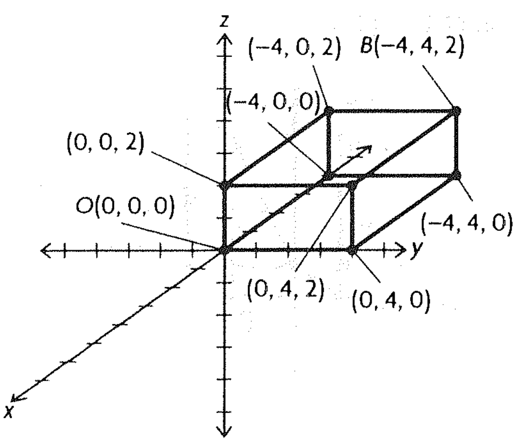
b. From part **a.**, $A(5, -3, 8)$, so $\overrightarrow{OA} = (5, -3, 8)$. Here is a depiction of this vector.

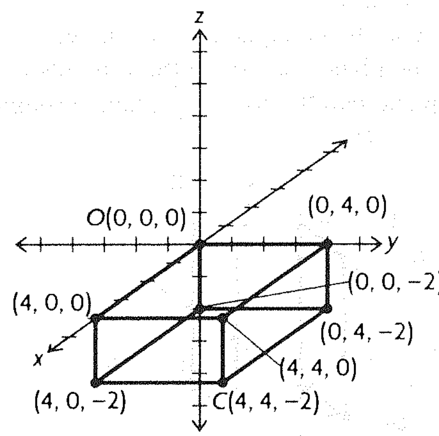


4. This is not an acceptable vector in I^3 as the z-coordinate is not an integer. However, since all of the coordinates are real numbers, this is acceptable as a vector in R^3.

5.







6. a. $A(0, -1, 0)$ is located on the y-axis. $B(0, -2, 0)$, $C(0, 2, 0)$, and $D(0, 10, 0)$ are three other points on this axis.

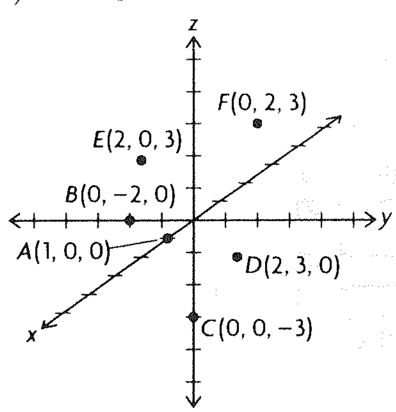
b. $\overrightarrow{OA} = (0, -1, 0)$, the vector with tail at the origin $O(0, 0, 0)$ and head at A.

7. a. Answers may vary. For example:
$\overrightarrow{OA} = (0, 0, 1)$, $\overrightarrow{OB} = (0, 0, -1)$,
$\overrightarrow{OC} = (0, 0, -5)$

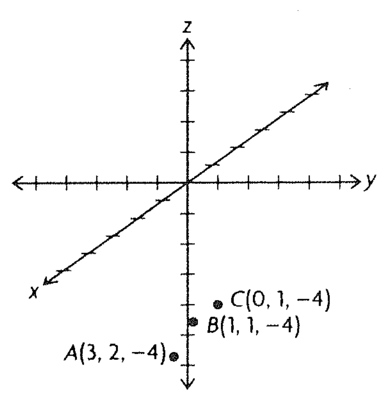
b. Yes, these vectors are collinear (parallel), as they all lie on the same line, in this case the z-axis.

c. A general vector lying on the z-axis would be of the form $\overrightarrow{OA} = (0, 0, a)$ for any real number a. Therefore, this vector would be represented by placing the tail at O, and the head at the point $(0, 0, a)$ on the z-axis.

8.

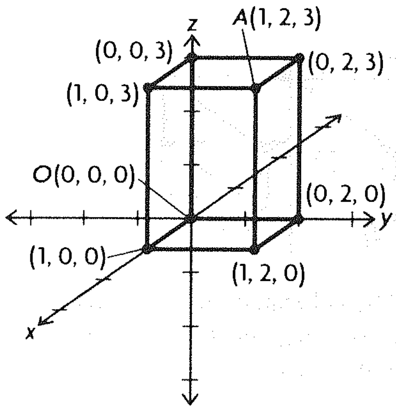


9. a.

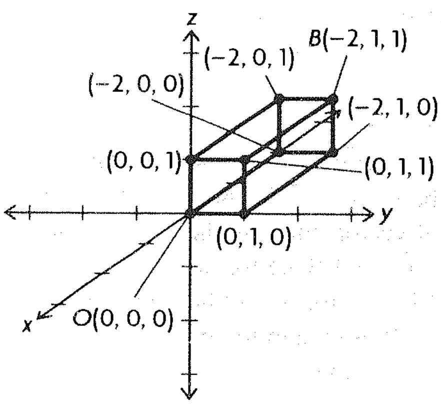


b. Every point on the plane containing points A, B, and C has z-coordinate equal to -4. Therefore, the equation of the plane containing these points is $z = -4$ (a plane parallel to the xy-plane through the point $z = -4$).

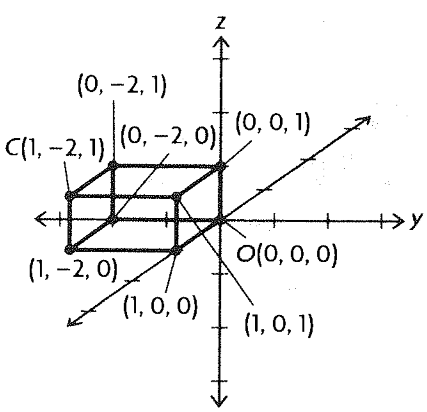
10. a.



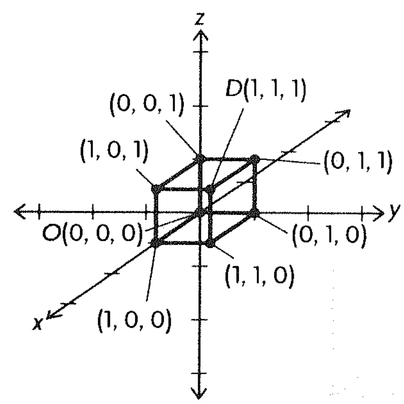
b.



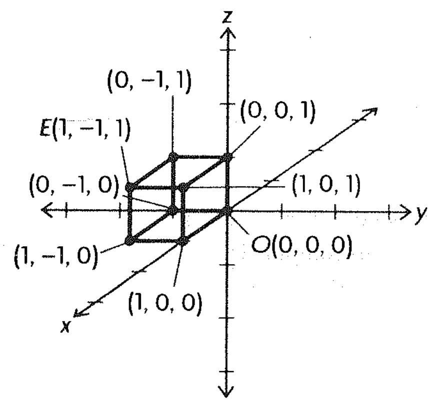
c.



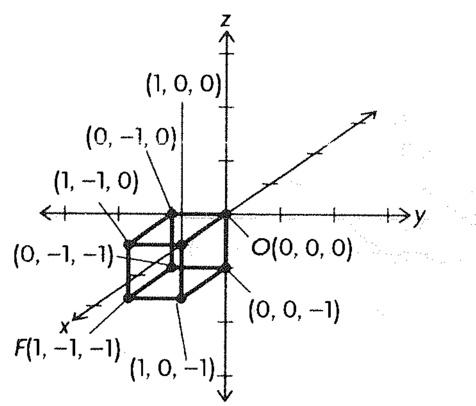
d.



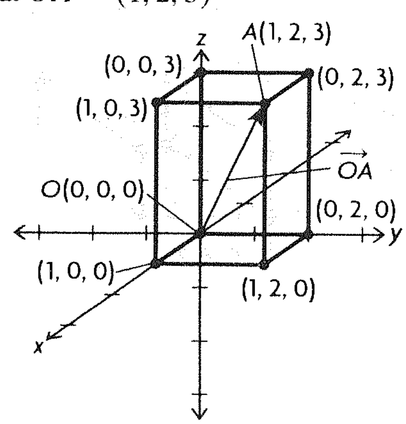
e.



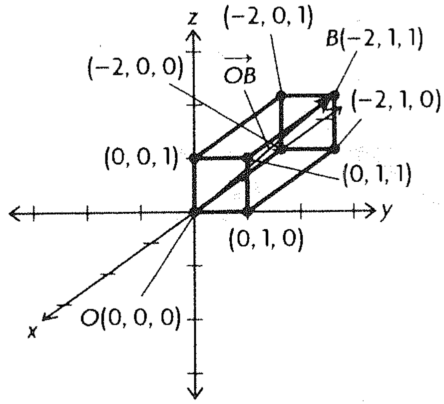
f.



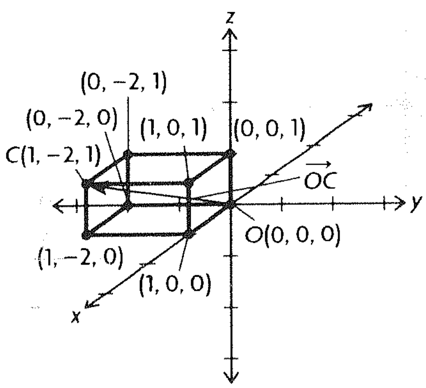
11. a. $\overrightarrow{OA} = (1, 2, 3)$



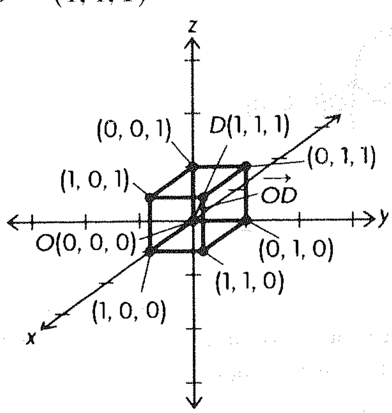
b. $\overrightarrow{OB} = (-2, 1, 1)$



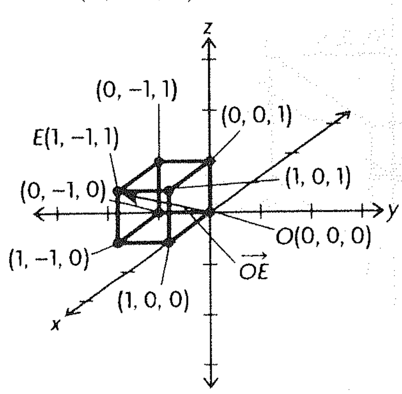
c. $\overrightarrow{OC} = (1, -2, 1)$



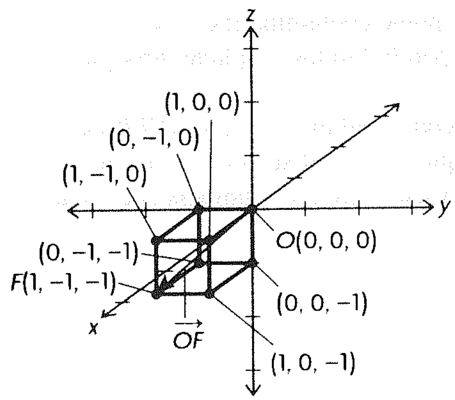
d. $\overrightarrow{OD} = (1, 1, 1)$



e. $\overrightarrow{OE} = (1, -1, 1)$



f. $\overrightarrow{OF} = (1, -1, -1)$



12. a. Since P and Q represent the same point, we can equate their y- and z-coordinates to get the system of equations

$a - c = 6$

$a = 11$

Substituting this second equation into the first gives

$11 - c = 6$

$c = 5$

So $a = 11$ and $c = 5$.

b. Since P and Q represent the same point in R^3, they will have the same associated position vector, i.e. $\overrightarrow{OP} = \overrightarrow{OQ}$. So, since these vectors are equal, they will certainly have equal magnitudes, i.e. $|\overrightarrow{OP}| = |\overrightarrow{OQ}|$.

13. $P(x, y, 0)$ represents a general point on the xy-plane, since the z-coordinate is 0. Similarly, $Q(x, 0, z)$ represents a general point in the xz-plane, and $R(0, y, z)$ represents a general point in the yz-plane.

14. a. Every point on the plane containing points M, N, and P has y-coordinate equal to 0. Therefore, the equation of the plane containing these points is $y = 0$ (this is just the xz-plane).

b. The plane $y = 0$ contains the origin $O(0, 0, 0)$, and so since it also contains the points M, N, and P as well, it will contain the position vectors associated with these points joining O (tail) to the given point (head). That is, the plane $y = 0$ contains the vectors $\overrightarrow{OM}$, $\overrightarrow{ON}$, and $\overrightarrow{OP}$.

15. a. $A(-2, 0, 0)$, $B(-2, 4, 0)$, $C(0, 4, 0)$, $D(0, 0, -7)$, $E(0, 4, -7)$, $F(-2, 0, -7)$

b. $\overrightarrow{OA} = (-2, 0, 0)$, $\overrightarrow{OB} = (-2, 4, 0)$,

$\overrightarrow{OC} = (0, 4, 0)$, $\overrightarrow{OD} = (0, 0, -7)$,

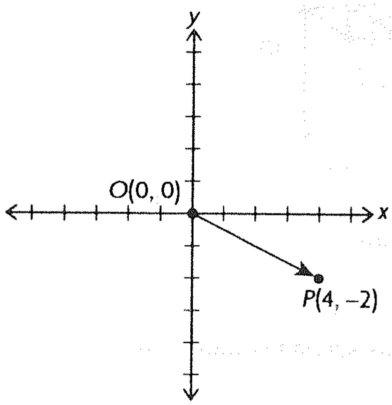
$\overrightarrow{OE} = (0, 4, -7)$, $\overrightarrow{OF} = (-2, 0, -7)$

c. Rectangle $DEPF$ is 7 units below the xy-plane.

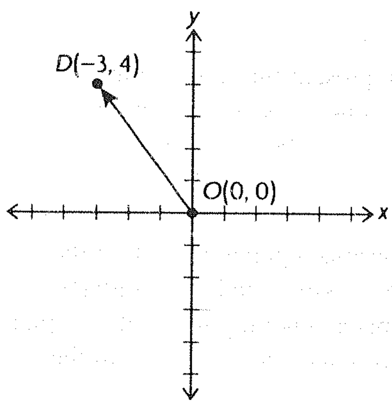
d. Every point on the plane containing points B, C, E, and P has y-coordinate equal to 4. Therefore, the equation of the plane containing these points is $y = 4$ (a plane parallel to the xz-plane through the point $y = 4$).

e. Every point contained in rectangle $BCEP$ has y-coordinate equal to 4, and so is of the form $(x, 4, z)$ where x and z are real numbers such that $-2 \le x \le 0$ and $-7 \le z \le 0$.

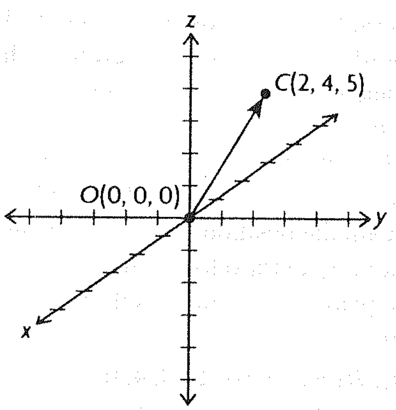
16. a.



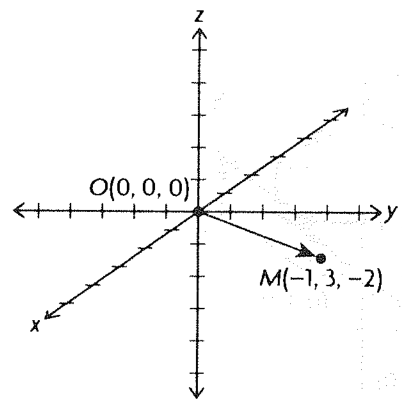
b.



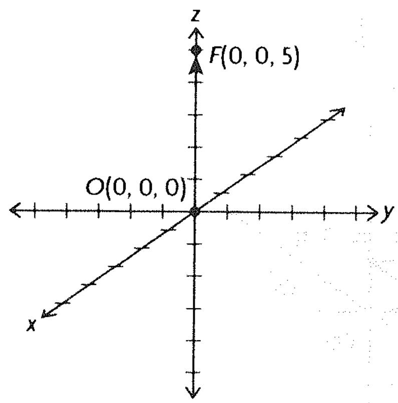
c.



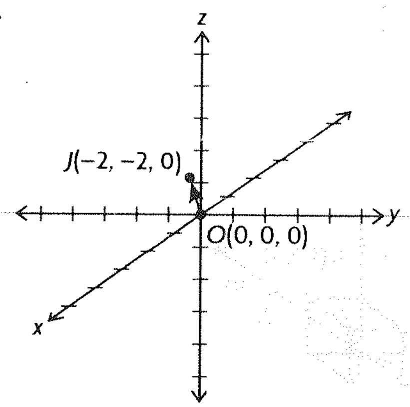
d.



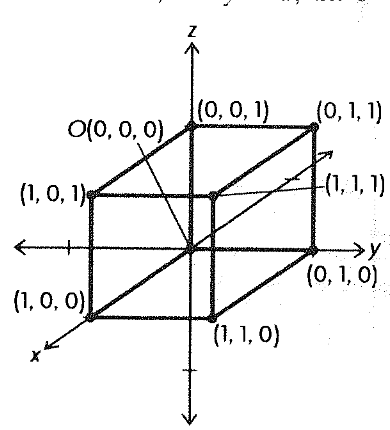
e.



f.



17. The following box illustrates the three dimensional solid consisting of the set of all points (x, y, z) such that $0 \le x \le 1$, $0 \le y \le 1$, and $0 \le z \le 1$.



18. First, $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{OB}$ by the triangle law of vector addition, where $\overrightarrow{OA} = (5, -10, 0)$, $\overrightarrow{OB} = (0, 0, -10)$. $\overrightarrow{OP}$ and $\overrightarrow{OA}$ are drawn in standard position (starting from the origin $O(0, 0, 0)$), and $\overrightarrow{OB}$ is drawn starting from the head of $\overrightarrow{OA}$. Notice that $\overrightarrow{OA}$ lies in the xy-plane, and $\overrightarrow{OB}$ is perpendicular to the xy-plane (so is perpendicular to $\overrightarrow{OA}$). So, $\overrightarrow{OP}$, $\overrightarrow{OA}$, and $\overrightarrow{OB}$ form a right triangle and, by the Pythagorean theorem, $|\overrightarrow{OP}|^2 = |\overrightarrow{OA}|^2 + |\overrightarrow{OB}|^2$
Similarly, $\overrightarrow{OA} = \vec{a} + \vec{b}$ by the triangle law of vector addition, where $\vec{a} = (5, 0, 0)$ and $\vec{b} = (0, -10, 0)$, and these three vectors form a right triangle as well. So,

$$\begin{aligned}|\overrightarrow{OA}|^2 &= |\vec{a}|^2 + |\vec{b}|^2\\ &= 25 + 100\\ &= 125\end{aligned}$$

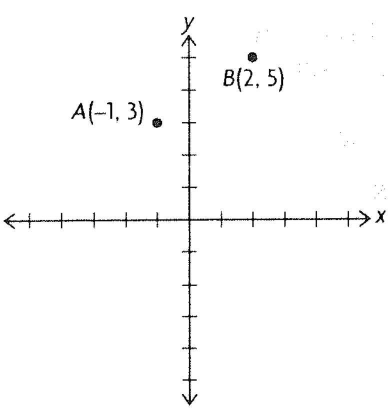
Obviously $|\overrightarrow{OB}|^2 = 100$, and so substituting gives

$$\begin{aligned}|\overrightarrow{OP}|^2 &= |\overrightarrow{OA}|^2 + |\overrightarrow{OB}|^2\\ &= 125 + 100\\ &= 225\\ |\overrightarrow{OP}| &= \sqrt{225}\\ &= 15\end{aligned}$$

19. To find a vector $\overrightarrow{AB}$ equivalent to $\overrightarrow{OP} = (-2, 3, 6)$, where $B(4, -2, 8)$, we need to move 2 units to the right of the x-coordinate for B (to $4 + 2 = 6$), 3 units to the left of the y-coordinate for B (to $-2 - 3 = -5$), and 6 units below the z-coordinate for B (to $8 - 6 = 2$). So we get the point $A(6, -5, 2)$. Indeed, notice that to get from A to B (which describes vector $\overrightarrow{AB}$), we move 2 units left in the x-coordinate, 3 units right in the y-coordinate, and 6 units up in the z-coordinate. This is equivalent to vector $\overrightarrow{OP} = (-2, 3, 6)$.

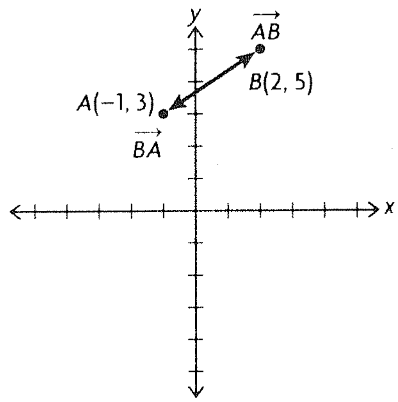
6.6 Operations with Algebraic Vectors in R^2, pp. 324–326

1.



a.
$$\begin{aligned}\overrightarrow{AB} &= (2, 5) - (-1, 3)\\ &= (3, 2)\\ \overrightarrow{BA} &= -\overrightarrow{AB}\\ &= -(3, 2)\\ &= (-3, -2)\end{aligned}$$

Here is a sketch of these two vectors in the xy-coordinate plane.



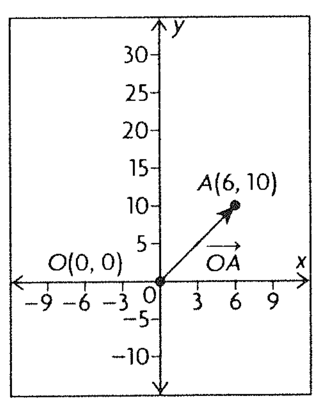
b.
$$\begin{aligned}|\overrightarrow{OA}| &= \sqrt{(-1)^2 + 3^2}\\ &= \sqrt{10}\\ &\doteq 3.16\\ |\overrightarrow{OB}| &= \sqrt{2^2 + 5^2}\\ &= \sqrt{29}\\ &\doteq 5.39\end{aligned}$$

c.
$$\begin{aligned}|\overrightarrow{AB}| &= \sqrt{3^2 + 2^2}\\ &= \sqrt{13}\\ &\doteq 3.61\end{aligned}$$

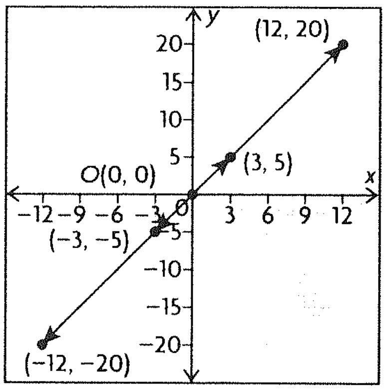
Also, since $\overrightarrow{BA} = -\overrightarrow{AB}$,

$$\begin{aligned}|\overrightarrow{BA}| &= |-\overrightarrow{AB}|\\ &= |-1| \cdot |\overrightarrow{AB}|\\ &= |\overrightarrow{AB}|\\ &= \sqrt{13}\\ &\doteq 3.61\end{aligned}$$

2.



a.



b. The vectors with the same magnitude are

$\frac{1}{2}\overrightarrow{OA}$ and $-\frac{1}{2}\overrightarrow{OA}$,

$2\overrightarrow{OA}$ and $-2\overrightarrow{OA}$

3. $|\overrightarrow{OA}| = \sqrt{3^2 + (-4)^2}$
$= \sqrt{25}$
$= 5$

4. a. The $\vec{i}$-component will be equal to the first coordinate in component form, and so $a = -3$. Similarly, the $\vec{j}$-component will be equal to the second coordinate in component form, and so $b = 5$.

b. $|(-3, b)| = |(-3, 5)|$
$= \sqrt{(-3)^2 + 5^2}$
$= \sqrt{34}$
$\doteq 5.83$

5. a. $|\vec{a}| = \sqrt{(-60)^2 + 11^2}$
$= \sqrt{3721}$
$= 61$

$|\vec{b}| = \sqrt{(-40)^2 + (-9)^2}$
$= \sqrt{1681}$
$= 41$

b. $\vec{a} + \vec{b} = (-60, 11) + (-40, -9)$
$= (-100, 2)$

$|\vec{a} + \vec{b}| = \sqrt{(-100)^2 + 2^2}$
$= \sqrt{10\,004}$
$\doteq 100.02$

$\vec{a} - \vec{b} = (-60, 11) - (-40, -9)$
$= (-20, 20)$

$|\vec{a} - \vec{b}| = \sqrt{(-20)^2 + 20^2}$
$= \sqrt{800}$
$\doteq 28.28$

6. a. $2(-2, 3) + (2, 1) = (2(-2) + 2, 2(3) + 1)$
$= (-2, 7)$

b. $-3(4, -9) - 9(2, 3)$
$= (-3(4) - 9(2), -3(-9) - 9(3))$
$= (-30, 0)$

c. $-\frac{1}{2}(6, -2) + \frac{2}{3}(6, 15)$
$= \left(-\frac{1}{2}(6) + \frac{2}{3}(6), -\frac{1}{2}(-2) + \frac{2}{3}(15)\right)$
$= (1, 11)$

7. $\vec{x} = 2\vec{i} - \vec{j}, \vec{y} = -\vec{i} + 5\vec{j}$

a. $3\vec{x} - \vec{y} = 3(2\vec{i} - \vec{j}) - (-\vec{i} + 5\vec{j})$
$= (6 + 1)\vec{i} + (-3 - 5)\vec{j}$
$= 7\vec{i} - 8\vec{j}$

b. $-(\vec{x} + 2\vec{y}) + 3(-\vec{x} - 3\vec{y})$
$= -4\vec{x} - 11\vec{y}$
$= -4(2\vec{i} - \vec{j}) - 11(-\vec{i} + 5\vec{j})$
$= 3\vec{i} - 51\vec{j}$

c. $2(\vec{x} + 3\vec{y}) - 3(\vec{y} + 5\vec{x})$
$= -13\vec{x} + 3\vec{y}$
$= -13(2\vec{i} - \vec{j}) + 3(-\vec{i} + 5\vec{j})$
$= -29\vec{i} + 28\vec{j}$

8. a. $\vec{x} + \vec{y} = (2\vec{i} - \vec{j}) + (-\vec{i} + 5\vec{j})$
$= \vec{i} + 4\vec{j}$

$|\vec{x} + \vec{y}| = |\vec{i} + 4\vec{j}|$
$= \sqrt{1^2 + 4^2}$
$= \sqrt{17}$
$\doteq 4.12$

b. $\vec{x} - \vec{y} = (2\vec{i} - \vec{j}) - (-\vec{i} + 5\vec{j})$
$= 3\vec{i} - 6\vec{j}$

$|\vec{x} - \vec{y}| = |3\vec{i} - 6\vec{j}|$
$= \sqrt{3^2 + (-6)^2}$
$= \sqrt{45}$
$\doteq 6.71$

c. $2\vec{x} - 3\vec{y} = 2(2\vec{i} - \vec{j}) - 3(-\vec{i} + 5\vec{j})$
$= 7\vec{i} - 17\vec{j}$

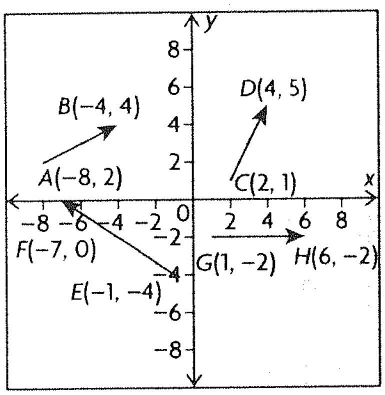
$|2\vec{x} - 3\vec{y}| = |7\vec{i} - 17\vec{j}|$
$= \sqrt{7^2 + (-17)^2}$
$= \sqrt{338}$
$\doteq 18.38$

d. $|3\vec{y} - 2\vec{x}| = |-(2\vec{x} - 3\vec{y})|$
$= |-1||2\vec{x} - 3\vec{y}|$
$= |2\vec{x} - 3\vec{y}|$

so, from part c.,

$|3\vec{y} - 2\vec{x}| = |2\vec{x} - 3\vec{y}|$
$= \sqrt{338}$
$\doteq 18.38$

9.



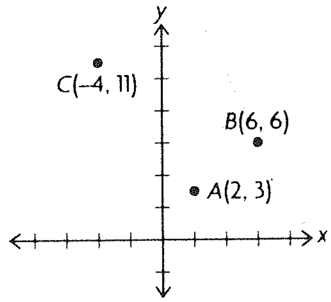
a. $\overrightarrow{AB} = (-4, 4) - (-8, 2)$
$= (4, 2)$
$\overrightarrow{CD} = (4, 5) - (2, 1)$
$= (2, 4)$
$\overrightarrow{EF} = (-7, 0) - (-1, -4)$
$= (-6, 4)$
$\overrightarrow{GH} = (6, -2) - (1, -2)$
$= (5, 0)$

b. $|\overrightarrow{AB}| = \sqrt{4^2 + 2^2}$
$= \sqrt{20}$
$\doteq 4.47$
$|\overrightarrow{CD}| = \sqrt{2^2 + 4^2}$
$= \sqrt{20}$
$\doteq 4.47$
$|\overrightarrow{EF}| = \sqrt{(-6)^2 + 4^2}$
$= \sqrt{52}$
$\doteq 7.21$
$|\overrightarrow{GH}| = \sqrt{5^2 + 0^2}$
$= \sqrt{25}$
$= 5$

10. a. By the parallelogram law of vector addition,
$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OB}$
$= (6, 3) + (11, -6)$
$= (17, -3)$
For the other vectors,
$\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB}$
$= (6, 3) - (11, -6)$
$= (-5, 9)$
$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$
$= (17, -3) - (11, -6)$
$= (6, 3)$

b. $\overrightarrow{OA} = (6, 3)$
$= \overrightarrow{BC},$
so obviously we will have $|\overrightarrow{OA}| = |\overrightarrow{BC}|$.
(It turns out that their common magnitude is $\sqrt{6^2 + 3^2} = \sqrt{45}$.)

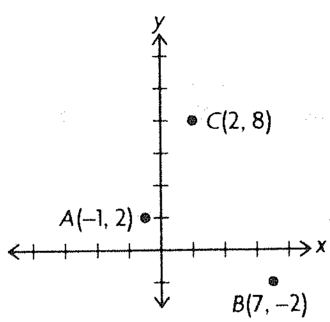
11. a.



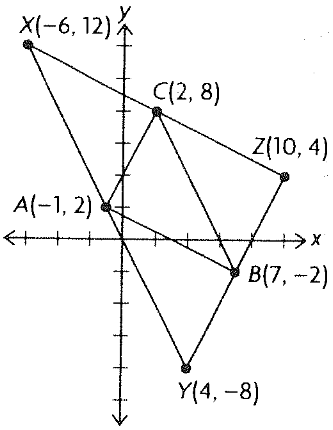
b. $\overrightarrow{AB} = (6, 6) - (2, 3)$
$= (4, 3)$
$|\overrightarrow{AB}| = \sqrt{4^2 + 3^2}$
$= \sqrt{25}$
$= 5$
$\overrightarrow{AC} = (-4, 11) - (2, 3)$
$= (-6, 8)$
$|\overrightarrow{AC}| = \sqrt{(-6)^2 + 8^2}$
$= \sqrt{100}$
$= 10$
$\overrightarrow{CB} = (6, 6) - (-4, 11)$
$= (10, -5)$
$|\overrightarrow{CB}| = \sqrt{10^2 + (-5)^2}$
$= \sqrt{125}$
$\doteq 11.18$

c. $|\overrightarrow{CB}|^2 = 125 |\overrightarrow{AC}|^2 = 100, |\overrightarrow{AB}|^2 = 25$
Since $|\overrightarrow{CB}|^2 = |\overrightarrow{AC}|^2 + |\overrightarrow{AB}|^2$, the triangle is a right triangle.

12. a.



b.



c. As a first possibility for the fourth vertex, there is $X(x_1, x_2)$. From the sketch in part b., we see that we would then have

$$\overrightarrow{CX} = \overrightarrow{BA}$$
$$(x_1 - 2, x_2 - 8) = (-1 - 7, 2 - (-2))$$
$$= (-8, 4)$$
$$x_1 - 2 = -8$$
$$x_2 - 8 = 4$$

So $X(-6, 12)$. By similar reasoning for the other points labelled in the sketch in part b.,

$$\overrightarrow{AY} = \overrightarrow{CB}$$
$$(y_1 - (-1), y_2 - 2) = (7 - 2, -2 - 8)$$
$$= (5, -10)$$
$$y_1 + 1 = 5$$
$$y_2 - 2 = -10$$

So $Y(4, -8)$. Finally,

$$\overrightarrow{BZ} = \overrightarrow{AC}$$
$$(z_1 - 7, z_2 - (-2)) = (2 - (-1), 8 - 2)$$
$$= (3, 6)$$
$$z_1 - 7 = 3$$
$$z_2 + 2 = 6$$

So $Z(10, 4)$. In conclusion, the three possible locations for a fourth vertex in a parallelogram with vertices A, B, and C are $X(-6, 12)$, $Y(4, -8)$, and $Z(10, 4)$.

13. a. $3(x, 1) - 5(2, 3y) = (11, 33)$
$$(3x - 5(2), 3 - 5(3y)) = (11, 33)$$
$$(3x - 10, 3 - 15y) = (11, 33)$$
$$3x - 10 = 11$$
$$3 - 15y = 33$$

So $x = 7$ and $y = -2$.

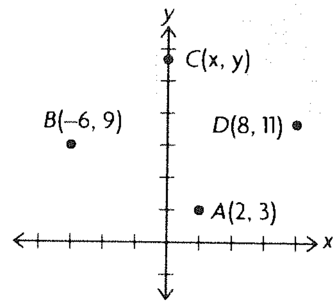
b. $-2(x, x + y) - 3(6, y) = (6, 4)$
$$(-2x - 18, -2x - 5y) = (6, 4)$$
$$-2x - 18 = 6$$
$$-2x - 5y = 4$$

To solve for x, use
$$-2x - 18 = 6$$
$$x = -12$$

Substituting this into the last equation above, we can now solve for y.
$$-2(-12) - 5y = 4$$
$$y = 4$$

So $x = -12$ and $y = 4$.

14. a.



b. Because $ABCD$ is a rectangle, we will have
$$\overrightarrow{BC} = \overrightarrow{AD}$$
$$(x, y) - (-6, 9) = (8, 11) - (2, 3)$$
$$(x + 6, y - 9) = (6, 8)$$
$$x + 6 = 6$$
$$y - 9 = 8$$

So, $x = 0$ and $y = 17$, i.e., $C(0, 17)$.

15. a. Since $|\overrightarrow{PA}| = |\overrightarrow{PB}|$, and
$$\overrightarrow{PA} = (5, 0) - (a, 0)$$
$$= (5 - a, 0),$$
$$\overrightarrow{PB} = (0, 2) - (a, 0)$$
$$= (-a, 2),$$

this means that
$$(5 - a)^2 = (-a)^2 + 2^2$$
$$25 - 10a + a^2 = a^2 + 4$$
$$10a = 21$$
$$a = \frac{21}{10}$$

So $P\left(\frac{21}{10}, 0\right)$.

b. This point Q on the y-axis will be of the form $Q(0, b)$ for some real number b. Reasoning similarly to part a., we have
$$\overrightarrow{QA} = (5, 0) - (0, b)$$
$$= (5, -b)$$
$$\overrightarrow{QB} = (0, 2) - (0, b)$$
$$= (0, 2 - b)$$

So since $|\overrightarrow{QA}| = |\overrightarrow{QB}|$,
$$(-b)^2 + 5^2 = (2 - b)^2$$
$$b^2 + 25 = 4 - 4b + b^2$$
$$4b = -21$$
$$b = -\frac{21}{4}$$

So $Q\left(0, -\frac{21}{4}\right)$.

16. $\overrightarrow{QP}$ is in the direction opposite to $\overrightarrow{PQ}$, and

$$\begin{aligned}\overrightarrow{QP} &= \overrightarrow{OP} - \overrightarrow{OQ}\\ &= (11, 19) - (2, -21)\\ &= (9, 40)\end{aligned}$$

$$\begin{aligned}|\overrightarrow{QP}| &= \sqrt{9^2 + 40^2}\\ &= \sqrt{1681}\\ &= 41\end{aligned}$$

A unit vector in the direction of $\overrightarrow{QP}$ is

$$\begin{aligned}\vec{u} &= \frac{1}{41}\overrightarrow{QP}\\ &= \left(\frac{9}{41}, \frac{40}{41}\right)\end{aligned}$$

Indeed, $\vec{u}$ is obviously in the same direction as $\overrightarrow{QP}$ (since $\vec{u}$ is a positive scalar multiple of $\overrightarrow{QP}$), and notice that

$$\begin{aligned}|\vec{u}| &= \sqrt{\left(\frac{9}{41}\right)^2 + \left(\frac{40}{41}\right)^2}\\ &= \sqrt{\frac{81 + 1600}{1681}}\\ &= 1\end{aligned}$$

17. a. O, P, and R can be thought of as the vertices of a triangle.

$$\begin{aligned}\overrightarrow{PR} &= \overrightarrow{OR} - \overrightarrow{OP}\\ &= (-8, -1) - (-7, 24)\\ &= (-1, -25)\end{aligned}$$

$$\begin{aligned}|\overrightarrow{PR}|^2 &= (-1)^2 + (-25)^2\\ &= 626\end{aligned}$$

$$\begin{aligned}|\overrightarrow{OR}|^2 &= (-8)^2 + (-1)^2\\ &= 65\end{aligned}$$

$$\begin{aligned}|\overrightarrow{OP}|^2 &= (-7)^2 + 24^2\\ &= 625\end{aligned}$$

By the cosine law, the angle, θ, between $\overrightarrow{OR}$ and $\overrightarrow{OP}$ satisfies

$$\begin{aligned}\cos\theta &= \frac{|\overrightarrow{OR}|^2 + |\overrightarrow{OP}|^2 - |\overrightarrow{PR}|^2}{2|\overrightarrow{OR}| \cdot |\overrightarrow{OP}|}\\ &= \frac{65 + 625 - 626}{2\sqrt{65} \cdot \sqrt{625}}\\ \theta &= \cos^{-1}\left(\frac{65 + 625 - 626}{2\sqrt{65} \cdot \sqrt{625}}\right)\\ &\doteq 80.9°\end{aligned}$$

So the angle between $\overrightarrow{OR}$ and $\overrightarrow{OP}$ is about 80.86°.

b. We found the vector $\overrightarrow{PR} = (-1, -25)$ in part a., so $\overrightarrow{RP} = -\overrightarrow{PR} = (1, 25)$ and

$$\begin{aligned}|\overrightarrow{RP}|^2 &= |\overrightarrow{PR}|^2\\ &= 626\end{aligned}$$

Also, by the parallelogram law of vector addition,

$$\begin{aligned}\overrightarrow{OQ} &= \overrightarrow{OR} + \overrightarrow{OP}\\ &= (-8, -1) + (-7, 24)\\ &= (-15, 23)\end{aligned}$$

$$\begin{aligned}|\overrightarrow{OQ}|^2 &= (-15)^2 + 23^2\\ &= 754\end{aligned}$$

Placing $\overrightarrow{RP} = (1, 25)$ and $\overrightarrow{OQ} = (-15, 23)$ with their tails at the origin, a triangle is formed by joining the heads of these two vectors. The third side of this triangle is the vector

$$\begin{aligned}\vec{v} &= \overrightarrow{RP} - \overrightarrow{OQ}\\ &= (1, 25) - (-15, 23)\\ &= (16, 2)\end{aligned}$$

$$\begin{aligned}|\vec{v}|^2 &= 16^2 + 2^2\\ &= 260\end{aligned}$$

Now by reasoning similar to part a., the cosine law implies that the angle, θ, between $\overrightarrow{RP}$ and $\overrightarrow{OQ}$ satisfies

$$\begin{aligned}\cos\theta &= \frac{|\overrightarrow{RP}|^2 + |\overrightarrow{OQ}|^2 - |\vec{v}|^2}{2|\overrightarrow{RP}| \cdot |\overrightarrow{OQ}|}\\ &= \frac{626 + 754 - 260}{2\sqrt{626} \cdot \sqrt{754}}\\ \theta &= \cos^{-1}\left(\frac{626 + 754 - 260}{2\sqrt{626} \cdot \sqrt{754}}\right)\\ &\doteq 35.4°\end{aligned}$$

So the angle between $\overrightarrow{RP}$ and $\overrightarrow{OQ}$ is about 35.40°. However, since we are discussing the diagonals of parallelogram $OPQR$ here, it would also have been appropriate to report the supplement of this angle, or about 180° − 35.40° = 144.60°, as the angle between these vectors.

6.7 Operations with Vectors in R^3, pp. 332–333

1. a. $\overrightarrow{OA} = -1\vec{i} + 2\vec{j} + 4\vec{k}$

b. $|\overrightarrow{OA}| = \sqrt{(-1)^2 + 2^2 + 4^2} = \sqrt{21} \doteq 4.58$

2. $\overrightarrow{OB} = (3, 4, -4)$

$|\overrightarrow{OB}| = \sqrt{3^2 + 4^2 + (-4)^2} = \sqrt{41} \doteq 6.40$

3. $$\begin{aligned}\vec{a} + \frac{1}{3}\vec{b} - \vec{c} &= (1, 3, -3) + (-1, 2, 4) - (0, 8, 1)\\ &= (1 + (-1) - 0, 3 + 2 - 8, (-3) + 4 - 1)\\ &= (0, -3, 0)\end{aligned}$$

$$\begin{aligned}\left|\vec{a} + \frac{1}{3}\vec{b} - \vec{c}\right| &= \sqrt{0^2 + (-3)^2 + 0^2}\\ &= 3\end{aligned}$$

4. a. $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{OB}$
$= ((-3) + 2, 4 + 2, 12 + (-1))$
$= (-1, 6, 11)$

b. $|\overrightarrow{OA}| = \sqrt{(-3)^2 + 4^2 + 12^2} = 13$
$|\overrightarrow{OB}| = \sqrt{2^2 + 2^2 + (-1)^2} = 3$
$|\overrightarrow{OP}| = \sqrt{(-1)^2 + 6^2 + 11^2} \doteq 12.57$

c. $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$
$= (2, 2, -1) - (-3, 4, 12)$
$= (2 - (-3), 2 - 4, (-1) - 12)$
$= (5, -2, -13)$
$|\overrightarrow{AB}| = \sqrt{5^2 + (-2)^2 + (-13)^2} = \sqrt{198} \doteq 14.07$
$\overrightarrow{AB}$ represents the vector from the tip of $\overrightarrow{OA}$ to the tip of $\overrightarrow{OB}$. It is the difference between the two vectors.

5. a. $\vec{x} - 2\vec{y} - \vec{z}$
$= (1, 4, -1) - 2(1, 3, -2) - (-2, 1, 0)$
$= (1, 4, -1) - (2, 6, -4) - (-2, 1, 0)$
$= (1 - 2 - (-2), 4 - 6 - 1, -1 - (-4) - 0)$
$= (1, -3, 3)$

b. $-2\vec{x} - 3\vec{y} + \vec{z}$
$= -2(1, 4, -1) - 3(1, 3, -2) + (-2, 1, 0)$
$= (-2, -8, 2) - (3, 9, -6) + (-2, 1, 0)$
$= (-2 - 3 - 2, -8 - 9 + 1, 2 + 6 + 0)$
$= (-7, -16, 8)$

c. $\frac{1}{2}\vec{x} - \vec{y} + 3\vec{z}$
$= \frac{1}{2}(1, 4, -1) - (1, 3, -2) + 3(-2, 1, 0)$
$= \left(\frac{1}{2}, 2, -\frac{1}{2}\right) - (1, 3, -2) + (-6, 3, 0)$
$= \left(\frac{1}{2} - 1 + (-6), 2 - 3 + 3, -\frac{1}{2} - (-2) + 0\right)$
$= \left(-\frac{13}{2}, 2, \frac{3}{2}\right)$

d. $3\vec{x} + 5\vec{y} + 3\vec{z}$
$= 3(1, 4, -1) + 5(1, 3, -2) + 3(-2, 1, 0)$
$= (3, 12, -3) + (5, 15, -10) + (-6, 3, 0)$
$= (3 + 5 - 6, 12 + 15 + 3, -3 - 10 + 0)$
$= (2, 30, -13)$

6. a. $\vec{p} + \vec{q} = (2\vec{i} - \vec{j} + \vec{k}) + (-\vec{i} - \vec{j} + \vec{k})$
$= (2 - 1)\vec{i} + (-1 - 1)\vec{j} + (1 + 1)\vec{k}$
$= \vec{i} - 2\vec{j} + 2\vec{k}$

b. $\vec{p} - \vec{q} = (2\vec{i} - \vec{j} + \vec{k}) - (-\vec{i} - \vec{j} + \vec{k})$
$= (2 + 1)\vec{i} + (-1 + 1)\vec{j} + (1 - 1)\vec{k}$
$= 3\vec{i} + 0\vec{j} + 0\vec{k}$

c. $2\vec{p} - 5\vec{q} = 2(2\vec{i} - \vec{j} + \vec{k}) - 5(-\vec{i} - \vec{j} + \vec{k})$
$= (4\vec{i} - 2\vec{j} + 2\vec{k}) - (-5\vec{i} - 5\vec{j} + 5\vec{k})$
$= (4 + 5)\vec{i} + (-2 + 5)\vec{j} + (2 - 5)\vec{k}$
$= 9\vec{i} + 3\vec{j} - 3\vec{k}$

d. $-2\vec{p} + 5\vec{q} = -2(2\vec{i} - \vec{j} + \vec{k}) + 5(-\vec{i} - \vec{j} + \vec{k})$
$= (-4\vec{i} + 2\vec{j} - 2\vec{k}) + (-5\vec{i} - 5\vec{j} + 5\vec{k})$
$= (-4 - 5)\vec{i} + (2 - 5)\vec{j} + (-2 + 5)\vec{k}$
$= -9\vec{i} - 3\vec{j} + 3\vec{k}$

7. a. $\vec{m} - \vec{n} = (2\vec{i} - \vec{k}) - (-2\vec{i} + \vec{j} + 2\vec{k})$
$= (2 - (-2))\vec{i} + (-1)\vec{j} + (-1 - 2)\vec{k}$
$= 4\vec{i} - \vec{j} - 3\vec{k}$
$|\vec{m} - \vec{n}| = \sqrt{4^2 + (-1)^2 + (-3)^2} = \sqrt{26} \doteq 5.10$

b. $\vec{m} + \vec{n} = (2\vec{i} - \vec{k}) + (-2\vec{i} + \vec{j} + 2\vec{k})$
$= (2 + (-2))\vec{i} + \vec{j} + (-1 + 2)\vec{k}$
$= 0\vec{i} + \vec{j} + \vec{k}$
$|\vec{m} + \vec{n}| = \sqrt{0^2 + 1^2 + 1^2} = \sqrt{2} \doteq 1.41$

c. $2\vec{m} + 3\vec{n} = 2(2\vec{i} - \vec{k}) + 3(-2\vec{i} + \vec{j} + 2\vec{k})$
$= (4\vec{i} - 2\vec{k}) + (-6\vec{i} + 3\vec{j} + 6\vec{k})$
$= (4 + (-6))\vec{i} + 3\vec{j} + (-2 + 6)\vec{k}$
$= -2\vec{i} + 3\vec{j} + 4\vec{k}$
$|2\vec{m} + 3\vec{n}| = \sqrt{(-2)^2 + 3^2 + 4^2} = \sqrt{29} \doteq 5.39$

d. $-5\vec{m} = -5(2\vec{i} - \vec{k}) = -10\vec{i} + 5\vec{k}$
$|-5\vec{m}| = \sqrt{(-10)^2 + (5)^2} = \sqrt{125} \doteq 11.18$

8. $\vec{x} + \vec{y} = -\vec{i} + 2\vec{j} + 5\vec{k}$
$+\vec{x} - \vec{y} = 3\vec{i} + 6\vec{j} - 7\vec{k}$
$2\vec{x} = 2\vec{i} + 8\vec{j} - 2\vec{k}$
Divide by 2 on both sides to get:
$\vec{x} = \vec{i} + 4\vec{j} - \vec{k}$
Plug this equation into the first given equation:
$\vec{i} + 4\vec{j} - \vec{k} + \vec{y} = -\vec{i} + 2\vec{j} + 5\vec{k}$
$\vec{y} = -\vec{i} + 2\vec{j} + 5\vec{k} - (\vec{i} + 4\vec{j} - \vec{k})$
$\vec{y} = (-1 - 1)\vec{i} + (2 - 4)\vec{j} + (5 + 1)\vec{k}$
$\vec{y} = -2\vec{i} - 2\vec{j} + 6\vec{k}$

9. a. The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OC}$ represent the xy-plane, xz-plane, and yz-plane, respectively. They are also the vector from the origin to points $(a, b, 0)$, $(a, 0, c)$, and $(0, b, c)$, respectively.

b. $\overrightarrow{OA} = a\vec{i} + b\vec{j} + 0\vec{k}$
$\overrightarrow{OB} = a\vec{i} + 0\vec{j} + c\vec{k}$
$\overrightarrow{OC} = 0\vec{i} + b\vec{j} + c\vec{k}$

c. $|\overrightarrow{OA}| = \sqrt{a^2 + b^2}$
$|\overrightarrow{OB}| = \sqrt{a^2 + c^2}$
$|\overrightarrow{OB}| = \sqrt{b^2 + c^2}$

d. $\overrightarrow{AB} = (a, 0, c) - (a, b, 0) = (0, -b, c)$
$\overrightarrow{AB}$ is a direction vector from A to B.

10. a. $|\overrightarrow{OA}| = \sqrt{(-2)^2 + (-6)^2 + 3^2} = \sqrt{49} = 7$

b. $|\overrightarrow{OB}| = \sqrt{(3)^2 + (-4)^2 + 12^2} = \sqrt{169} = 13$

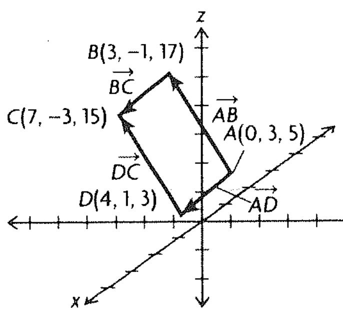
c. $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$
$= (3, -4, 12) - (-2, -6, 3)$
$= (3 - (-2), -4 - (-6), 12 - 3)$
$= (5, 2, 9)$

d. $|\overrightarrow{AB}| = \sqrt{5^2 + 2^2 + 9^2} = \sqrt{110} \doteq 10.49$

e. $\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB}$
$= (-2, -6, 3) - (3, -4, 12)$
$= (-5, -2, -9)$

f. $|\overrightarrow{BA}| = \sqrt{(-5)^2 + (-2)^2 + (-9)^2}$
$= \sqrt{110} \doteq 10.49$

11.



In order to show that $ABCD$ is a parallelogram, we must show that $\overrightarrow{AB} = \overrightarrow{DC}$ or $\overrightarrow{BC} = \overrightarrow{AD}$. This will show they have the same direction, thus the opposite sides are parallel. By showing the vectors are equal they will have the same magnitude, implying the opposite sides having congruency.

$\overrightarrow{AB} = (3, -1, 17) - (0, 3, 5)$
$= (3 - 0, -1 - 3, 17 - 5)$
$= (3, -4, 12)$

$\overrightarrow{DC} = (7, -3, 15) - (4, 1, 3)$
$= (7 - 4, -3 - 1, 15 - 3)$
$= (3, -4, 12)$

Thus $\overrightarrow{AB} = \overrightarrow{DC}$. Do the calculations for the other pair as a check.

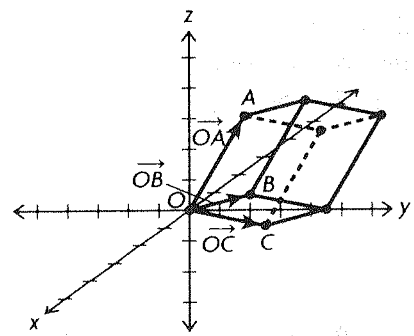
$\overrightarrow{BC} = (7, -3, 15) - (3, -1, 17)$
$= (7 - 3, -3 - (-1), 15 - 17)$
$= (4, -2, -2)$

$\overrightarrow{AD} = (4, 1, 3) - (0, 3, 5)$
$= (4 - 0, 1 - 3, 3 - 5)$
$= (4, -2, -2)$

So $\overrightarrow{BC} = \overrightarrow{AD}$.
We have shown $\overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{BC} = \overrightarrow{AD}$, so $ABCD$ is a parallelogram.

12. $2\vec{x} + \vec{y} - 2\vec{z}$
$= 2(-1, b, c) + (a, -2, c) - 2(-a, 6, c)$
$= (-2, 2b, 2c) + (a, -2, c) - (-2a, 12, 2c)$
$= (-2 + a + 2a, 2b - 2 - 12, 2c + c - 2c)$
$= (-2 + 3a, 2b - 14, c)$
$= (0, 0, 0)$
$-2 + 3a = 0; 2b - 14 = 0; c = 0$
$3a = 2; a = \frac{2}{3}$
$2b = 14; b = 7$
$c = 0$

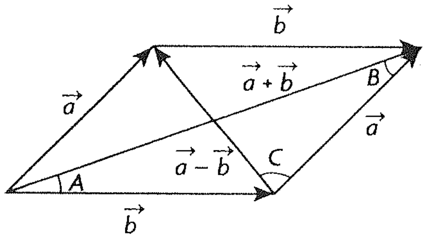
13. a.



b. $V_1 = (0, 0, 0)$, the origin
$V_2 =$ end point of $\overrightarrow{OA} = (-2, 2, 5)$
$V_3 =$ end point of $\overrightarrow{OB} = (0, 4, 1)$
$V_4 =$ end point of $\overrightarrow{OC} = (0, 5, -1)$
$V_5 = \overrightarrow{OA} + \overrightarrow{OB} = (-2, 2, 5) + (0, 4, 1)$
$= (-2 + 0, 2 + 4, 5 + 1)$
$= (-2, 6, 6)$
$V_6 = \overrightarrow{OA} + \overrightarrow{OC} = (-2, 2, 5) + (0, 5, -1)$
$= (-2 + 0, 2 + 5, 5 - 1)$
$= (-2, 7, 4)$
$V_7 = \overrightarrow{OB} + \overrightarrow{OC} = (0, 4, 1) + (0, 5, -1)$
$= (0 + 0, 4 + 5, 1 - 1)$
$= (0, 9, 0)$
$V_8 = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$
$= (-2, 2, 5) + (0, 9, 0)$ (by V_7)
$= (-2 + 0, 2 + 9, 5 + 0)$
$= (-2, 11, 5)$

14. Any point on the x-axis has y-coordinate 0 and z-coordinate 0. The z-coordinate of each of A and B is 3, so the z-component of the distance from the desired point is the same for each of A and B. The y-component of the distance from the desired point will be 1 for each of A and B, $12 = (-1)^2$. So, the x-coordinate of the desired point has to be halfway between the x-coordinates of A and B. The desired point is $(1, 0, 0)$.

15.



To solve this problem, we must first consider the triangle formed by $\vec{a}$, $\vec{b}$, and $\vec{a}+\vec{b}$. We will use their magnitudes to solve for angle A, which will be used to solve for $\frac{1}{2}\vec{a}-\vec{b}$ in the triangle formed by $\vec{b}$, $\frac{1}{2}\vec{a}+\vec{b}$, and $\frac{1}{2}\vec{a}-\vec{b}$.

Using the cosine law, we see that:

$$\cos(A) = \frac{|\vec{b}|^2 + |\vec{a}+\vec{b}|^2 - |\vec{a}|^2}{2|\vec{b}||\vec{a}+\vec{b}|}$$

$$= \frac{25+49-9}{70}$$

$$= \frac{13}{14}$$

Now, consider the triangle formed by $\vec{b}$, $\frac{1}{2}\vec{a}+\vec{b}$, and $\frac{1}{2}\vec{a}-\vec{b}$. Using the cosine law again:

$$\cos(A) = \frac{|\vec{b}|^2 + \left(\frac{1}{2}|\vec{a}+\vec{b}|\right)^2 - \left(\frac{1}{2}|\vec{a}-\vec{b}|\right)^2}{|\vec{b}||\vec{a}+\vec{b}|}$$

$$\frac{13}{14} = \frac{\frac{149}{4} - \left(\frac{1}{2}|\vec{a}-\vec{b}|\right)^2}{35}$$

$$|\vec{a}-\vec{b}|^2 = -4\left(\frac{65}{2} - \frac{149}{4}\right)$$

$$|\vec{a}-\vec{b}|^2 = 19$$

$$|\vec{a}-\vec{b}| = \sqrt{19} \text{ or } 4.36$$

6.8 Linear Combinations and Spanning Sets, pp. 340–341

1. They are collinear, thus a linear combination is not applicable.

2. It is not possible to use $\vec{0}$ in a spanning set. Therefore, the remaining vectors only span R^2.

3. The set of vectors spanned by $(0, 1)$ is $m(0, 1)$. If we let $m = -1$, then $m(0, 1) = (0, -1)$.

4. $\vec{i}$ spans the set $m(1, 0, 0)$. This is any vector along the x-axis. Examples: $(2, 0, 0)$, $(-21, 0, 0)$

5. As in question 2, it is not possible to use $\vec{0}$ in a spanning set.

6. $\{(-1, 2), (-1, 1)\}$, $\{(2, -4), (-1, 1)\}$, $\{(-1, 1), (-3, 6)\}$ are all the possible spanning sets for R^2 with 2 vectors.

7. a. $2(2\vec{a} - 3\vec{b} + \vec{c}) = 4\vec{a} - 6\vec{b} + 2\vec{c}$

$= 4\vec{i} - 8\vec{j} - 6\vec{j} + 18\vec{k} + 2\vec{i} - 6\vec{j} + 4k$

$= 6\vec{i} - 20\vec{j} + 22\vec{k}$

$4(-\vec{a} + \vec{b} - \vec{c}) = -4\vec{a} + 4\vec{b} - 4\vec{c}$

$= -4\vec{i} + 8\vec{j} + 4\vec{j} - 12\vec{k} - 4\vec{i} + 12\vec{j} - 8k$

$= -8\vec{i} + 24\vec{j} - 20\vec{k}$

$(\vec{a} - \vec{c}) = \vec{i} - 2\vec{j} - \vec{i} + 3\vec{j} - 2\vec{k}$

$= \vec{j} - 2\vec{k}$

$2(2\vec{a} - 3\vec{b} + \vec{c}) - 4(-\vec{a} + \vec{b} - \vec{c}) + (\vec{a} - \vec{c})$

$= 6\vec{i} - 20\vec{j} + 22\vec{k} + 8\vec{i} - 24\vec{j} + 20\vec{k} + \vec{j} - 2\vec{k}$

$= 14\vec{i} - 43\vec{j} + 40\vec{k}$

b. $\frac{1}{2}(2\vec{a} - 4\vec{b} - 8\vec{c}) = \vec{a} - 2\vec{b} - 4\vec{c}$

$= \vec{i} - 2\vec{j} - 2\vec{j} + 6\vec{k} - 4\vec{i} + 12\vec{j} - 8\vec{k}$

$= -3\vec{i} + 8\vec{j} - 2\vec{k}$

$\frac{1}{3}(3\vec{a} - 6\vec{b} + 9\vec{c}) = \vec{a} - 2\vec{b} + 3\vec{c}$

$= \vec{i} - 2\vec{j} - 2\vec{j} + 6\vec{k} + 3\vec{i} - 9\vec{j} + 6\vec{k}$

$= 4\vec{i} - 15\vec{j} + 12\vec{k}$

$\frac{1}{2}(2\vec{a} - 4\vec{b} - 8\vec{c}) - \frac{1}{3}(3\vec{a} - 6\vec{b} + 9\vec{c})$

$= -3\vec{i} + 8\vec{j} - 2\vec{k} - 4\vec{i} + 15\vec{j} - 12\vec{k}$

$= -7\vec{i} + 23\vec{j} - 14\vec{k}$

8. $\{(1, 0, 0), (0, 1, 0)\}$:

$(-1, 2, 0) = -1(1, 0, 0) + 2(0, 1, 0)$

$(3, 4, 0) = 3(1, 0, 0) + 4(0, 1, 0)$

$\{(1, 1, 0), (0, 1, 0)\}$

$(-1, 2, 0) = -1(1, 1, 0) + 3(0, 1, 0)$

$(3, 4, 0) = 3(1, 1, 0) + (0, 1, 0)$

9. a. It is the set of vectors in the xy-plane.

b. $(-2, 4, 0) = -2(1, 0, 0) + 4(0, 1, 0)$

c. By part a. the vector is not in the xy-plane. There is no combination that would produce a number other than 0 for the z-component.

d. It would still only span the xy-plane. There would be no need for that vector.

10. Looking at the x-component:

$2a + 3c = 5$

The y-component:

$6 + 21 = b + c$

The z-component:

$2c + 3c = 15$

$5c = 15$

$c = 3$

Substituting this into the first and second equation:
$2a + 9 = 5$
$a = -2$
$27 = b + 3$
$b = 24$
11. $(-10, -34) = a(-1, 3) + b(1, 5)$
Looking at the x-component:
$-10 = -a + b \quad a = 10 + b$
Looking at the y-component:
$-34 = 3a + 5b$
Substituting in a:
$-34 = 30 + 3b + 5b$
$b = -8$
Substituting b into x-component equation:
$-10 = -a + (-8)$
$a = -2$
$(-10, -34) = -2(-1, 3) - 8(1, 5)$
12. a. $a(2, -1) + b(-1, 1) = (x, y)$
$x = 2a - b$
$b = 2a - x$
$y = -a + b$
Substitute in b:
$y = -a + 2a - x$
$a = x + y$
Substitute this back into the first equation:
$b = 2x + 2y - x$
$b = x + 2y$
b. Using the formulas in part a:
For $(2, -3)$:
$a = x + y = 2 - 3 = -1$
$b = x + 2y = 2 - 6 = -4$
$(2, -3) = -1(2, -1) - 4(-1, 1)$
For $(124, -5)$:
$a = 124 - 5 = 119$
$b = 124 - 10 = 114$
$(124, -5) = 119(2, -1) + 114(-1, 1)$
For $(4, -11)$
$a = 4 - 11 = -7$
$b = 4 - 22 = -18$
$(4, -11) = -7(2, -1) - 18(-1, 1)$
13. Try: $a(-1, 2, 3) + b(4, 1, -2)$
$= (-14, -1, 16)$
x components:
$-a + 4b = -14$
$a = 14 + 4b$
y components:
$2a + b = -1$
Substitute in a:
$28 + 8b + b = -1$
$b = -3$
Substitute this result into the x-components:
$a = 14 - 3 = 11$
Check by substituting into z-components:
$3a - 2b = 16$
$33 + 5 = 16$
Therefore:
$a(-1, 2, 3) + b(4, 1, -2) \neq (-14, -1, 16)$ for any a and b. They do not lie on the same plane.
b. $a(-1, 3, 4) + b(0, -1, 1) = (-3, 14, 7)$
x components:
$-a = -3$
$a = 3$
y components:
$3a - b = 14$
Substitute in a:
$9 - b = 14$
$b = -5$
Check with z components:
$4a + b = 7$
$12 - 5 = 7$
Since there exists an a and b to form a linear combination of 2 of the vectors to form the third, they lie on the same plane.
$3(-1, 3, 4) - 5(0, -1, 1) = (-3, 14, 7)$
14. Let vector $\vec{a} = (-1, 3, 4)$ and $\vec{b} = (-2, 3, -1)$ (vectors from the origin to points A and B, respectively). To determine x, we let $\vec{c}$ (vector from origin to C) be a linear combination of $\vec{a}$ and $\vec{b}$.
$a(-1, 3, 4) + b(-2, 3, -1) = (-5, 6, x)$
x components:
$-a - 2b = -5$
$a = 5 - 2b$
y components:
$3a + 3b = 6$
Substitute in a:
$15 - 6b + 3b = 6$
$b = 3$
Substitute in b in x component equation:
$a = 5 - 6 = -1$
z components:
$4a - b = x$
Substitute in a and b:
$x = -4 - 3 = -7$
15. $m = 2, n = 3$. Non-parallel vectors cannot be equal, unless their magnitudes equal 0.
16. Answers may vary. For example:
Try linear combinations of the 2 vectors such that

the z component equals 5. Then calculate what p and q would equal.

$-1(4, 1, 7) + 2(-1, 1, 6) = (-6, 1, 5)$

So $p = -6$ and $q = 1$

$5(4, 1, 5) - 5(-1, 1, 6) = (25, 0, 5)$

So $p = 25$ and $q = 0$

$(4, 1, 7) - \frac{1}{3}(-1, 1, 6) = \left(\frac{13}{3}, \frac{2}{3}, 5\right)$

So $p = \frac{13}{3}$ and $q = \frac{2}{3}$

17. As in question 15, non-parallel vectors. Their magnitudes must be 0 again to make the equality true.

$m^2 + 2m - 3 = (m - 1)(m + 3)$

$m = 1, -3$

$m^2 + m - 6 = (m - 2)(m + 3)$

$m = 2, -3$

So, when $m = -3$, their sum will be 0.

Review Exercise, pp. 344–347

1. a. false; Let $\vec{b} = -\vec{a} \neq 0$ then:

$|\vec{a} + \vec{b}| = |\vec{a} + (-\vec{a})|$
$= |0|$
$= 0 < |\vec{a}|$

b. true; $|\vec{a} + \vec{b}|$ and $|\vec{a} + \vec{c}|$ both represent the lengths of the diagonal of a parallelogram, the first with sides $\vec{a}$ and $\vec{b}$ and the second with sides $\vec{a}$ and $\vec{c}$; since both parallelograms have $\vec{a}$ as a side and diagonals of equal length $|\vec{b}| = |\vec{c}|$.

c. true; Subtracting $\vec{a}$ from both sides shows that $\vec{b} = \vec{c}$

d. true; Draw the parallelogram formed by $\overrightarrow{RF}$ and $\overrightarrow{SW}$. $\overrightarrow{FW}$ and $\overrightarrow{RS}$ are the opposite sides of a parallelogram and must be equal.

e. true; The distributive law for scalars

f. false; Let $\vec{b} = -\vec{a}$ and let $\vec{c} = \vec{d} \neq 0$. Then,

$|\vec{a}| = |-\vec{a}| = |\vec{b}|$ and $|\vec{c}| = |\vec{d}|$

but $|\vec{a} + \vec{b}| = |\vec{a} + (-\vec{a})| = 0$

$|\vec{c} + \vec{d}| = |\vec{c} + \vec{c}| = |2\vec{c}|$

so $|\vec{a} + \vec{b}| \neq |\vec{c} + \vec{d}|$

2. a. Substitute the given values of $\vec{x}$, $\vec{y}$, and $\vec{z}$ into the expression $2\vec{x} - 3\vec{y} + 5\vec{z}$

$2\vec{x} - 3\vec{y} + 5\vec{z}$

$= 2(2\vec{a} - 3\vec{b} - 4\vec{c}) - 3(-2\vec{a} + 3\vec{b} + 3\vec{c})$
$+ 5(2\vec{a} - 3\vec{b} + 5\vec{c})$

$= 4\vec{a} - 6\vec{b} - 8\vec{c} + 6\vec{a} - 9\vec{b} - 9\vec{c} + 10\vec{a} - 15\vec{b}$
$+ 25\vec{c}$

$= 4\vec{a} + 6\vec{a} + 10\vec{a} - 6\vec{b} - 9\vec{b} - 15\vec{b} - 8\vec{c} - 9\vec{c}$
$+ 25\vec{c}$

$= 20\vec{a} - 30\vec{b} + 8\vec{c}$

b. Simplify the expression before substituting the given values of $\vec{x}$, $\vec{y}$, and $\vec{z}$

$3(-2\vec{x} - 4\vec{y} + \vec{z}) - (2\vec{x} - \vec{y} + \vec{z})$
$- 2(-4\vec{x} - 5\vec{y} + \vec{z})$

$= -6\vec{x} - 12\vec{y} + 3\vec{z} - 2\vec{x} + \vec{y} - \vec{z} + 8\vec{x}$
$+ 10\vec{y} - 2\vec{z}$

$= -6\vec{x} - 2\vec{x} + 8\vec{x} - 12\vec{y} + \vec{y} + 10\vec{y} + 3\vec{z}$
$- \vec{z} - 2\vec{z}$

$= 0\vec{x} - \vec{y} + 0\vec{z}$

$= -\vec{y}$

$= 2\vec{a} - 3\vec{b} - 3\vec{c}$

3. a. $\overrightarrow{XY} = \overrightarrow{OY} - \overrightarrow{OX}$
$= (x_2, y_2, z_2) - (x_1, y_1, z_1)$
$= (x_2 - x_1, y_2 - y_1, z_2 - z_1)$
$= (-4 - (-2), 4 - 1, 8 - 2)$
$= (-2, 3, 6)$

$|\overrightarrow{XY}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
$= \sqrt{(-2)^2 + (3)^2 + (6)^2}$
$= \sqrt{4 + 9 + 36}$
$= \sqrt{49}$
$= 7$

b. The components of a unit vector in the same direction as $\overrightarrow{XY}$ are $\frac{1}{7}(-2, 3, 6) = \left(-\frac{2}{7}, \frac{3}{7}, \frac{6}{7}\right)$.

4. a. The position vector $\overrightarrow{OP}$ is equivalent to $\overrightarrow{YX}$.

$\overrightarrow{OP} = \overrightarrow{YX}$
$= (x_2, y_2, z_2) - (x_1, y_1, z_1)$
$= (x_2 - x_1, y_2 - y_1, z_2 - z_1)$
$= (-1 - 5, 2 - 5, 6 - 12)$
$= (-6, -3, -6)$

b. $|\overrightarrow{YX}| = \sqrt{(-6)^2 + (-3)^2 + (-6)^2}$
$= \sqrt{81}$
$= 9$

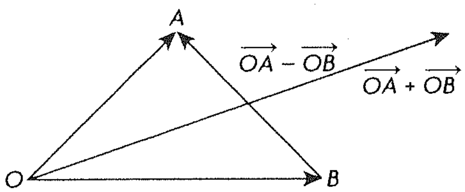
The components of a unit vector in the same direction as $\overrightarrow{YX}$ are $\frac{1}{9}(-6, -3, -6) = \left(-\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\right)$

5. $-\overrightarrow{MN} = \overrightarrow{NM}$
$= (x_2, y_2, z_2) - (x_1, y_1, z_1)$
$= (x_2 - x_1, y_2 - y_1, z_2 - z_1)$
$= (2 - 8, 3 - 1, 5 - 2)$
$= (-6, 2, 3)$

$$|\overrightarrow{NM}| = \sqrt{(-6)^2 + (2)^2 + (3)^2}$$
$$= \sqrt{49}$$
$$= 7$$

The components of the unit vector with the opposite direction to $\overrightarrow{MN}$ are $\frac{1}{7}(-6, 2, 3) = \left(-\frac{6}{7}, \frac{2}{7}, \frac{3}{7}\right)$

6. a. The two diagonals can be found by calculating $\overrightarrow{OA} + \overrightarrow{OB}$ and $\overrightarrow{OA} - \overrightarrow{OB}$.



$$\overrightarrow{OA} + \overrightarrow{OB} = (3, 2, -6) + (-6, 6, -2)$$
$$= (3 + -6, 2 + 6, -6 + -2)$$
$$= (-3, 8, -8)$$
$$\overrightarrow{OA} - \overrightarrow{OB} = (3, 2, -6) + (-6, 6, -2)$$
$$= (3 - (-6), 2 - 6, -6 - (-2))$$
$$= (9, -4, -4)$$

b. To determine the angle between the sides of the parallelogram, calculate $|\overrightarrow{OA}|$, $|\overrightarrow{OB}|$, and $|\overrightarrow{OA} - \overrightarrow{OB}|$ and apply the cosine law.

$$|\overrightarrow{OA}| = \sqrt{(3)^2 + (2)^2 + (-6)^2}$$
$$= \sqrt{49}$$
$$= 7$$
$$|\overrightarrow{OB}| = \sqrt{(-6)^2 + (6)^2 + (-2)^2}$$
$$= \sqrt{76}$$
$$= 2\sqrt{19}$$
$$|\overrightarrow{OA} - \overrightarrow{OB}| = \sqrt{(9)^2 + (-4)^2 + (-4)^2}$$
$$= \sqrt{113}$$
$$\cos\theta = \frac{|\overrightarrow{OA}|^2 + |\overrightarrow{OB}|^2 - |\overrightarrow{OA} - \overrightarrow{OB}|^2}{2|\overrightarrow{OA}||\overrightarrow{OB}|}$$
$$\cos\theta = \frac{(7)^2 + (2\sqrt{19})^2 - (\sqrt{113})^2}{2(7)(2\sqrt{19})}$$
$$\cos\theta \doteq 0.098$$
$$\theta \doteq 84.4°$$

7. a. $|\overrightarrow{AB}| = \sqrt{(2 - (-1))^2 + (0 - 1)^2 + (3 - 1)^2}$
$$= \sqrt{(3)^2 + (-1)^2 + (2)^2}$$
$$= \sqrt{9 + 1 + 4}$$
$$= \sqrt{14}$$

$$|\overrightarrow{BC}| = \sqrt{(3 - 2)^2 + (3 - 0)^2 + (-4 - 3)^2}$$
$$= \sqrt{(1)^2 + (3)^2 + (-7)^2}$$
$$= \sqrt{1 + 9 + 49}$$
$$= \sqrt{59}$$
$$|\overrightarrow{CA}| = \sqrt{(-1 - 3)^2 + (1 - 3)^2 + (1 - (-4))^2}$$
$$= \sqrt{(-4)^2 + (-2)^2 + (5)^2}$$
$$= \sqrt{16 + 4 + 25}$$
$$= \sqrt{45}$$

Triangle ABC is a right triangle if and only if $|\overrightarrow{AB}|^2 + |\overrightarrow{CA}|^2 = |\overrightarrow{BC}|^2$.

$$|\overrightarrow{AB}|^2 + |\overrightarrow{CA}|^2 = (\sqrt{14})^2 + (\sqrt{45})^2$$
$$= 14 + 45$$
$$= 59$$
$$|\overrightarrow{BC}|^2 = (\sqrt{59})^2$$
$$= 59$$

So triangle ABC is a right triangle.

b. Area of a triangle $= \frac{1}{2}bh$. For triangle ABC the longest side $\overrightarrow{BC}$ is the hypotenuse, so $\overrightarrow{AB}$ and $\overrightarrow{CA}$ are the base and height of the triangle.

$$\text{Area} = \frac{1}{2}(|\overrightarrow{AB}|)(|\overrightarrow{CA}|)$$
$$= \frac{1}{2}\sqrt{14}\sqrt{45}$$
$$= \frac{1}{2}\sqrt{630}$$
$$= \frac{3}{2}\sqrt{70} \text{ or } 12.5$$

c. Perimeter of a triangle equals the sum of the sides.

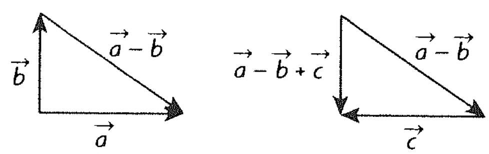
$$\text{Perimeter} = |\overrightarrow{AB}| + |\overrightarrow{BC}| + |\overrightarrow{CA}|$$
$$= \sqrt{14} + \sqrt{59} + \sqrt{45}$$
$$\doteq 18.13$$

d. The fourth vertex D is the head of the diagonal vector from A. To find $\overrightarrow{AD}$ take $\overrightarrow{AB} + \overrightarrow{AC}$.

$$\overrightarrow{AB} = (2 - (-1), 0 - 1, 3 - 1) = (3, -1, 2)$$
$$\overrightarrow{AC} = (3 - (-1), 3 - 1, -4 - 1) = (4, 2, -5)$$
$$\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{AC}$$
$$= (3 + 4, -1 + 2, 2 + (-5))$$
$$= (7, 1, -3)$$

So the fourth vertex is $D(-1 + 7, 1 + 1, 1 + (-3))$ or $D(6, 2, -2)$.

8. a.



b. Since the vectors $\vec{a}$ and $\vec{b}$ are perpendicular, $|\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2$. So,

$$|\vec{a}+\vec{b}|^2 = (4)^2 + (3)^2 = 16 + 9 = 25$$

$$|\vec{a}+\vec{b}| = \sqrt{25} = 5$$

9. Express $\vec{r}$ as a linear combination of $\vec{p}$ and $\vec{q}$:
Solve for a and b:

$$\vec{r} = a\vec{p} + b\vec{q}$$
$$(-1,2) = a(-11,7) + b(-3,1)$$
$$(-1,2) = (-11a, 7a) + (-3b, b)$$
$$(-1,2) = (-11a - 3b, 7a + b)$$

Solve the system of equations:

$$-1 = -11a - 3b$$
$$2 = 7a + b$$

Use the method of elimination:

$$3(2) = 3(7a + b)$$
$$6 = 21a + 3b$$
$$+\ -1 = -11a - 3b$$
$$5 = 10a$$
$$\frac{1}{2} = a$$

By substitution, $b = -\frac{3}{2}$

Therefore $\frac{1}{2}(-11,7) + -\frac{3}{2}(-3,1) = (-1,2)$

Express $\vec{q}$ as a linear combination of $\vec{p}$ and $\vec{r}$.
Solve for a and b:

$$\vec{q} = a\vec{p} + b\vec{r}$$
$$(-3,1) = a(-11,7) + b(-1,2)$$
$$(-3,1) = (-11a, 7a) + (-b, 2b)$$
$$(-3,1) = (-11a - b, 7a + 2b)$$

Solve the system of equations:

$$-3 = -11a - b$$
$$1 = 7a + 2b$$

Use the method of elimination:

$$2(-3) = 2(-11a - b)$$
$$-6 = -22a - 2b$$
$$+\ 1 = 7a + 2b$$
$$-5 = -15a$$
$$\frac{1}{3} = a$$

By substitution, $-\frac{2}{3} = b$
Therefore $\frac{1}{3}(-11,7) + -\frac{2}{3}(-1,2) = (-3,1)$
Express $\vec{p}$ as a linear combination of $\vec{q}$ and $\vec{r}$.
Solve for a and b:

$$\vec{p} = a\vec{q} + b\vec{r}$$
$$(-11,7) = a(-3,1) + b(-1,2)$$
$$(-11,7) = (-3a, a) + (-b, 2b)$$
$$(-11,7) = (-3a - b, a + 2b)$$

Solve the system of equations:

$$-11 = -3a - b$$
$$7 = a + 2b$$

Use the method of elimination:

$$2(-11) = 2(-3a - b)$$
$$-22 = -6a - 2b$$
$$+\ 7 = a + 2b$$
$$-15 = -5a$$
$$3 = a$$

By substitution, $2 = b$
Therefore $3(-3,1) + 2(-1,2) = (-11,7)$

10. a. Let $P(x, y, z)$ be a point equidistant from A and B. Then $|\vec{PA}| = |\vec{PB}|$.

$$(x-2)^2 + (y-(-1))^2 + (z-3)^2 = (x-1)^2 + (y-2)^2 + (z-(-3))^2$$
$$x^2 - 4x + 4 + y^2 + 2y + 1 + z^2 - 6z + 9 = x^2 - 2x + 1 + y^2 - 4y + 4 + z^2 + 6z + 9$$
$$-2x + 6y - 12z = 0$$
$$x - 3y + 6z = 0$$

b. $(0, 0, 0)$ and $\left(1, \frac{1}{3}, 0\right)$ clearly satisfy the equation and are equidistant from A and B.

11. a.

$$(-24, 3, 25) = 2(a, b, 4) + \frac{1}{2}(6, 8, c) - 3(7, c, -4)$$
$$(-24, 3, 25) = (2a, 2b, 8) + \left(3, 4, \frac{c}{2}\right) - (21, 3c, -12)$$
$$(-24, 3, 25) = \left(2a - 18, 2b + 4 - 3c, \frac{c}{2} + 20\right)$$

Solve the equations:

i. $-24 = 2a - 18$
$-6 = 2a$
$-3 = a$

ii. $25 = \frac{c}{2} + 20$
$5 = \frac{c}{2}$
$10 = c$

iii. $3 = 2b + 4 - 3c$
$3 = 2b + 4 - 3(18)$
$3 = 2b - 50$
$53 = 2b$
$26.5 = b$

b. $(3, -22, 54)$

$= 2\left(a, a, \frac{1}{2}a\right) + (3b, 0, -5c) + 2\left(c, \frac{3}{2}c, 0\right)$

$(3, -22, 54)$
$= (2a, 2a, a) + (3b, 0, -5c) + (2c, 3c, 0)$

$(3, -22, 54) = (2a + 3b + 2c, 2a + 3c, a - 5c)$

Solve the system of equations:

$-22 = 2a + 3c$

$54 = a - 5c$

Use the method of elimination:

$-2(54) = -2(a - 5c)$

$-108 = -2a + 10c$

$+ \; -22 = \;\; 2a + \;\; 3c$

$-130 = 13c$

$-10 = c$

By substitution, $8 = a$

Solve the equation:

$3 = 2a + 3b + 2c$

$3 = 2(8) + 3b + 2(-10)$

$3 = 16 + 3b - 20$

$3 = 3b - 4$

$7 = 3b$

$\frac{7}{3} = b$

12. a. Find $|\vec{AB}|, |\vec{BC}|, |\vec{CA}|$

$|\vec{AB}| = \sqrt{(2 - 1)^2 + (2 - (-1))^2 + (2 - 1)^2}$
$= \sqrt{(1)^2 + (3)^2 + (1)^2}$
$= \sqrt{11}$

$|\vec{BC}| = \sqrt{(4 - 2)^2 + (-2 - 2)^2 + (1 - 2)^2}$
$= \sqrt{(2)^2 + (-4)^2 + (-1)^2}$
$= \sqrt{21}$

$|\vec{CA}| = \sqrt{(4 - 1)^2 + (-2 - (-1))^2 + (1 - 1)^2}$
$= \sqrt{(3)^2 + (-1)^2}$
$= \sqrt{10}$

Test $|\vec{AB}|, |\vec{BC}|, |\vec{CA}|$ in the Pythagorean theorem:

$|\vec{AB}|^2 + |\vec{CA}|^2 = (\sqrt{11})^2 + (\sqrt{10})^2$
$= 11 + 10$
$= 21$

$|\vec{BC}|^2 = (\sqrt{21})^2$
$= 21$

So triangle ABC is a right triangle.

b. Yes, $P(1, 2, 3)$, $Q(2, 4, 6)$, and $R(-1, -2, -3)$ are collinear because:

$2P = (2, 4, 6)$

$1Q = (2, 4, 6)$

$-2R = (2, 4, 6)$

13. a. Find $|\vec{AB}|, |\vec{BC}|, |\vec{CA}|$

$|\vec{AB}| = \sqrt{(1 - 3)^2 + (2 - 0)^2 + (5 - 4)^2}$
$= \sqrt{(-2)^2 + (2)^2 + (1)^2}$
$= \sqrt{9}$
$= 3$

$|\vec{BC}| = \sqrt{(2 - 1)^2 + (1 - 2)^2 + (3 - 5)^2}$
$= \sqrt{(1)^2 + (-1)^2 + (-2)^2}$
$= \sqrt{6}$

$|\vec{CA}| = \sqrt{(2 - 3)^2 + (1 - 0)^2 + (3 - 4)^2}$
$= \sqrt{(-1)^2 + (1)^2 + (-1)^2}$
$= \sqrt{3}$

Test $|\vec{AB}|, |\vec{BC}|, |\vec{CA}|$ in the Pythagorean theorem:

$|\vec{BC}|^2 + |\vec{CA}|^2 = (\sqrt{6})^2 + (\sqrt{3})^2$
$= 6 + 3$
$= 9$

$|\vec{AB}|^2 = (3)^2$
$= 9$

So triangle ABC is a right triangle.

b. Since triangle ABC is a right triangle,

$\cos\angle ABC = \sqrt{\frac{6}{3}}$

14. a. $\vec{DA}, \vec{BC}$ and $\vec{EB}, \vec{ED}$

b. $\vec{DC}, \vec{AB}$ and $\vec{CE}, \vec{EA}$

c. $|\vec{AD}|^2 + |\vec{DC}|^2 = |\vec{AC}|^2$.

But $|\vec{AC}|^2 = |\vec{DB}|^2$

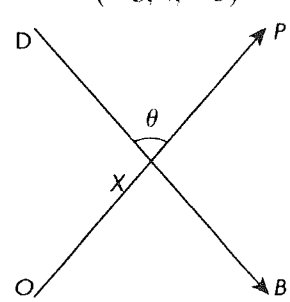
Therefore, $|\vec{AD}|^2 + |\vec{DC}|^2 = |\vec{DB}|^2$

15. a. $C(3, 0, 5)$; $P(3, 4, 5)$; $E(0, 4, 5)$; $F(0, 4, 0)$

b. $\vec{DB} = (3 - 0, 4 - 0, 0 - 5)$
$= (3, 4, -5)$

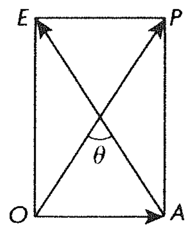
$\vec{CF} = (0 - 3, 4 - 0, 0 - 5)$
$= (-3, 4, -5)$

c.



$|\overrightarrow{OD}| = 5$
$|\overrightarrow{DP}| = 5$ by the Pythagorean theorem
Thus $ODPB$ is a square and $\cos\theta = 0$, so the angle between the vectors is 90°.

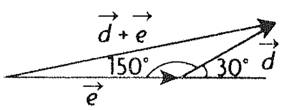
d.



$|\overrightarrow{OA}| = 3, |\overrightarrow{OP}| = \sqrt{5}$

$$\theta = 180^\circ - 2(m\angle POA)$$
$$= 180^\circ - 2\left(\cos^{-1}\left(\frac{3}{\sqrt{50}}\right)\right)$$
$$\doteq 50.2^\circ$$

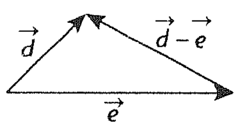
16. a.



Use the cosine law to evaluate $|\vec{d} + \vec{e}|$

$$|\vec{d} + \vec{e}|^2 = |\vec{d}|^2 + |\vec{e}|^2 - 2|\vec{d}||\vec{e}|\cos\theta$$
$$= (3)^2 + (5)^3 - 2(3)(5)\cos 150^\circ$$
$$= 9 + 25 - 30\frac{-\sqrt{3}}{2}$$
$$\doteq 59.98$$
$$|\vec{d} + \vec{e}| \doteq \sqrt{59.98}$$
$$\doteq 7.74$$

b.

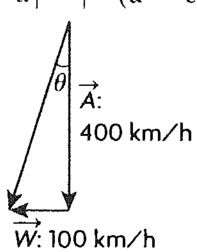


Use the cosine law to evaluate $|\vec{d} - \vec{e}|$

$$|\vec{d} - \vec{e}|^2 = |\vec{d}|^2 + |\vec{e}|^2 - 2|\vec{d}||\vec{e}|\cos\theta$$
$$= (3)^2 + (5)^3 - 2(3)(5)\cos 30^\circ$$
$$= 9 + 25 - 30\frac{\sqrt{3}}{2}$$
$$\doteq 8.02$$
$$|\vec{d} - \vec{e}| \doteq \sqrt{8.02}$$
$$\doteq 2.83$$

c. $|\vec{e} - \vec{d}| = |-(\vec{d} - \vec{e})| = |\vec{d} - \vec{e}| \doteq 2.83$

17. a.



Let $\vec{A}$ represent the air speed of the airplane and let $\vec{W}$ represent the velocity of the wind. In one hour, the plane will travel $|\vec{A} + \vec{W}|$ kilometers. Because $\vec{A}$ and $\vec{W}$ make a right angle, use the Pythagorean theorem:

$$|\vec{A} + \vec{W}|^2 = |\vec{A}|^2 + |\vec{W}|^2$$
$$= (400)^2 + (100)^2$$
$$= 170000$$
$$|\vec{A} + \vec{W}| = \sqrt{170000}$$
$$\doteq 412.3 \text{ km}$$

So in 3 hours, the plane will travel
3(412.3)km $\doteq$ 1236.9 km

b. $$\tan\theta = \frac{|\vec{W}|}{|\vec{A}|}$$
$$= \frac{100}{400}$$
$$\theta = \tan^{-1}\left(\frac{1}{4}\right)$$
$$\doteq 14.0^\circ$$

The direction of the airplane is S14.0°W.

18. a. Any pair of nonzero, noncollinear vectors will span R^2. To show that (2, 3) and (3, 5) are noncollinear, show that there does not exist any number k such that $k(2, 3) = (3, 5)$. Solve the system of equations:

$2k = 3$
$3k = 5$

Solving both equations gives two different values for k, $\frac{3}{2}$ and $\frac{5}{3}$, so (2, 3) and (3, 5) are noncollinear and thus span R^2

b. $(323, 795) = m(2, 3) + n(3, 5)$
$(323, 795) = (2m, 3m) + (3n, 5n)$
$(323, 795) = (2m + 3n, 3m + 5n)$

Solve the system of equations:

$323 = 2m + 3n$
$795 = 3m + 5n$

Use the method of elimination:

$$-3(323) = -3(2m + 3n)$$
$$2(795) = 2(3m + 5n)$$
$$-969 = -6m - 9n$$
$$+ \quad 1590 = 6m + 10n$$
$$621 = n$$

By substitution, $m = -770$.

19. a. Find a and b such that

$(5, 9, 14) = a(-2, 3, 1) + b(3, 1, 4)$

$(5, 9, 14) = (-2a, 3a, a) + (3b, b, 4b)$

$(5, 9, 14) = (-2a + 3b, 3a + b, a + 4b)$

i. $5 = -2a + 3b$

ii. $9 = 3a + b$

iii. $14 = a + 4b$

Use the method of elimination with **i.** and **iii.**

$$\begin{aligned} 2(14) &= 2(a + 4b) \\ 28 &= 2a + 8b \\ +\quad 5 &= -2a + 3b \\ \hline 33 &= 11b \\ 3 &= b \end{aligned}$$

By substitution, $a = 2$.

$\vec{a}$ lies in the plane determined by $\vec{b}$ and $\vec{c}$ because it can be written as a linear combination of $\vec{b}$ and $\vec{c}$.

b. If vector $\vec{a}$ is in the span of $\vec{b}$ and $\vec{c}$, then $\vec{a}$ can be written as a linear combination of $\vec{b}$ and $\vec{c}$. Find m and n such that

$$\begin{aligned} (-13, 36, 23) &= m(-2, 3, 1) + n(3, 1, 4) \\ &= (-2m, 3m, m) + (3n, n, 4n) \\ &= (-2m + 3n, 3m + n, m + 4n) \end{aligned}$$

Solve the system of equations:

$-13 = -2m + 3n$

$36 = 3m + n$

$23 = m + 4n$

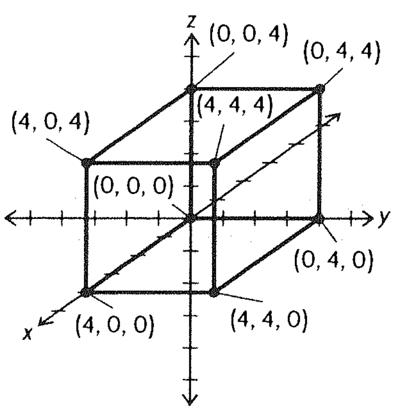
Use the method of elimination:

$$\begin{aligned} 2(23) &= 2(m + 4n) \\ 46 &= 2m + 8n \\ + -13 &= -2m + 3n \\ \hline 33 &= 11n \\ 3 &= n \end{aligned}$$

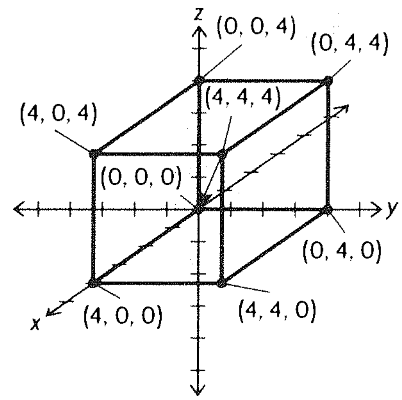
By substitution, $m = 11$.

So, vector $\vec{a}$ is in the span of $\vec{b}$ and $\vec{c}$.

20. a.



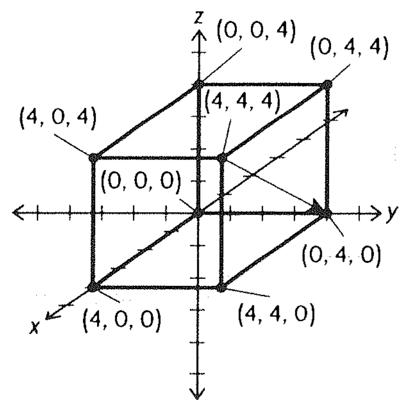
b.



$\overrightarrow{PO} = (4, 4, 4)$ so.

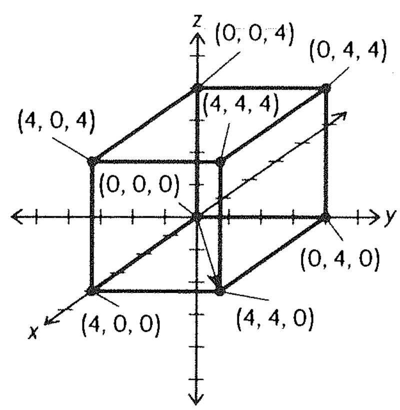
$\overrightarrow{OP} = -\overrightarrow{PO} = -(4, 4, 4) = (-4, -4, -4)$

c.



The vector $\overrightarrow{PQ}$ from $P(4, 4, 4)$ to $Q(0, 4, 0)$ can be written as $\overrightarrow{PQ} = (-4, 0, -4)$.

d.



The vector with the coordinates (4, 4, 0).

21. $|2(\vec{a} + \vec{b} - \vec{c}) - (\vec{a} + 2\vec{b}) + 3(\vec{a} - \vec{b} + \vec{c})|$

$= |2\vec{a} + 2\vec{b} - 2\vec{c} - \vec{a} - 2\vec{b} + 3\vec{a} - 3\vec{b} + 3\vec{c}|$

$= |4\vec{a} - 3\vec{b} + \vec{c}|$

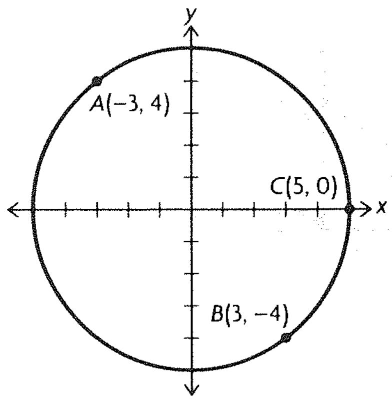
$= |4(1, 1, -1) - 3(2, -1, 3) + (2, 0, 13)|$

$= |(4, 4, -4) + (-6, 3, -9) + (2, 0, 13)|$

$= |(0, 7, 0)|$

$= 7$

22.



a. $|\overrightarrow{AB}| = 10$ because it is the diameter of the circle.

$$|\overrightarrow{BC}| = \sqrt{(5-3)^2 + (0-(-4))^2} = \sqrt{(2)^2 + (4)^2} = \sqrt{20} = 2\sqrt{5} \text{ or } 4.47$$

$$|\overrightarrow{CA}| = \sqrt{(5-(-3))^2 + (0-4)^2} = \sqrt{(8)^2 + (-4)^2} = \sqrt{80} \text{ or } 8.94$$

b. If A, B, and C are vertices of a right triangle, then

$$|\overrightarrow{BC}|^2 + |\overrightarrow{CA}|^2 = |\overrightarrow{AB}|^2$$

$$|\overrightarrow{BC}|^2 + |\overrightarrow{CA}|^2 = (2\sqrt{5})^2 + (\sqrt{80})^2 = 20 + 80 = 100$$

$$|\overrightarrow{AB}|^2 = 10^2 = 100$$

So, triangle ABC is a right triangle.

23. a. $\overrightarrow{FL} = \overrightarrow{FG} + \overrightarrow{GH} + \overrightarrow{HL} = \vec{a} + \vec{b} + \vec{c}$

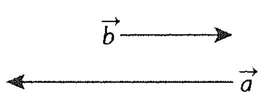
b. $\overrightarrow{MK} = \overrightarrow{JK} - \overrightarrow{JM} = \vec{a} - \vec{b}$

c. $\overrightarrow{HJ} = \overrightarrow{HG} + \overrightarrow{GF} + \overrightarrow{FJ} = -\vec{b} - \vec{a} + \vec{c}$

d. $\overrightarrow{IH} + \overrightarrow{KJ} = \overrightarrow{FG} + \overrightarrow{GF} = 0$

e. $\overrightarrow{IK} - \overrightarrow{IH} = \overrightarrow{HK} = \overrightarrow{IJ} = \vec{b} - \vec{c}$

24.



25. a. $\sqrt{|\vec{a}|^2 + |\vec{b}|^2}$ by the Pythagorean theorem

b. $\sqrt{|\vec{a}|^2 + |\vec{b}|^2}$ by the Pythagorean theorem

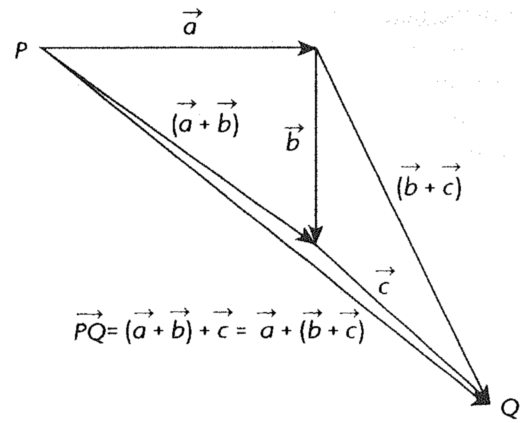
c. $\sqrt{4|\vec{a}|^2 + 9|\vec{b}|^2}$ by the Pythagorean theorem

26. Case 1 If $\vec{b}$ and $\vec{c}$ are collinear, then $2\vec{b} + 4\vec{c}$ is also collinear with both $\vec{b}$ and $\vec{c}$. But $\vec{a}$ is perpendicular to $\vec{b}$ and $\vec{c}$, so $\vec{a}$ is perpendicular to $2\vec{b} + 4\vec{c}$.

Case 2 If $\vec{b}$ and $\vec{c}$ are not collinear, then by spanning sets, $\vec{b}$ and $\vec{c}$ span a plane in R^3, and $2\vec{b} + 4\vec{c}$ is in that plane. If $\vec{a}$ is perpendicular to $\vec{b}$ and $\vec{c}$, then it is perpendicular to the plane and all vectors in the plane. So, $\vec{a}$ is perpendicular to $2\vec{b} + 4\vec{c}$.

Chapter 6 Test, p. 348

1. Let P be the tail of $\vec{a}$ and let Q be the head of $\vec{c}$. The vector sums $[\vec{a} + (\vec{b} + \vec{c})]$ and $[(\vec{a} + \vec{b}) + \vec{c}]$ can be depicted as in the diagram below, using the triangle law of addition. We see that $\overrightarrow{PQ} = \vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$. This is the associative property for vector addition.



2. a. $\overrightarrow{AB} = (6-(-2), 7-3, 3-(-5)) = (8, 4, 8)$

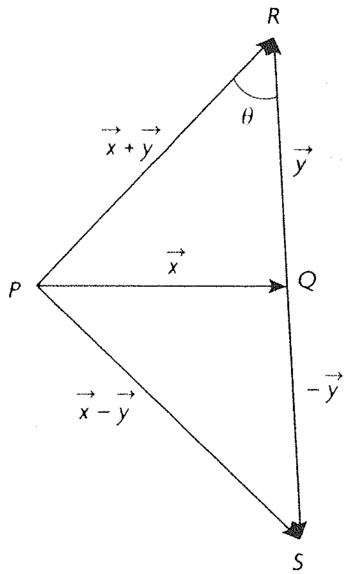
b. $|\overrightarrow{AB}| = \sqrt{8^2 + 4^2 + 8^2} = 12$

c. $\overrightarrow{BA} = (-1)\overrightarrow{AB} = (-8, -4, -8)$;

$|\overrightarrow{BA}| = |\overrightarrow{AB}| = 12$; unit vector in direction of

$$|\overrightarrow{BA}| = \frac{1}{|\overrightarrow{BA}|}\overrightarrow{BA} = \frac{1}{12}(-8, -4, -8) = \left(-\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\right)$$

3. Let $\vec{x} = \overrightarrow{PQ}$, $\vec{y} = \overrightarrow{QR}$, and $-\vec{y} = \overrightarrow{QS}$, as in the diagram below. Note that $|\overrightarrow{RS}| = |2\vec{y}| = 6$ and that triangle PQR and triangle PRS share angle θ.



By the cosine law:

$$\cos\theta = \frac{|\vec{y}|^2 + |\vec{x} + \vec{y}|^2 - |\vec{x}|^2}{2|\vec{y}||\vec{x} + \vec{y}|} \text{ and}$$

$$\cos\theta = \frac{|2\vec{y}|^2 + |\vec{x} + \vec{y}|^2 - |\vec{x} - \vec{y}|^2}{2|2\vec{y}||\vec{x} + \vec{y}|}.$$

Hence,

$$\frac{|\vec{y}|^2 + |\vec{x} + \vec{y}|^2 - |\vec{x}|^2}{2|\vec{y}||\vec{x} + \vec{y}|}$$

$$= \frac{|2\vec{y}|^2 + |\vec{x} + \vec{y}|^2 - |\vec{x} - \vec{y}|^2}{2|2\vec{y}||\vec{x} + \vec{y}|}$$

$$2(|\vec{y}|^2 + |\vec{x} + \vec{y}|^2 - |\vec{x}|^2)$$

$$= |2\vec{y}|^2 + |\vec{x} + \vec{y}|^2 - |\vec{x} - \vec{y}|^2$$

$$|\vec{x} - \vec{y}|^2 = 2|\vec{y}|^2 - |\vec{x} + \vec{y}|^2 + 2|\vec{x}|^2$$

$$|\vec{x} - \vec{y}| = \sqrt{2|\vec{y}|^2 - |\vec{x} + \vec{y}|^2 + \sqrt{2|\vec{x}|^2}}$$

$$|\vec{x} - \vec{y}| = \sqrt{2(3)^2 - (\sqrt{17})^2 + \sqrt{2(3)^2}}$$

$$|\vec{x} - \vec{y}| = \sqrt{19}$$

4. a. We have $3\vec{x} - 2\vec{y} = \vec{a}$ and $5\vec{x} - 3\vec{y} = \vec{b}$. Multiplying the first equation by -3 and the second equation by 2 yields: $-9\vec{x} + 6\vec{y} = -3\vec{a}$ and $10\vec{x} - 6\vec{y} = 2\vec{b}$. Adding these equations, we have: $\vec{x} = 2\vec{b} - 3\vec{a}$. Substituting this into the first equation yields: $3(2\vec{b} - 3\vec{a}) - 2\vec{y} = \vec{a}$. Simplifying, we have: $\vec{y} = 3\vec{b} - 5\vec{a}$.

b. First, conduct scalar multiplication on the third vector, yielding:
$(2, -1, c) + (a, b, 1) - (6, 3a, 12) = (-3, 1, 2c)$.
Now, each of the three components corresponds to an equation. First, $2 + a - 6 = -3$, which implies $a = 1$. Second, $-1 + b - 3a = 1$. Substituting $a = 1$ and simplifying yields $b = 5$. Third, $c + 1 - 12 = 2c$, so $c = -11$.

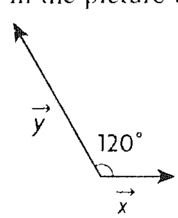
5. a. $\vec{a}$ and $\vec{b}$ span R^2, because any vector (x, y) in R^2 can be written as a linear combination of $\vec{a}$ and $\vec{b}$. These two vectors are not multiples of each other.

b. First, conduct scalar multiplication on the vectors, yielding: $(-2p, 3p) + (3q, -q) = (13, -9)$. Now, each component corresponds to an equation. First, $-2p + 3q = 13$. Second, $3p - q = -9$. Multiplying the second equation by 3 and adding the result to the first equation yields: $7p = -14$, which implies $p = -2$. Substituting this into the first equation and simplifying yields $q = 3$.

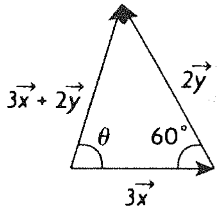
6. a. $\vec{a} = m\vec{b} + n\vec{c}$
$(1, 12, -29) = m(3, 1, 4) + n(1, 2, -3)$
$(1, 12, -29) = (3m, m, 4m) + (n, 2n, -3n)$
Each of the three components corresponds to an equation. First, $1 = 3m + n$. Second, $12 = m + 2n$. Third, $-29 = 4m - 3n$. Multiplying the first equation by -2 and adding the result to the second equation yields $m = -2$. Substituting $m = -2$ into the first equation yields $n = 7$. Since $m = -2$ and $n = 7$ also solves the third component's equation, $\vec{a} = m\vec{b} + n\vec{c}$ for $m = -2$ and $n = 7$. Hence, $\vec{a}$ can be written as a linear combination of $\vec{b}$ and $\vec{c}$.

b. $\vec{r} = m\vec{p} + n\vec{q}$
$(16, 11, -24) = m(-2, 3, 4) + n(4, 1, -6)$
$(16, 11, -24) = (-2m, 3m, 4m) + (4n, n, -6n)$
Each of the three components corresponds to an equation. First, $16 = -2m + 4n$. Second, $11 = 3m + n$. Third, $-24 = 4m - 6n$. Multiplying the first equation by 2 and adding the result to the third equation yields $n = 4$. Substituting $n = 4$ into the first equation yields $m = 0$. We have that $n = 4$ and $m = 0$ is the *unique* solution to the first and third equations, but $n = 4$ and $m = 0$ does not solve the second equation. Hence, this system of equations has no solution, and $\vec{r}$ cannot be written as a linear combination of $\vec{p}$ and $\vec{q}$. In other words, $\vec{r}$ does not lie in the plane determined by $\vec{p}$ and $\vec{q}$.

7. $\vec{x}$ and $\vec{y}$ have magnitudes of 1 and 2, respectively, and have an angle of 120° between them, as depicted in the picture below.



Since 60° is the complement of 120° $3\vec{x} + 2\vec{y}$ can be depicted as below.



By the cosine law:

$|3\vec{x} + 2\vec{y}|^2 = |3\vec{x}|^2 + |2\vec{y}|^2 - 2|3\vec{x}||2\vec{y}| \cos 60$

$|3\vec{x} + 2\vec{y}|^2 = 9|\vec{x}|^2 + 4|\vec{y}|^2 - 6|\vec{x}||\vec{y}|$

$|3\vec{x} + 2\vec{y}|^2 = 9 + 16 - 12$

$|3\vec{x} + 2\vec{y}| = \sqrt{13}$ or 3.61

The direction of $3\vec{x} + 2\vec{y}$ is θ, the angle from $\vec{x}$. This can be computed from the sine law:

$$\frac{|3\vec{x} + 2\vec{y}|}{\sin 60} = \frac{|2\vec{y}|}{\sin \theta}$$

$$\sin \theta = \frac{|2\vec{y}| \sin 60}{|3\vec{x} + 2\vec{y}|}$$

$$\theta = \sin^{-1}\left(\frac{|2\vec{y}| \sin 60}{|3\vec{x} + 2\vec{y}|}\right)$$

$$\theta = \sin^{-1}\left(\frac{(4) \sin 60}{\sqrt{13}}\right)$$

$\theta \doteq 73.9°$ relative to x

8. $\overrightarrow{DE} = \overrightarrow{CE} - \overrightarrow{CD}$

$\overrightarrow{DE} = \vec{b} - \vec{a}$

Also,

$\overrightarrow{BA} = \overrightarrow{CA} - \overrightarrow{CB}$

$\overrightarrow{BA} = 2\vec{b} - 2\vec{a}$

Thus,

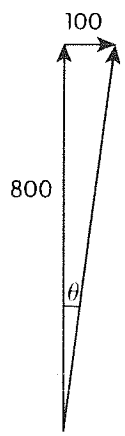
$\overrightarrow{DE} = \frac{1}{2}\overrightarrow{BA}$

CHAPTER 7
Applications of Vectors

Review of Prerequisite Skills, p. 350

1. The velocity relative to the ground has a magnitude equivalent to the hypotenuse of a triangle with sides 800 and 100. So, by the Pythagorean theorem we can find the magnitude of the velocity.

$$\begin{aligned} v^2 &= 800^2 + 100^2 \\ &= 640\,000 + 10\,000 \\ &= 650\,000 \\ v &= \sqrt{650\,000} \\ &\doteq 806 \text{ km/h} \end{aligned}$$



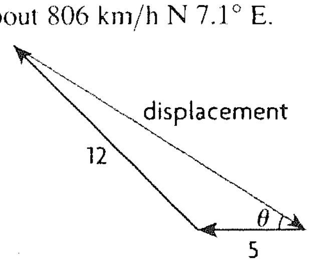
$$\tan\theta = \frac{100}{800}$$

$$\theta = \tan^{-1}\left(\frac{100}{800}\right)$$

$$\theta \doteq 7.1°$$

The velocity of the airplane relative to the ground is about 806 km/h N 7.1° E.

2.

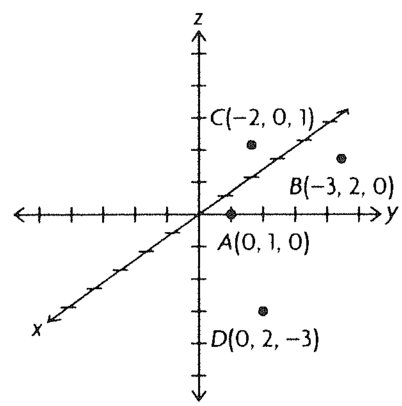


The angle between the two displacements is 135°. The magnitude, m, and the angle, θ, of the displacement can be found using the cosine law.

$$\begin{aligned} m^2 &= 5^2 + 12^2 - 2(5)(12)\cos 135 \\ &= 25 + 144 - 120\left(\frac{-\sqrt{2}}{2}\right) \\ &= 169 + 84.85 \\ &= 253.85 \\ m &= \sqrt{253.85} \\ &\doteq 15.93 \text{ units} \\ 12^2 &= 15.93^2 + 5^2 - 2(15.93)(5)\cos\theta \\ 144 &= 253.76 + 25 - 159.3\cos\theta \\ -134.76 &= -159.3\cos\theta \\ \cos\theta &= \frac{134.76}{159.3} \\ \theta &= \cos^{-1}\left(\frac{134.76}{159.3}\right) \\ &\doteq 32.2° \end{aligned}$$

So the displacement is 15.93 units, W 32.2° N.

3.



4. a. $(3, -2, 7)$

$$\begin{aligned} l &= \text{magnitude} \\ &= \sqrt{3^2 + (-2)^2 + 7^2} \\ &= \sqrt{9 + 4 + 49} \\ &= \sqrt{62} \\ &\doteq 7.87 \end{aligned}$$

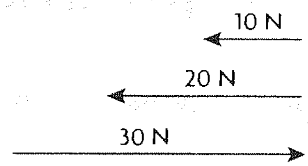
b. $(-9, 3, 14)$

$$\begin{aligned} l &= \text{magnitude} \\ &= \sqrt{(-9)^2 + 3^2 + 14^2} \\ &= \sqrt{81 + 9 + 196} \\ &= \sqrt{286} \\ &\doteq 16.91 \end{aligned}$$

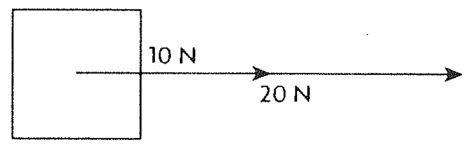
c. $(1, 1, 0)$
l = magnitude
$= \sqrt{1^2 + 1^2 + 0^2}$
$= \sqrt{2}$
$\doteq 1.41$
d. $(2, 0, -9)$
l = magnitude
$= \sqrt{2^2 + 0^2 + (-9)^2}$
$= \sqrt{4 + 0 + 81}$
$= \sqrt{85}$
$\doteq 9.22$
5. a. $A(x, y, 0)$
In the xy-plane at the point (x, y).
b. $B(x, 0, z)$
In the xz-plane at the point (x, z).
c. $C(0, y, z)$
In the yz-plane at the point (y, z).
6. a. $(-6, 0) + 7(1, -1)$
$= (-6\vec{i} + 0\vec{j}) + 7(\vec{i} - \vec{j})$
$= (-6\vec{i} + 0\vec{j}) + (7\vec{i} - 7\vec{j})$
$= \vec{i} - 7\vec{j}$
b. $(4, -1, 3) - (-2, 1, 3)$
$= (4\vec{i} - \vec{j} + 3\vec{k}) - (-2\vec{i} + \vec{j} + 3\vec{k})$
$= 6\vec{i} - 2\vec{j}$
c. $2(-1, 1, 3) + 3(-2, 3, -1)$
$= 2(-\vec{i} + \vec{j} + 3\vec{k}) + 3(-2\vec{i} + 3\vec{j} - \vec{k})$
$= (-2\vec{i} + 2\vec{j} + 6\vec{k}) + (-6\vec{i} + 9\vec{j} - 3\vec{k})$
$= -8\vec{i} + 11\vec{j} + 3\vec{k}$
d. $-\frac{1}{2}(4, -6, 8) + \frac{3}{2}(4, -6, 8)$
$= -\frac{1}{2}(4\vec{i} - 6\vec{j} + 8\vec{k}) + \frac{3}{2}(4\vec{i} - 6\vec{j} + 8\vec{k})$
$= (-2\vec{i} + 3\vec{j} - 4\vec{k}) + (6\vec{i} - 9\vec{j} + 12\vec{k})$
$= 4\vec{i} - 6\vec{j} + 8\vec{k}$
7. a. $\vec{a} + \vec{b}$
$= (3\vec{i} + 2\vec{j} - \vec{k}) + (-2\vec{i} + \vec{j})$
$= \vec{i} + 3\vec{j} - \vec{k}$
b. $\vec{a} - \vec{b}$
$= (3\vec{i} + 2\vec{j} - \vec{k}) - (-2\vec{i} + \vec{j})$
$= (3\vec{i} + 2\vec{j} - \vec{k}) + (2\vec{i} - \vec{j})$
$= 5\vec{i} + \vec{j} - \vec{k}$
c. $2\vec{a} - 3\vec{b}$
$= 2(3\vec{i} + 2\vec{j} - \vec{k}) - 3(-2\vec{i} + \vec{j})$
$= (6\vec{i} + 4\vec{j} - 2\vec{k}) + (6\vec{i} - 3\vec{j})$
$= 12\vec{i} + \vec{j} - 2\vec{k}$

7.1 Vectors as Forces, pp. 362–364

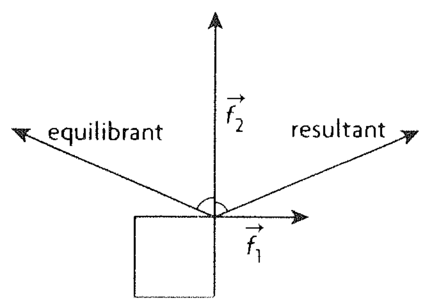
1. a. 10 N is a melon, 50 N is a chair, 100 N is a computer
b. Answers will vary.
2. a.



b. 180°
3.



The forces should be placed in a line along the same direction.
4. For three forces to be in equilibrium, they must form a triangle, which is a planar figure.
5.



a. The resultant is equivalent in magnitude to the hypotenuse, h, of the triangle with 5 and 12 as sides and is directed northeast at an angle of $\sin^{-1}\frac{12}{h}$. Thus, the resultant is $\sqrt{5^2 + 12^2} = 13$ N at an angle of $\sin^{-1}\frac{12}{13} =$ N 22.6° E. The equlibrant is equal in magnitude and opposite in direction of the resultant. Thus, the equilibrant is 13 N at an angle of S 22.6° W.
b. The resultant is $\sqrt{9^2 + 12^2} = 15$ N at an angle of $\sin^{-1}\frac{12}{15} =$ S 36.9° W. The equilibrant, then, is 15 N at N 36.9° E.
6. For three forces to form equilibrium, they must be able to form a triangle or a balanced line, so
a. Yes, since $3 + 4 > 7$ these can form a triangle.
b. Yes, since $9 + 40 > 41$ these can form a triangle.
c. No, since $\sqrt{5} + 6 < 9$ these cannot form a triangle.
d. Yes, since $9 + 10 = 19$, placing the 9 N and 10 N force in a line directly opposing the 19 N force achieves equilibrium.

7. Arms 90 cm apart will yield a resultant with a smaller magnitude than at 30 cm apart. A resultant with a smaller magnitude means less force to counter your weight, hence a harder chin-up.

8. Using the cosine law, the resultant has a magnitude, r, of

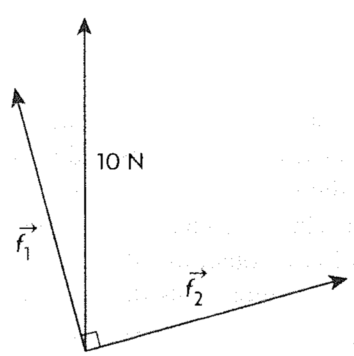
$$r^2 = |\vec{f_1}|^2 + |\vec{f_2}|^2 - 2|\vec{f_1}||\vec{f_2}|\cos 120^\circ$$
$$= 6^2 + 8^2 - 2(6)(8)\left(-\frac{1}{2}\right)$$
$$= 36 + 64 + 48$$
$$= 148$$
$$r = \sqrt{148}$$
$$\doteq 12.17 \text{ N}$$

Using the sine law, the resultant's angle, θ, can be found by

$$\frac{\sin\theta}{8} = \frac{\sin 120^\circ}{12.17}$$
$$\sin\theta = 8\frac{\frac{\sqrt{3}}{2}}{12.17}$$
$$\theta = \sin^{-1} 8\frac{\frac{\sqrt{3}}{2}}{12.17}$$

$\doteq 34.7^\circ$ from the 6 N force toward the 8 N force. The equilibrant, then, would be 12.17 N at $180^\circ - 34.7^\circ = 145.3^\circ$ from the 6 N force away from the 8 N force.

9.



$\vec{f_1}$ = force 15° from the 10 N force
$\vec{f_2}$ = force perpendicular to $\vec{f_1}$
x_1 = component of $\vec{f_1}$ parallel to the 10 N force
x_2 = component of $\vec{f_2}$ parallel to the 10 N force

We know that the components of $\vec{f_1}$ and $\vec{f_2}$ perpendicular to the 10 N force must be equal, so we can write

$$|\vec{f_1}|\cos 15 = |\vec{f_2}|\cos 75$$
$$|\vec{f_1}| = |\vec{f_2}|\frac{\cos 75}{\cos 15}$$

Now we look at x_1 and x_2. We know

$$x_1 = |\vec{f_1}|\sin 15$$
$$x_2 = |\vec{f_2}|\sin 75$$
$$x_1 + x_2 = 10$$

So $|\vec{f_1}|\sin 15 + |\vec{f_2}|\sin 75 = 10$

Substituting then solving for $\vec{f_2}$ yields

$$|\vec{f_2}|\frac{\cos 75}{\cos 15}\sin 15 + |\vec{f_2}|\sin 75 = 10$$
$$|\vec{f_2}|\left(\frac{\cos 75}{\cos 15}\sin 15 + \sin 75\right) = 10$$
$$|\vec{f_2}|(1.035) = 10$$
$$|\vec{f_2}| = 9.66 \text{ N}$$

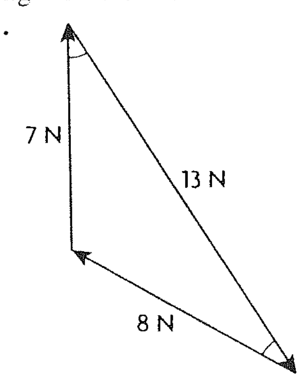
Now we solve for $\vec{f_1}$:

$$|\vec{f_1}| = |\vec{f_2}|\frac{\cos 75}{\cos 15}$$
$$|\vec{f_1}| = (9.66)\frac{\cos 75}{\cos 15}$$
$$|\vec{f_1}| = (9.66)(0.268)$$
$$|\vec{f_1}| = 2.59 \text{ N}$$

So the force 15° from the 10 N force is 9.66 N and the force perpendicular to it is 2.59 N.

10. The force of the block is $(10 \text{ kg})(9.8 \text{ N/kg}) = 98$ N. The component of this force parallel to the ramp is $(98)\sin 30^\circ = (98)(\frac{1}{2}) = 49$ N, directed down the ramp. So the force preventing this block from moving would be 49 N directed up the ramp.

11. a.



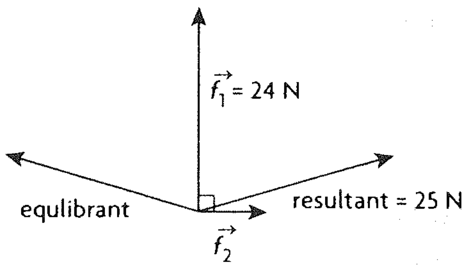
b. Using the cosine law for the angle, θ, we have

$$13^2 = 8^2 + 7^2 - 2(8)(7)\cos\theta$$
$$169 = 64 + 49 - 112\cos\theta$$
$$56 = -112\cos\theta$$
$$\cos\theta = \frac{-56}{112}$$
$$\theta = \cos^{-1}\frac{-1}{2}$$
$$= 120$$

This is the angle between the vectors when placed head to tail. So the angle between the vectors when placed tail to tail is $180° - 120° = 60°$.

12. The 10 N force and the 5 N force result in a 5 N force east. The 9 N force and the 14 N force result in a 5 N force south. The resultant of these is now equivalent to the hypotenuse of the right triangle with 5 N as both bases and is directed 45° south of east. So the resultant is $\sqrt{5^2 + 5^2} = \sqrt{50} \doteq 7.1$ N 45° south of east.

13.



a. Using the Pythagorean theorem,

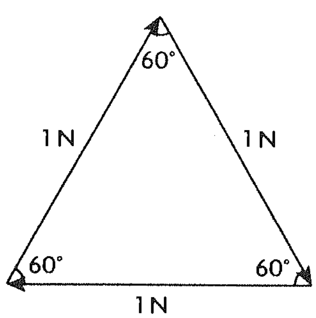
$$|\vec{f_1}|^2 + |\vec{f_2}|^2 = 25^2$$
$$|\vec{f_2}|^2 = 25^2 - |\vec{f_1}|^2$$
$$= 25^2 - 24^2$$
$$= 49$$
$$|\vec{f_2}| = 7$$

b. The angle, θ, between $\vec{f_1}$ and the resultant is given by

$$\sin\theta = \frac{|\vec{f_2}|}{25}$$
$$\sin\theta = \frac{7}{25}$$
$$\theta = \sin^{-1}\frac{7}{25}$$
$$\doteq 16.3°$$

So the angle between $\vec{f_1}$ and the equilibrant is $180° - 16.3° = 163.7°$.

14. a.



For these three equal forces to be in equilibrium, they must form an equilateral triangle. Since the resultant will lie along one of these lines, and since all angles of an equilateral triangle are 60°, the resultant will be at a 60° angle with the other two vectors.

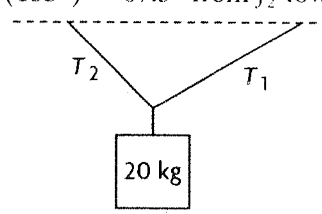
b. Since the equilibrant is directed opposite the resultant, the angle between the equilibrant and the other two vectors is $180° - 60° = 120°$.

15. Since $\vec{f_1}$ and $\vec{f_2}$ act opposite one another, they net a 10 N force directed west. Since $\vec{f_3}$ and $\vec{f_4}$ act opposite one another, they net a 10 N force directed 45° north of east. So using the cosine law to find the resultant, $\vec{f_r}$,

$$|\vec{f_r}|^2 = 10^2 + 10^2 - 2(10)(10)\cos 45°$$
$$= 200 - 200\cos 45°$$
$$= 200 - 200\left(\frac{\sqrt{2}}{2}\right)$$
$$|\vec{f_r}| = \sqrt{200 - 200\left(\frac{\sqrt{2}}{2}\right)}$$
$$\doteq 7.65 \text{ N}$$

Since our net forces are equal at 10 N, the angle of the resultant is directed halfway between the two, or at $\frac{1}{2}(135°) = 67.5°$ from $\vec{f_2}$ toward $\vec{f_3}$.

16.



Let T_1 be the tension in the 30° rope and T_2 be the tension in the 45° rope.

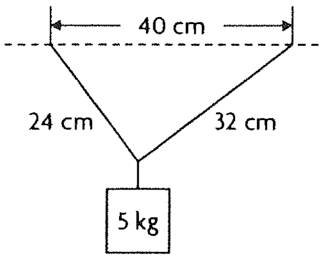
Since this system is in equilibrium, we know that the horizontal components of T_1 and T_2 are equal and opposite and the vertical components add to be opposite the action of the mass. Also, the force produced by the mass is $(20 \text{ kg})(9.8 \text{ N/kg}) = 196$ N. So we have a system of two equations: the first, $(T_1)\cos 30° = (T_2)\cos 45°$ represents the balance of the horizontal components, and the second, $(T_1)\sin 30° + (T_2)\sin 45° = 196$ represents the balance of the vertical components with the mass. So solving this system of two equations with two variable gives the desired tensions.

$$T_1\cos 30° = T_2\cos 45°$$
$$T_1 = T_2\frac{\cos 45°}{\cos 30°}$$
$$T_1\sin 30° + T_2\sin 45° = 196$$
$$\left(T_2\frac{\cos 45°}{\cos 30°}\right)\sin 30° + T_2\sin 45° = 196$$
$$T_2\left(\left(\frac{\cos 45°}{\cos 30°}\right)\sin 30° + \sin 45°\right) = 196$$

$$T_2(1.12) = 196$$
$$T_2 \doteq 175.73 \text{ N}$$
$$T_1 = (175.73)\frac{\cos 45°}{\cos 30°}$$
$$\doteq 143.48 \text{ N}$$

Thus the tension in the 45° rope is 175.73 N and the tension in the 30° rope is 143.48 N.

17.



First, use the Cosine Law to find the angles the strings make at the point of suspension. Let θ_1 be the angle made by the 32 cm string and θ_2 be the angle made by the 24 cm string.

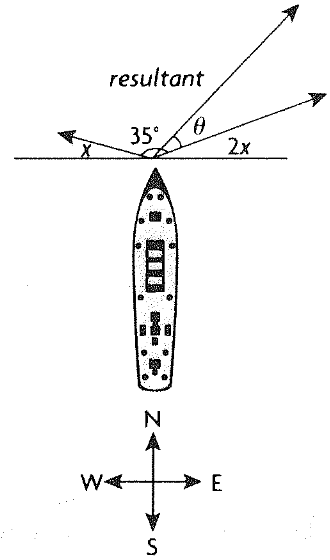
$$24^2 = 32^2 + 40^2 - 2(32)(40)\cos\theta_1$$
$$-2048 = -2560\cos\theta_1$$
$$\theta_1 = \cos^{-1}\frac{2048}{2560}$$
$$\doteq 36.9°$$
$$32^2 = 24^2 + 40^2 - 2(24)(40)\cos\theta_2$$
$$-1152 = -1920\cos\theta_2$$
$$\theta_2 = \cos^{-1}\frac{1152}{1920}$$
$$\doteq 53.1°$$

A keen eye could have recognized this triangle as a 3-4-5 right triangle and simply used the Pythagorean theorem as well. Now we set up the same system of equations as in problem 16, with T_1 being the tension in the 32 cm string and T_2 being the tension in the 24 cm string, and the force of the mass being $(5 \text{ kg})(9.8 \text{ N/kg}) = 49$ N.

$$T_1\cos 36.9° = T_2\cos 53.1°$$
$$T_1 = T_2\frac{\cos 53.1°}{\cos 36.9°}$$
$$T_1\sin 36.9° + T_2\sin 53.1° = 49$$
$$\left(T_2\frac{\cos 53.1°}{\cos 36.9°}\right)\sin 36.9° + T_2\sin 53.1° = 49$$
$$T_2\left(\left(\frac{\cos 53.1°}{\cos 36.9°}\right)\sin 36.9° + \sin 53.1°\right) = 49$$
$$T_2(1.25) = 49$$
$$T_2 \doteq 39.2 \text{ N}$$
$$T_1 = (39.2)\frac{\cos 53.1°}{\cos 36.9°}$$
$$\doteq 29.4 \text{ N}$$

Thus the tension in the 24 cm string is 39.2 N and the tension in the 32 cm string is 29.4 N.

18.



(Port means left and starboard means right.) We are looking for the resultant of these two force vectors that are 35° apart. We don't know the exact value of the force, so we will call it x. So the small tug is pulling with a force of x and the large tug is pulling with a force of $2x$. To find the magnitude of the resultant, r, in terms of x, we use the cosine law.

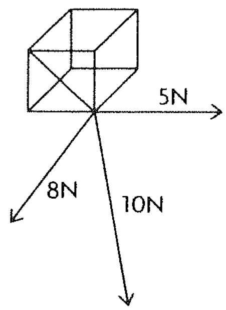
$$r^2 = x^2 + (2x)^2 - 2(x)(2x)\cos 145°$$
$$= x^2 + 4x^2 - 4x^2\cos 145°$$
$$\doteq 5x^2 - 4x^2(-0.8192)$$
$$\doteq 5x^2 + 3.2768x^2$$
$$\doteq 8.2768x^2$$
$$r \doteq \sqrt{8.2768x^2}$$
$$\doteq 2.8769x$$

Now we use the cosine law again to find the angle, θ, made by the resultant.

$$x^2 = r^2 + (2x)^2 - 2(2.8769x)(2x)\cos\theta$$
$$x^2 = 8.2768x^2 + 4x^2 - 11.5076x^2\cos\theta$$
$$x^2 = 12.2768x^2 - 11.5076x^2\cos\theta$$
$$-11.2768x^2 = -11.5076x^2\cos\theta$$
$$\cos\theta = \frac{11.2768}{11.5076}$$
$$\theta = \cos^{-1}\left(\frac{11.2768}{11.5076}\right)$$

$\doteq 11.5°$ from the large tug toward the small tug, for a net of 8.5° to the starboard side.

19.



a. First we will find the resultant of the 5 N and 8 N forces. Use the Pythagorean theorem to find the magnitude, m.

$$m^2 = 5^2 + 8^2 = 25 + 64 = 89$$

$$m = \sqrt{89} \doteq 9.4$$

Next we use the Pythagorean theorem again to find the magnitude, M, of the resultant of this net force and the 10 N force.

$$M^2 = m^2 + 10^2 = 89 + 100 = 189$$

$$M = \sqrt{189} \doteq 13.75$$

Since the equilibrant is equal in magnitude to the resultant, we have the magnitude of the equilibrant equal to approximately 13.75 N.

b. To find each angle, use the definition of cosine with respect each force as a leg and the resultant as the hypotenuse. Let θ_{5N} be the angle from the 5 N force to the resultant, θ_{8N} be the angle from the 8 N force to the resultant, and θ_{10N} be the angle from the 10 N force to the resultant.

Let the sign of the resultant be negative, since it is in a direction away from the head of each of the given forces.

$$\cos\theta_{5N} = \frac{5}{-13.75}$$

$$\theta_{5N} = \cos^{-1}\left(\frac{5}{-13.75}\right) \doteq 111.3°$$

$$\cos\theta_{8N} = \frac{8}{-13.75}$$

$$\theta_{8N} = \cos^{-1}\left(\frac{8}{-13.75}\right) \doteq 125.6°$$

$$\cos\theta_{10N} = \frac{10}{-13.75}$$

$$\theta_{10N} = \cos^{-1}\left(\frac{10}{-13.75}\right) \doteq 136.7°$$

20. We know that the resultant of these two forces is equal in magnitude and angle to the diagonal line of the parallelogram formed with $\vec{f_1}$ and $\vec{f_2}$ as legs and has diagonal length $|\vec{f_1} + \vec{f_2}|$. We also know from the cosine law that

$$|\vec{f_1} + \vec{f_2}|^2 = |\vec{f_1}|^2 + |\vec{f_2}|^2 - 2|\vec{f_1}||\vec{f_2}|\cos\phi$$

where ϕ is the supplement to θ in our parallelogram. Since we know $\phi = 180 - \theta$, then $\cos\phi = \cos(180 - \theta) = -\cos\theta$.

Thus we have

$$|\vec{f_1} + \vec{f_2}|^2 = |\vec{f_1}|^2 + |\vec{f_2}|^2 - 2|\vec{f_1}||\vec{f_2}|\cos\phi = |\vec{f_1}|^2 + |\vec{f_2}|^2 + 2|\vec{f_1}||\vec{f_2}|\cos\theta$$

$$|\vec{f_1} + \vec{f_2}| = \sqrt{|\vec{f_1}|^2 + |\vec{f_2}|^2 + 2|\vec{f_1}||\vec{f_2}|\cos\theta}$$

7.2 Velocity, pp. 367–370

1. a. Both the woman and the train's velocities are in the same direction, so we add them.

80 km/h + 4 km/h = 84 km/h

b. The woman's velocity is directed opposite that of train, so we subtract her velocity from the train's.

80 km/h − 4 km/h = 76 km/h. The resultant is in the same direction as the train's movement.

2. a. The velocity of the wind is directed opposite that of the airplane, so we subtract the wind's velocity from the airplane's.

600 km/h − 100 km/h = 500 km/h north.

b. Both the wind and the airplane's velocities are in the same direction, so we add them.

600 km/h + 100 km/h = 700 km/h north.

3. We use the Pythagorean theorem to find the magnitude, m, of the resultant velocity and we use the definition of sine to find the angle, θ, made.

$$m^2 = 300^2 + 50^2 = 90\,000 + 2500 = 92\,500$$

$$m = \sqrt{92\,500} \doteq 304.14 \text{ km/h}$$

$$\tan\theta = \frac{50}{300}$$

$$\theta = \tan^{-1}\frac{50}{300}$$

$\doteq 9.5°$. The resultant is 304.14 km/h, W 9.5° S.

4. Adam must swim at an angle, θ, upstream so as to counter the 1 km/h velocity of the stream. This is equivalent to Adam swimming along the hypotenuse of a right traingle with 1 km/h leg and a 2 km/h hypotenuse. So the angle is found using the definition of cosine.

$$\cos\theta = \frac{1}{2}$$
$$\theta = \cos^{-1}\frac{1}{2}$$
$$= 60^\circ \text{ upstream}$$

5. a. 2 m/s forward

b. 20 m/s + 2 m/s = 22 m/s in the direction of the car

6. Since the two velocities are at right angles we can use the Pythagorean theorem to find the magnitude, m, of the resultant velocity and we use the definition of sine to find the angle, θ, made.

$$m^2 = 12^2 + 5^2$$
$$= 144 + 25$$
$$= 169$$
$$m = \sqrt{169}$$
$$= 13 \text{ m/s}$$
$$\sin\theta = \frac{5}{13}$$
$$\theta = \sin^{-1}\frac{5}{13}$$

$\doteq 22.6^\circ$ from the direction of the boat toward the direction of the current. This results in a net of $22.6^\circ + 15^\circ = 37.6^\circ$, or N 37.6° W.

7. a. First we find the components of the resultant directed north and directed west. The component directed north is the velocity of the airplane, 800, minus 100 sin 45°, since the wind forms a 45° angle south of west. The western component of the resultant is simply 100 cos 45°. So we use the Pythagorean theorem to find the magnitude, m, of the resultant and the definition of sine to find the angle, θ, of the resultant.

$$m^2 = (800 - 100\sin 45^\circ)^2 + (100\cos 45^\circ)^2$$
$$\doteq (729.29)^2 + (71.71)^2$$
$$\doteq 536\,863.8082$$
$$m \doteq 732.71 \text{ km/h}$$

Use the sine law to determine the direction.

$$\frac{\sin\theta}{100} = \frac{\sin 45^\circ}{732.71}$$
$$\theta \doteq 5.5^\circ$$

The direction is N 5.5° W.

b. The airplane is travelling at approximately 732.71 km/h, so in 1 hour the airplane will travel about 732.71 km.

8. a. First we find the velocity of the airplane. We use the Pythagorean theorem to find the magnitude, m, of the resultant.

$$m^2 = 450^2 + 100^2$$
$$= 202\,500 + 10\,000$$
$$= 212\,500$$
$$m = \sqrt{212\,500}$$
$$\doteq 461 \text{ km/h}$$

So in 3 hours, the airplane will travel about (461 km/h)(3 h) = 1383 km.

b. To find the angle, θ, the airplane travels, we use the definition of sine.

$$\sin\theta = \frac{100}{461}$$
$$\theta = \sin^{-1}\frac{100}{461}$$

$\doteq 12.5^\circ$ east of north.

9. a. To find the angle, θ, at which to fly is the equivalent of the angle of a right triangle with 44 as the opposite leg and 244 as the hypotenuse. So we use the definition of sine to find this angle.

$$\sin\theta = \frac{44}{244}$$
$$\theta = \sin^{-1}\frac{44}{244}$$

$\doteq 10.4^\circ$ south of west.

b. By the Pythagorean Theorem, the resultant ground speed of the airplane is $\sqrt{(244^2 - 44^2)} = 240$ km/h. Since time = distance/rate, the duration of the flight is simply (480 km)/(240 km/h) = 2 h.

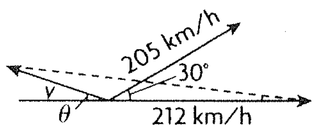
10. a. Since Judy is swimming perpendicular to the flow of the river, her resultant velocity is simply the hypotenuse of a right triangle with 3 and 4 as bases, which is a 3-4-5 right triangle. Thus, Judy's resultant velocity is 5 km/h. The direction is determined by $\tan\theta = \frac{4}{3}$. $\theta \doteq 53.1^\circ$ downstream

b. Judy's distance traveled down the river would be the "4" leg of the 3-4-5 triangle formed by the vectors, but scaled down so that 1m (the width of the river) is equivalent to the "3" leg. So her distance traveled is $\frac{4}{3} \doteq 1.33$ km. This makes her about 0.67 km from Helen's cottage.

c. While in the river, Judy is swimming at 5 km/h for a distance of $\frac{5}{3}$ km. Since time = distance/rate, her time taken is

$$\frac{\frac{5}{3}\text{ km}}{5 \text{ km/h}} = \frac{1}{3} \text{ hours} = 20 \text{ minutes.}$$

11.



a. and **b.** Here, 205 km/h directed 30° north of east is the resultant of 212 km/h directed east, and the wind speed, v, directed at some angle. This problem is more easily approached finding the wind speed, v, first. So we will do that using the cosine law.

$v^2 = 205^2 + 212^2 - 2(205)(212)\cos 30°$
$= 42\,025 + 44\,944 - 86\,920 \cos 30°$
$= 86\,969 - 75\,275$
$= 11\,694$
$v = \sqrt{11\,694}$
$\doteq 108$ km/h

Now to find the wind's direction, we simply find the angle supplementary to the lesser angle, θ, formed by the parallelogram of these three velocities. We can use the sine law for this.

$\frac{\sin\theta}{205} = \frac{\sin 30°}{108}$

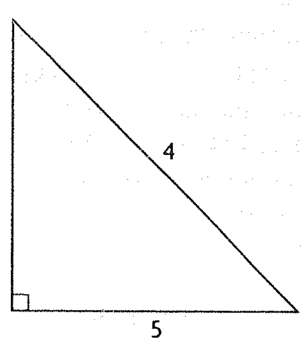
$\sin\theta = 205\left(\frac{\sin 30°}{108}\right)$

$\theta = \sin^{-1}\left(205\left(\frac{\sin 30°}{108}\right)\right)$

$\doteq 71.6°$

Thus, the direction of v is the angle supplementary to θ in the parallelogram:
$180° - 71.6° = 108.4° = 18.4°$ west of north.

12.



Since her swimming speed is a maximum of 4 km/h, this is her maximum resultant magnitude, which is also the hypotenuse of the triangle formed by her and the river's velocity vector. Since one of these legs is 5 km/h, we have a triangle with a leg larger than its hypotenuse, which is impossible.

13. a. First we need to find Mary's resultant velocity, v. Since this resultant is the diagonal of the parallelogram formed by hers and the river's velocity, we can use the cosine law with the angle, θ, of the parallelogram adjacent 30°.

$v^2 = 3^2 + 4^2 - 2(3)(4)\cos 150°$
$= 9 + 16 - 24\cos 150°$
$= 25 + 20.8$
$= 45.8$
$v = \sqrt{45.8}$
$\doteq 6.8$ m/s

So in 10 seconds, Mary travels about (6.8 m/s)(10 s) = 68 m.

b. Since Mary is travelling at 3 m/s at an angle of 30°, to find the component of her velocity, v, perpendicular to the current, we use the definition of sine.

$v = 3\sin 30$
$= 3\left(\frac{1}{2}\right)$
$= 1.5$ m/s perpendicular to the current.

So since time = distance/rate, the time taken is (150 m)/(1.5 m/s) = 100 s.

14. a. So we have a 5.5 m/s vector and a 4 m/s vector with a resultant vector that is directed 45° south of west. Letting θ be the angle between the 4 km/h vector and the resultant, we can construct a parallelogram using these three vectors and a subsequent triangle with θ opposite the 5.5 m/s vector and 45° opposite the 4 m/s vector. We now use the sine law to find θ.

$\frac{\sin\theta}{5.5} = \frac{\sin 45°}{4}$

$\sin\theta = 5.5\left(\frac{\sin 45°}{4}\right)$

$\theta = \sin^{-1}\left(5.5\left(\frac{\sin 45°}{4}\right)\right)$

$\doteq 76.5°$ from the resultant.

Since the resultant is 45° west of south, Dave's direction is $76.5° + 45° = 121.5°$ west of south, which is equivalent to about $180° - 121.5° = 58.5°$ upstream.

b. First, we find the magnitude, m, of Dave's 4 m/s velocity in the direction perpendicular to the river. This is done using the definition of sine.

$m = 4\sin 58.5°$
$\doteq 3.41$ m/s perpendicular to the river.

Since time is distance/rate, we have (200 m)/(3.41 m/s) $\doteq$ 58.6 s.

15. Let b represent the speed of the steamboat and c represent the speed of the current. On the way downstream, the effective speed is $b + c$, and upstream is $b - c$. The distance upstream and downstream is the same, so $5(b + c) = 7(b - c)$. So, $b = 6c$. This means that the speed of the boat is 6 times the speed of the current. So, $(6c + c) \cdot 5$

or $35c$ is the distance. This means that it would take a raft 35 hours moving with the speed of the current to get from A to B.

7.3 The Dot Product of Two Geometric Vectors, pp. 377–378

1. $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta = 0$. This means $|\vec{a}| = 0$, or $|\vec{b}| = 0$, or $\cos\theta = 0$. To be guaranteed that the two vectors are perpendicular, the vectors must be nonzero.

2. $\vec{a} \cdot \vec{b}$ is a scalar, and a dot product is only defined for vectors, so $(\vec{a} \cdot \vec{b}) \cdot \vec{c}$ is meaningless.

3. Answers may vary. Let $\vec{a} = \vec{i}, \vec{b} = \vec{j}, \vec{c} = -\vec{i}$.
$\vec{a} \cdot \vec{b} = 0, \vec{b} \cdot \vec{c} = 0$, but $\vec{a} = -\vec{c}$.

4. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} = \vec{b} \cdot \vec{c}$ because $\vec{c} = \vec{a}$

5. Since $\vec{a}$ and $\vec{b}$ are unit vectors, $|\vec{a}| = |\vec{b}| = 1$ and since they are pointing in opposite directions then $\theta = 180°$ so $\cos\theta = -1$. Therefore $\vec{a} \cdot \vec{b} = -1$.

6. a. $\vec{p} \cdot \vec{q} = |\vec{p}||\vec{q}|\cos\theta$
$= (4)(8)\cos(60°)$
$= (32)(.5)$
$= 16$

b. $\vec{x} \cdot \vec{y} = |\vec{x}||\vec{y}|\cos\theta$
$= (2)(4)\cos(150°)$
$= (8)\left(-\frac{\sqrt{3}}{2}\right)$
$\doteq -6.93$

c. $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$
$= (0)(8)\cos(100°)$
$= 0$

d. $\vec{p} \cdot \vec{q} = |\vec{p}||\vec{q}|\cos\theta$
$= (1)(1)\cos(180°)$
$= (1)(-1)$
$= -1$

e. $\vec{m} \cdot \vec{n} = |\vec{m}||\vec{n}|\cos\theta$
$= (2)(5)\cos(90°)$
$= (10)(0)$
$= 0$

f. $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\cos\theta$
$= (4)(8)\cos 145°$
$\doteq -26.2$

7. a. $\vec{x} \cdot \vec{y} = |\vec{x}||\vec{y}|\cos\theta$
$12\sqrt{3} = (8)(3)\cos\theta$
$\frac{\sqrt{3}}{2} = \cos\theta$
$\theta = 30°$

b. $\vec{m} \cdot \vec{n} = |\vec{m}||\vec{n}|\cos\theta$
$(6) = (6)(6)\cos\theta$
$\frac{1}{6} = \cos\theta$
$\theta \doteq 80°$

c. $\vec{p} \cdot \vec{q} = |\vec{p}||\vec{q}|\cos\theta$
$3 = (5)(1)\cos\theta$
$\frac{3}{5} = \cos\theta$
$\theta \doteq 53°$

d. $\vec{p} \cdot \vec{q} = |\vec{p}||\vec{q}|\cos\theta$
$-3 = (5)(1)\cos\theta$
$-\frac{3}{5} = \cos\theta$
$\theta \doteq 127°$

e. $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$
$10.5 = (7)(3)\cos\theta$
$\frac{1}{2} = \cos\theta$
$\theta = 60°$

f. $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\cos\theta$
$-50 = (10)(10)\cos\theta$
$-\frac{1}{2} = \cos\theta$
$\theta = 120°$

8. $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$
$= (7.5)(6)\cos(180° - 120°)$
$= (45)\left(\frac{1}{2}\right)$
$= 22.5$

Note: θ is the angle between the two vectors when they are tail to tail, so $\theta \neq 120°$.

9. a. $(\vec{a} + 5\vec{b}) \cdot (2\vec{a} - 3\vec{b}) = \vec{a} \cdot 2\vec{a} - \vec{a} \cdot 3\vec{b}$
$+ 5\vec{b} \cdot 2\vec{a} - 5\vec{b} \cdot 3\vec{b}$
$= 2|\vec{a}|^2 - 15|\vec{b}|^2$
$- 3\vec{a} \cdot \vec{b} + 10\vec{a} \cdot \vec{b}$
$= 2|\vec{a}|^2 - 15|\vec{b}|^2$
$+ 7\vec{a} \cdot \vec{b}$

b. $3\vec{x} \cdot (\vec{x} - 3\vec{y}) - (\vec{x} - 3\vec{y}) \cdot (-3\vec{x} + \vec{y})$
$= 3|\vec{x}|^2 - 3\vec{x} \cdot 3\vec{y} + 3|\vec{x}|^2 - \vec{x} \cdot \vec{y} - (-3\vec{y} \cdot -3\vec{x})$
$+ 3|\vec{y}|^2$
$= 6|\vec{x}|^2 - 9\vec{x} \cdot \vec{y} - \vec{x} \cdot \vec{y} - 9\vec{x} \cdot \vec{y} + 3|\vec{y}|^2$
$= 6|\vec{x}|^2 - 19\vec{x} \cdot \vec{y} + 3|\vec{y}|^2$

10. $|\vec{0}| = 0$ so the dot product of any vector with $\vec{0}$ is 0.

11. $(\vec{a} - 5\vec{b}) \cdot (\vec{a} - \vec{b}) = |\vec{a} - 5\vec{b}||\vec{a} - \vec{b}|\cos(90°)$

$$|\vec{a}|^2 - \vec{a} \cdot \vec{b} - 5\vec{b} \cdot \vec{a} + 5|\vec{b}|^2 = 0$$

$$|\vec{a}|^2 + 5|\vec{b}|^2 = 6\vec{a} \cdot \vec{b}$$

$$\vec{a} \cdot \vec{b} = \frac{1}{6}(|\vec{a}|^2 + 5|\vec{b}|^2)$$

$$= 1$$

12. a. $(\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) = \vec{a} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b}$

$$= |\vec{a}|^2 + \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{b} + |\vec{b}|^2$$

$$= |\vec{a}|^2 + 2\vec{a} \cdot \vec{b} + |\vec{b}|^2$$

b. $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = \vec{a} \cdot \vec{a} - \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{a} - \vec{b} \cdot \vec{b}$

$$= |\vec{a}|^2 - \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{b} - |\vec{b}|^2$$

$$= |\vec{a}|^2 - |\vec{b}|^2$$

13. a. $|\vec{a}|^2 = \vec{a} \cdot \vec{a}$

$$= (\vec{b} + \vec{c}) \cdot (\vec{b} + \vec{c})$$

$$= |\vec{b}|^2 + 2\vec{b} \cdot \vec{c} + |\vec{c}|^2$$

b. $\vec{b} \cdot \vec{c} = |\vec{b}||\vec{c}|\cos(90°) = 0$

Therefore $|\vec{a}|^2 = |\vec{b}|^2 + |\vec{c}|^2$.

This is just what the Pythagorean theorem says, where $\vec{b}$ and $\vec{c}$ are the legs of the right triangle.

14. $(\vec{u} + \vec{v} + \vec{w}) \cdot (\vec{u} + \vec{v} + \vec{w})$

$$= \vec{u} \cdot \vec{u} + \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{v} + \vec{v} \cdot \vec{w} + \vec{w} \cdot \vec{u} + \vec{w} \cdot \vec{v} + \vec{w} \cdot \vec{w}$$

$$= |\vec{u}|^2 + |\vec{v}|^2 + |\vec{w}|^2 + 2|\vec{u}||\vec{v}|\cos(90°) + 2|\vec{u}||\vec{w}|\cos(90°) + 2|\vec{v}||\vec{w}|\cos(90°)$$

$$= (1)^2 + (2)^2 + (3)^2$$

$$= 14$$

15. $|\vec{u} + \vec{v}|^2 + |\vec{u} - \vec{v}|^2$

$$= (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) + (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v})$$

$$= |\vec{u}|^2 + 2\vec{u} \cdot \vec{v} + |\vec{v}|^2 + |\vec{u}|^2 - 2\vec{u} \cdot \vec{v} + |\vec{v}|^2$$

$$= 2|\vec{u}|^2 + 2|\vec{v}|^2$$

16. $(\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b} + \vec{c})$

$$= |\vec{a}|^2 + \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{a} + |\vec{b}|^2 + \vec{b} \cdot \vec{c}$$

$$= 1 + 2|\vec{a}||\vec{b}|\cos(60°) + |\vec{a}||\vec{c}|\cos(60°) + 1 + |\vec{b}||\vec{c}|\cos(120°)$$

$$= 2 + 2\left(\frac{1}{2}\right) + \frac{1}{2} - \frac{1}{2}$$

$$= 3$$

17. $\vec{a} + \vec{b} + \vec{c} = \vec{0}$

$$\vec{a} \cdot (\vec{a} + \vec{b} + \vec{c}) + \vec{b} \cdot (\vec{a} + \vec{b} + \vec{c}) + \vec{c} \cdot (\vec{a} + \vec{b} + \vec{c}) = 0$$

$$|\vec{a}|^2 + \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{a} + |\vec{b}|^2 + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} + \vec{c} \cdot \vec{b} + |\vec{c}|^2 = 0$$

$$1 + 4 + 9 + 2(\vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c}) = 0$$

$$2(\vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c}) = -14$$

$$\vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c} = -7$$

18. $\vec{d} = \vec{b} - \vec{c}$

$\vec{b} = \vec{d} + \vec{c}$

$\vec{c} \cdot \vec{a} = ((\vec{b} \cdot \vec{a})\vec{a}) \cdot \vec{a}$

$\vec{c} \cdot \vec{a} = (\vec{b} \cdot \vec{a})(\vec{a} \cdot \vec{a})$ because $\vec{b} \cdot \vec{a}$ is a scalar

$\vec{c} \cdot \vec{a} = (\vec{b} \cdot \vec{a})|\vec{a}|^2$

$\vec{c} \cdot \vec{a} = (\vec{d} + \vec{c}) \cdot \vec{a}$ because $|\vec{a}| = 1$

$\vec{c} \cdot \vec{a} = \vec{d} \cdot \vec{a} + \vec{c} \cdot \vec{a}$

$\vec{d} \cdot \vec{a} = 0$

7.4 The Dot Product for Algebraic Vectors, pp. 385–387

1. $\vec{a} \cdot \vec{b} = 0$

$$(-1)b_1 + b_2 = 0$$

$$b_2 = b_1$$

Any vector of the form (c, c) is perpendicular to $\vec{a}$. Therefore there are infinitely many vectors perpendicular to $\vec{a}$. Answers may vary. For example: (1, 1), (2, 2), (3, 3).

2. a. $\vec{a} \cdot \vec{b} = (-2)(1) + (1)(2)$

$$= 0$$

$$\theta = 90°$$

b. $\vec{a} \cdot \vec{b} = (2)(4) + (3)(3) + (-1)(-17)$

$$= 8 + 9 + 17$$

$$= 34 > 0$$

$\cos\theta > 0$

θ is acute

c. $\vec{a} \cdot \vec{b} = (1)(3) + (-2)(-2) + (5)(-2)$

$$= 3 + 4 - 10$$

$$= -3 < 0$$

$\cos\theta < 0$

θ is obtuse

3. Any vector in the xy-plane is of the form $\vec{a} = (a_1, a_2, 0)$. Let $\vec{b} = (0, 0, 1)$.

$$\vec{a} \cdot \vec{b} = (0)(a_1) + (0)(a_2) + (0)(1)$$

$$= 0$$

Therefore (0, 0, 1) is perpendicular to every vector in the xy-plane.

Any vector in the xz-plane is of the form $\vec{c} = (c_1, 0, c_3)$. Let $\vec{d} = (0, 1, 0)$.

$$\vec{c} \cdot \vec{d} = (0)(c_1) + (0)(1) + (0)(c_3)$$

$$= 0$$

Therefore (0, 1, 0) is perpendicular to every vector in the xz-plane.

Any vector in the yz-plane is of the form $\vec{e} = (0, e_2, e_3)$. Let $\vec{f} = (1, 0, 0)$.

$\vec{e} \cdot \vec{f} = (1)(0) + (0)(e_2) + (0)(e_3)$
$= 0$

Therefore $(1, 0, 0)$ is perpendicular to every vector in the yz-plane.

4. a. $(1, 2, -1) \cdot (4, 3, 10) = 4 + 6 - 10$
$= 0$

$$(-4, -5, -6) \cdot \left(5, -3, -\frac{5}{6}\right) = -20 + 15 + 5 = 0$$

b. If any of the vectors were collinear then one would be a scalar multiple of the other. Comparing the signs of the individual components of each vector eliminates $(1, 2, -1)$ and $\left(5, -3, -\frac{5}{6}\right)$. All of the components of $(-4, -5, -6)$ have the same sign and the same is true for $(4, 3, 10)$, but $(4, 3, 10)$ is not a scalar multiple of $(-4, -5, -6)$. Therefore none of the vectors are collinear.

5. a. Using the strategy of Example 5 yields
$(x, y) \cdot (1, -2) = 0$ and $(x, y) \cdot (1, 1) = 0$
$x - 2y = 0$ and $x + y = 0$
$3y = 0$
Therefore the only result is $x = y = 0$, or $(0, 0)$. This is because $(1, -2)$ and $(1, 1)$ both lie on the xy-plane and are not collinear, so any vector that is perpendicular to both vectors must be in R^3 which does not exist in R^2.

b. If we select any two vectors that are not collinear in R^2, then any vector that is perpendicular to both cannot be in R^2 and must be in R^3. This is not possible since R^3 does not exist in R^2.

6. a.
$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{(5)(-1) + (3)(-2)}{\sqrt{25 + 9}\sqrt{1 + 4}} = \frac{-11}{\sqrt{(34)(5)}} = \frac{-11}{\sqrt{170}}$$
$\theta \doteq 148°$

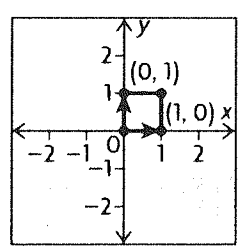
b.
$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{(-1)(6) + (4)(-2)}{\sqrt{1 + 16}\sqrt{36 + 4}} = \frac{-14}{\sqrt{680}}$$
$\theta \doteq 123°$

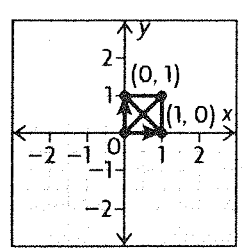
c.
$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{(2)(2) + (2)(1) + (1)(-2)}{\sqrt{4 + 4 + 1}\sqrt{4 + 1 + 4}} = \frac{4}{(3)(3)} = \frac{4}{9}$$
$\theta \doteq 64°$

d.
$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{(2)(-5) + (3)(0) + (-6)(12)}{\sqrt{4 + 9 + 36}\sqrt{25 + 144}} = \frac{-82}{(7)(13)} = \frac{-82}{91}$$
$\theta \doteq 154°$

7. a.
$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$$
$$(-1)(-6k) + (2)(-1) + (-3)(k) = |\vec{a}||\vec{b}|\cos(90°)$$
$$6k - 2 - 3k = 0$$
$$3k = 2$$
$$k = \frac{2}{3}$$

b.
$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$$
$$(1)(0) + (1)(k) = \sqrt{1 + 1}\sqrt{k^2}\cos(45°)$$
$$k = \sqrt{2}|k|\frac{1}{\sqrt{2}}$$
$$k = |k|$$
$$k \geq 0$$

8. a.


b.


The diagonals are $(1, 0) + (0, 1) = (1, 1)$ and
$(1, 0) - (0, 1) = (1, -1)$ or
$(1, 0) + (0, 1) = (1, 1)$ and
$(0, 1) - (1, 0) = (-1, 0)$.

c. $(1, 1) \cdot (1, -1)$
$= 1 - 1$
$= 0$
or $(1, 1) \cdot (-1, 1)$
$= -1 + 1$
$= 0$

9. a. $\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$

$$= \frac{(1 - \sqrt{2})(1) + (\sqrt{2} - 1)(1)}{|\vec{a}||\vec{b}|}$$
$$= 0$$
$$\theta = 90^\circ$$

b. $\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$

$$= \frac{\sqrt{2} - 1 + \sqrt{2} + 1 + \sqrt{2}}{\sqrt{(2 - 2\sqrt{2} + 1) + (2 + 2\sqrt{2} + 1) + 2}\sqrt{1 + 1 + 1}}$$
$$= \frac{3\sqrt{2}}{\sqrt{8}\sqrt{3}}$$
$$= \frac{\sqrt{3}}{2}$$
$$\theta = 30^\circ$$

10. a. i. $\vec{a} = k\vec{b}$
$$8 = 12k$$
$$k = \frac{2}{3}$$
$$p = 4\left(\frac{2}{3}\right)$$
$$p = \frac{8}{3}$$
$$2 = \frac{2}{3}q$$
$$q = 3$$

ii. Answers may vary. For example:
$$\vec{a} \cdot \vec{b} = 0$$
$$2q + 4p + 96 = 0$$
$$q = -2p - 48$$
Let $p = 1$
$$q = -50$$

b. In part **a.**, the values are unique because both vectors have their third component specified, and the ratios must be the same for each component $\vec{b}$. In part **b.** the values are not unique; any value of p could have been chosen, each resulting in a different value of q.

11. $\overrightarrow{AB} = (2, 6)$, $\overrightarrow{BC} = (-5, -5)$, $\overrightarrow{CA} = (3, -1)$

$$\cos(180^\circ - \theta_A) = \frac{\overrightarrow{AB} \cdot \overrightarrow{CA}}{|\overrightarrow{AB}||\overrightarrow{CA}|}$$
$$= \frac{6 - 6}{|\overrightarrow{AB}||\overrightarrow{CA}|}$$
$$= 0$$
$$180^\circ - \theta_A = 90^\circ$$
$$\theta_A = 90^\circ$$
$$\cos(180^\circ - \theta_B) = \frac{\overrightarrow{AB} \cdot \overrightarrow{BC}}{|\overrightarrow{AB}||\overrightarrow{BC}|}$$
$$= \frac{-10 - 30}{\sqrt{4 + 36}\sqrt{25 + 25}}$$
$$= \frac{-40}{\sqrt{(40)(50)}}$$
$$= -\sqrt{\frac{4}{5}}$$
$$180^\circ - \theta_B \doteq 153.4^\circ$$
$$\theta_B \doteq 26.6^\circ$$
$$\theta_C = 180^\circ - \theta_A - \theta_B$$
$$\theta_C \doteq 63.4^\circ$$

12. a. $O = (0, 0, 0)$, $A = (7, 0, 0)$, $B = (7, 4, 0)$, $C = (0, 4, 0)$, $D = (7, 0, 5)$, $E = (0, 4, 5)$, $F = (0, 0, 5)$

b. $\overrightarrow{AE} \cdot \overrightarrow{BF} = |\overrightarrow{AE}||\overrightarrow{BF}|\cos\theta$
$$(-7, 4, 5) \cdot (-7, -4, 5) = \sqrt{49 + 16 + 25} \times \sqrt{49 + 16 + 25}\cos\theta$$
$$49 - 16 + 25 = 90\cos\theta$$
$$\frac{58}{90} = \cos\theta$$
$$\theta \doteq 50^\circ$$

13. a. Answers may vary. For example:
$$(x, y, z) \cdot (-1, 3, 0) = 0$$
$$-x + 3y = 0$$
$$x = 3y$$
$$(x, y, z) \cdot (1, -5, 2) = 0$$
$$x - 5y + 2z = 0$$
$$-2y + 2z = 0$$
$$y = z$$
Let $y = 1$.
$(3, 1, 1)$ is perpendicular to $(-1, 3, 0)$ and $(1, -5, 2)$.

b. Answers may vary. For example:
$$(x, y, z) \cdot (1, 3, -4) = 0$$
$$x + 3y - 4z = 0$$
$$x = 4z - 3y$$
$$(x, y, z) \cdot (-1, -2, 3) = 0$$

$$-x - 2y + 3z = 0$$
$$3y - 4z - 2y + 3z = 0$$
$$y = z$$
$$\text{Let } y = 1.$$

(1, 1, 1) is perpendicular to (1, 3, −4) and (−1, −2, 3).

14. $(p, p, 1) \cdot (p, -2, -3) = 0$
$$p^2 - 2p - 3 = 0$$
$$p = \frac{2 \pm \sqrt{2^2 - 4(-3)}}{2}$$
$$p = 1 \pm 2$$
$$p = 3 \text{ or } -1$$

15. a. $(-3, p, -1) \cdot (1, -4, q) = 0$
$$-3 - 4p - q = 0$$
$$3 + 4p + q = 0$$

b. $3 + 4p - 3 = 0$
$$p = 0$$

16. Answers may vary. For example: Note that $\vec{s} = -2\vec{r}$, so they are collinear. Therefore any vector that is perpendicular to $\vec{s}$ is also perpendicular to $\vec{r}$.

$(x, y, z) \cdot (1, 2, -1) = 0$
$$x + 2y - z = 0$$
Let $x = z = 1$.

(1, 0, 1) is perpendicular to (1, 2, −1) and (−2, −4, 2).

Let $x = y = 1$.

(1, 1, 3) is perpendicular to (1, 2 − 1) and (−2, −4, 2).

17. $\vec{x} \cdot \vec{y} = |\vec{x}||\vec{y}|\cos\theta$

$(-4, p, -2) \cdot (-2, 3, 6)$

$= \sqrt{16 + p^2 + 4}\sqrt{4 + 9 + 36}\cos\theta$
$$8 + 3p - 12 = \sqrt{20 + p^2}(7)\cos\theta$$
$$(3p - 4)^2 = \left(7\sqrt{20 + p^2}\cos\theta\right)^2$$
$$9p^2 - 24p + 16 = 49(20 + p^2)\left(\frac{4}{21}\right)^2$$
$$9p^2 - 24p + 16 = \frac{320}{9} + \frac{16}{9}p^2$$
$$65p^2 - 216p - 176 = 0$$
$$p = \frac{216 \pm \sqrt{(-216)^2 - 4(65)(-176)}}{2(65)}$$
$$p = 4 \text{ or } -\frac{44}{65}$$

18. a. $\vec{a} \cdot \vec{b} = -3 + 3$
$$= 0$$
Therefore, since the two diagonals are perpendicular, all the sides must be the same length.

b. $\overrightarrow{AB} = \frac{1}{2}(\vec{a} + \vec{b})$
$$= (1, 2, -1)$$
$$\overrightarrow{BC} = \frac{1}{2}(\vec{a} - \vec{b})$$
$$= (2, 1, 1)$$
$$|\overrightarrow{AB}| = |\overrightarrow{BC}| = \sqrt{6}$$

c. $\overrightarrow{AB} \cdot \overrightarrow{BC} = |\overrightarrow{AB}||\overrightarrow{BC}|\cos\theta_1$
$$2 + 2 - 1 = 6\cos\theta_1$$
$$\frac{1}{2} = \cos\theta_1$$
$$\theta_1 = 60°$$
$$2\theta_1 + 2\theta_2 = 360°$$
$$\theta_2 = 120°$$

19. a. $\overrightarrow{AB} = (3, 4, -12)$, $\overrightarrow{DA} = (-4, 2 - q, -5)$
$$\overrightarrow{AB} \cdot \overrightarrow{DA} = 0$$
$$-12 + 8 - 4q + 60 = 0$$
$$-1 - q + 15 = 0$$
$$q = 14$$
$$\overrightarrow{DA} = \overrightarrow{CB}$$
$$(-4, -12, -5) = (2 - x, 6 - y, -9 - z)$$
$x = 6, y = 18, z = -4$

The coordinates of vertex C are (6, 18, −4).

b. $\overrightarrow{AC} \cdot \overrightarrow{BD} = |\overrightarrow{AC}||\overrightarrow{BD}|\cos\theta$
$$(7, 16, -7) \cdot (1, 8, 17) = \sqrt{49 + 256 + 49} \times \sqrt{1 + 64 + 289}\cos\theta$$
$$7 + 128 - 119 = 354\cos\theta$$
$$\frac{16}{354} = \cos\theta$$
$$\theta \doteq 87.4°$$

20. The two vectors representing the body diagonals are $(0 - 1, 1 - 0, 1 - 0) = (-1, 1, 1)$ and $(0 - 1, 0 - 1, 1 - 0) = (-1, -1, 1)$

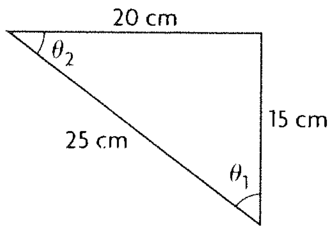
$(-1, 1, 1) \cdot (-1, -1, 1) = \sqrt{3}\sqrt{3}\cos\theta$
$$1 - 1 + 1 = 3\cos\theta$$
$$\frac{1}{3} = \cos\theta$$
$$\theta \doteq 70.5°$$
$$\alpha = 180° - \theta$$
$$\alpha \doteq 109.5°$$

Mid-Chapter Review, pp. 388–389

1. a. $\vec{a} \cdot \vec{b} = (3)(2)\cos(60°)$
$$= (6)\frac{1}{2}$$
$$= 3$$

b. $(3\vec{a} + 2\vec{b}) \cdot (4\vec{a} - 3\vec{b}) = 12|\vec{a}|^2 - 9\vec{a} \cdot \vec{b}$
$+ 8\vec{b} \cdot \vec{a} - 6|\vec{b}|^2$
$= 12(3)^2 - 3 - 6(2)^2$
$= 81$

2.



Let T_1 be the tension in the 15 cm cord and T_2 be the tension in the 20 cm cord. Let θ_1 be the angle the 15 cm cord makes with the ceiling and θ_2 be the angle the 20 cm cord makes with the ceiling. By the cosine law:

$(15)^2 = (20)^2 + (25)^2 - 2(20)(25)\cos(\theta_2)$
$\cos(\theta_2) = 0.8$
$\sin(\theta_2) = \sqrt{1 - \cos^2(\theta_2)}$
$\sin(\theta_2) = 0.6$
$(20)^2 = (15)^2 + (25)^2 - (2)(15)(25)\cos(\theta_1)$
$\cos(\theta_1) = 0.6$
$\sin(\theta_1) = 0.8$

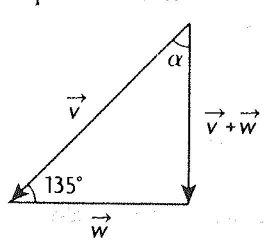
Horizontal Components:
$-T_1 \cos(\theta_1) + T_2 \cos(\theta_2) = 0$
$(0.8)T_2 = (0.6)T_1$
$T_2 = (0.75)T_1$

Vertical Components:
$T_1 \sin(\theta_1) + T_2 \sin(\theta_2) - (15)(9.8) = 0$
$(0.8)T_1 + (0.6)(0.75)T_1 = 147$
$(1.25)T_1 = 147$
$T_1 = 117.6$ N
$T_2 = (0.75)T_1$
$T_2 = 88.2$ N

Therefore the tension in the 15 cm cord is 117.60 N and the tension in the 20 cm cord is 88.20 N.

3. The diagonals of a square are perpendicular, so the dot product is 0.

4. a.



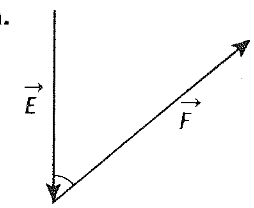
$|\vec{v}| = 500, |\vec{w}| = 100$
By the cosine law:
$|\vec{v} + \vec{w}|^2 = (500)^2 + (100)^2$
$- 2(500)(100)\cos(135°)$
$|\vec{v} + \vec{w}| \doteq 575.1$

By the cosine law:
$\frac{\sin(\alpha)}{100} = \frac{\sin(135°)}{575.1}$
$\sin(\alpha) \doteq 0.123$
$\alpha \doteq 7.06°$

The resultant velocity of the airplane is 575.1 km/h at $S7.06°E$

b. (distance) = (rate)(time)
$t \doteq \frac{1000}{575.1} \cdot \frac{\text{km}}{(\text{km/h})}$
$t \doteq 1.74$ hours

5. a.



$|\vec{E}_\perp| = |\vec{E}|\cos(40°)$
$|\vec{E}_\perp| = (9.8)(15)\cos(40°)$
$|\vec{E}_\perp| \doteq 112.61$ N

b. $|\vec{F}| = |\vec{E}|\sin(40°)$
$|\vec{F}| \doteq 94.49$ N

6. $6\theta = 360°$
$\theta = 60°$
$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos(60°)$
$= (3)(3)(0.5)$
$= 4.5$

7. a. $\vec{a} \cdot \vec{b} = (4)(1) + (-5)(2) + (20)(2)$
$= 34$

b. $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos(\theta)$
$34 = \sqrt{16 + 25 + 400}\sqrt{1 + 4 + 4}\cos(\theta)$
$\cos(\theta) = \frac{34}{63}$

8. a. $\vec{a} \cdot \vec{b} = (\vec{i} + 2\vec{j} + \vec{k}) \cdot (2\vec{i} - 3\vec{j} + 4\vec{k})$
$= 2 - 6 + 4$
$= 0$

b. $\vec{b} \cdot \vec{c} = (2\vec{i} - 3\vec{j} + 4\vec{k}) \cdot (3\vec{i} - \vec{j} - \vec{k})$
$= 6 + 3 - 4$
$= 5$

c. $\vec{b} + \vec{c} = (2\vec{i} - 3\vec{j} + 4\vec{k}) + (3\vec{i} - \vec{j} - \vec{k})$
$= 5\vec{i} - 4\vec{j} + 3\vec{k}$

d. $\vec{a} \cdot (\vec{b} + \vec{c}) = (\vec{i} + 2\vec{j} + \vec{k}) \cdot (5\vec{i} - 4\vec{j} + 3\vec{k})$
$= 5 - 8 + 3$
$= 0$

e. $(\vec{a} + \vec{b}) \cdot (\vec{b} + \vec{c}) = (3\vec{i} - \vec{j} + 5\vec{k})$
$\cdot (5\vec{j} - 4\vec{j} + 3\vec{k})$
$= 15 + 4 + 15$
$= 34$

f. $(2\vec{a} - 3\vec{b}) \cdot (2\vec{a} + \vec{c}) = ((2\vec{i} + 4\vec{j} + 2\vec{k})$
$- (6\vec{i} - 9\vec{j} + 12\vec{k}))$
$\cdot ((2\vec{i} + 4\vec{j} + 2\vec{k})$
$+ (3\vec{i} - \vec{j} + \vec{k}))$
$= (-4\vec{i} + 13\vec{j} - 10\vec{k})$
$\cdot (5\vec{i} + 3\vec{j} + \vec{k})$
$= -20 + 39 - 10$
$= 9$

9. a. $\vec{p} \cdot \vec{q} = 0$

$(x\vec{i} + \vec{j} + 3\vec{k}) \cdot (3x\vec{i} + 10x\vec{j} + \vec{k}) = 0$

$3x^2 + 10x + 3 = 0$

$$x = \frac{-10 \pm \sqrt{(10)^2 - 4(3)(3)}}{2(3)}$$

$$x = \frac{-10 \pm 8}{6}$$

$$x = -3 \text{ or } x = -\frac{1}{3}$$

b. If $\vec{p}$ and $\vec{q}$ are parallel then one is a scalar multiple of the other.

$\vec{p} = n\vec{q}$ where n is a constant

$x\vec{i} + \vec{j} + 3\vec{k} = n(3x\vec{i} + 10x\vec{j} + \vec{k})$

$n = 3$ by the $\vec{k}$ component

$x = 9x$ by the $\vec{i}$ component

$x = 0$

$1 = 30(0)$ by the $\vec{j}$ component

$1 \neq 0$

Therefore there is no value of x that will make these two vectors parallel.

10. a. $3\vec{x} - 2\vec{y} = (3\vec{i} - 6\vec{j} - 3\vec{k}) - (2\vec{i} - 2\vec{j} - 2\vec{k})$
$= \vec{i} - 4\vec{j} - \vec{k}$

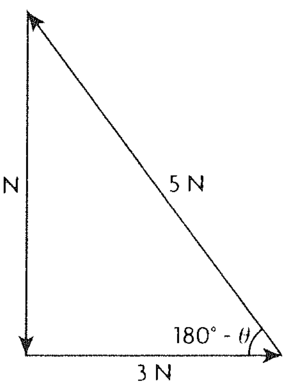
b. $3\vec{x} \cdot 2\vec{y} = (3\vec{i} - 6\vec{j} - 3\vec{k}) \cdot (2\vec{i} - 2\vec{j} - 2\vec{k})$
$= 6 + 12 + 6$
$= 24$

c. $|\vec{x} - 2\vec{y}| = |(\vec{i} - 2\vec{j} - \vec{k}) - (2\vec{i} - 2\vec{j} - 2\vec{k})|$
$= |-\vec{i} + \vec{k}|$
$= \sqrt{(-\vec{i} + \vec{k}) \cdot (-\vec{i} + \vec{k})}$
$= \sqrt{2}$ or 1.41

d. $(2\vec{x} - 3\vec{y}) \cdot (\vec{x} + 4\vec{y}) = ((2\vec{i} - 4\vec{j} - 2\vec{k})$
$- (3\vec{i} - 3\vec{j} - 3\vec{k})) \cdot$
$+ ((\vec{i} - 2\vec{j} - \vec{k})$
$+ (4\vec{i} - 4\vec{j} - 4\vec{k})$
$= (-\vec{i} - \vec{j} + \vec{k})$
$\cdot (5\vec{i} - 6\vec{j} - 5\vec{k})$
$= -5 + 6 - 5$
$= -4$

e. $2\vec{x} \cdot \vec{y} - 5\vec{y} \cdot \vec{x} = 2\vec{x} \cdot \vec{y} - 5\vec{x} \cdot \vec{y}$
$= -3\vec{x} \cdot \vec{y}$
$= -3(\vec{i} - 2\vec{j} - \vec{k}) \cdot (\vec{i} - \vec{j} - \vec{k})$
$= -3(1 + 2 + 1)$
$= -12$

11.



$(4)^2 = (5)^2 + (3)^2 - 2(3)(5)\cos(180° - \theta)$

$0.6 = \cos(180° - \theta)$

$180° - \theta \doteq 53.1$

$\theta \doteq 126.9°$

12. $(F)^2 = (3)^2 + (4)^2 - 2(3)(4)\cos(180° - 60°)$

$(F)^2 = 25 - 24\cos(120°)$

$(F)^2 = 37$

$F \doteq 6.08$ N

$(3)^2 = (4)^2 + (\sqrt{37})^2 - 2(4)(\sqrt{37})\cos\theta$

$$\cos\theta = \frac{44}{8\sqrt{37}}$$

$\theta \doteq 25.3°$

$\vec{F} \doteq 6.08$ N, 25.3° from the 4 N force towards the 3 N force.

$\vec{E} \doteq 6.08$ N, 180° − 25.3° = 154.7° from the 4 N force away from the 3 N force.

13. a. The diagonals are $\vec{m} + \vec{n}$ and $\vec{m} - \vec{n}$

$\vec{m} + \vec{n} = (1, 4, 10)$

$\vec{m} - \vec{n} = (3, -10, 0)$

$(\vec{m} + \vec{n}) \cdot (\vec{m} - \vec{n}) = |\vec{m} + \vec{n}||\vec{m} - \vec{n}|\cos\theta$

$3 - 40 = \sqrt{1 + 16 + 100}\sqrt{9 + 100}\cos\theta$

$\cos\theta \doteq -0.3276$

$\theta \doteq 109.1°$

b. $|\vec{m} - \vec{n}|^2 = |\vec{m}|^2 + |\vec{n}|^2 - 2|\vec{m}||\vec{n}|\cos\theta$

$(9 + 100) = (4 + 9 + 25) + (1 + 49 + 25)$
$- 2\sqrt{38}\sqrt{75}\cos\theta$

$\cos\theta \doteq 0.0374$

$\theta \doteq 87.9°$

14. a. $45 \sin(150°) = 500 \sin\theta$
$\theta \doteq \text{N } 2.6° \text{ E}$

b. $v = 500\cos(2.6°) - 45\cos(30°)$
$\doteq 460.5$ km/h
$t \doteq \dfrac{1000}{460.5}$
$t \doteq 2.17$ hours

15. $\vec{a} \cdot \vec{x} = 0$
$-x_1 + 2x_2 + 5x_3 = 0$
$x_1 = 2x_2 + 5x_3$
$\vec{b} \cdot \vec{x} = 0$
$x_1 + 3x_2 + 5x_3 = 0$
$2x_2 + 5x_3 + 3x_2 + 5x_3 = 0$
$x_2 + 2x_3 = 0$
choose $x_3 = 1$
$x_2 = -2$
$x_1 = 1$
$\vec{x} = \dfrac{1}{\sqrt{6}}(1, -2, 1)$
$\vec{x} = \left(\dfrac{1}{\sqrt{6}}, -\dfrac{2}{\sqrt{6}}, \dfrac{1}{\sqrt{6}}\right)$ or $\left(-\dfrac{1}{\sqrt{6}}, \dfrac{2}{\sqrt{6}}, -\dfrac{1}{\sqrt{6}}\right)$

16. a. $v = 4 + 3\cos(45°)$
$\doteq 6.12$ m/s
$d \doteq (6.12)(10)$
$\doteq 61.2$ m

b. $w = 3\sin(45°)$
$\doteq 2.12$ m/s
$t \doteq \dfrac{180}{2.12}$
$t \doteq 84.9$ seconds

17. a. $(\vec{x} + \vec{y}) \cdot (\vec{x} - \vec{y}) = 0$
$|\vec{x}|^2 - \vec{x} \cdot \vec{y} + \vec{y} \cdot \vec{x} - |\vec{y}|^2 = 0$
$|\vec{x}|^2 = |\vec{y}|^2$
$(\vec{x} + \vec{y}) \cdot (\vec{x} - \vec{y}) = 0$ when $\vec{x}$ and $\vec{y}$ have the same length.

b. Vectors $\vec{a}$ and $\vec{b}$ determine a parallelogram. Their sum $\vec{a} + \vec{b}$ is one diagonal of the parallelogram formed, with its tail in the same location as the tails of $\vec{a}$ and $\vec{b}$. Their difference $\vec{a} - \vec{b}$ is the other diagonal of the parallelogram.

18. $|\vec{F}| = 350\cos(40°)$
$\doteq 268.12$ N

7.5 Scalar and Vector Projections, pp. 398–400

1. a. Scalar projection of $\vec{a}$ on $\vec{b}$ is $\dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$ where $\vec{a} = (2, 3)$ and $\vec{b}$ is the positive x-axis $(X, 0)$.

$\vec{a} \cdot \vec{b} = (2X) + (3 \times 0)$
$= 2X + 0$
$= 2X$
$|\vec{b}| = \sqrt{X^2 + 0^2}$
$= X$
$\dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = \dfrac{2X}{X}$
$= 2;$

The vector projection is the scalar projection multiplied by $\dfrac{\vec{b}}{|\vec{b}|}$ where $\dfrac{\vec{b}}{|\vec{b}|}$ is the x-axis divided by the magnitude of the x-axis which is equal to $\vec{i}$. The scalar projection of 2 multiplied by $\vec{i}$ equals $2\vec{i}$.

b. Scalar projection of $\vec{a}$ on $\vec{b}$ is $\dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$ where $\vec{a} = (2, 3)$ and $\vec{b}$ is now the positive y-axis $(0, Y)$.

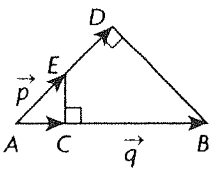
$\vec{a} \cdot \vec{b} = (2 \times 0) + (3Y)$
$= 0 + 3Y$
$= 3Y$
$|\vec{b}| = \sqrt{0^2 + Y^2}$
$= Y$
$\dfrac{\vec{a}|\vec{b}|}{|\vec{b}|} = \dfrac{3Y}{Y}$
$= 3;$

The vector projection is the scalar projection multiplied by $\dfrac{\vec{b}}{|\vec{b}|}$ where $\dfrac{\vec{b}}{|\vec{b}|}$ is the y-axis divided by the magnitude of the y-axis which is equal to $\vec{j}$. The scalar projection of 3 multiplied by $\vec{j}$ equals $3\vec{j}$.

2. Using the formula would cause a division by 0. Generally the $\vec{0}$ has any direction and 0 magnitude. You can not project onto nothing.

3. You are projecting $\vec{a}$ onto the tail of $\vec{b}$ which is a point with magnitude 0. Therefore it is $\vec{0}$; the projections of $\vec{b}$ onto the tail of $\vec{a}$ are also 0 and $\vec{0}$.

4. Answers may vary. For example: $\vec{p} = \overrightarrow{AE}$, $\vec{q} = \overrightarrow{AB}$



Scalar projection $\vec{p}$ on $\vec{q} = |\overrightarrow{AC}|$;
Vector projection $\vec{p}$ on $\vec{q} = \overrightarrow{AC}$;
Scalar projection $\vec{q}$ on $\vec{p} = |\overrightarrow{AD}|$;
Vector projection $\vec{q}$ on $\vec{p} = \overrightarrow{AD}$

5. When $\vec{a} = (-1, 2, 5)$ and $\vec{b} = (1, 0, 0)$ then

$$\vec{a} \cdot \vec{b} = (-1 \times 1 + 2 \times 0 + 5 \times 0) = -1$$

$$|\vec{b}| = \sqrt{1^2 + 0^2 + 0^2} = 1$$

Therefore the scalar projection is $\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = \frac{-1}{1} = -1$;

The vector equation is $-1 \times \frac{\vec{b}}{|\vec{b}|} = -1 \times \frac{(1, 0, 0)}{1} = -1$;

Under the same approach, when $\vec{a} = (-1, 2, 5)$ and $\vec{b} = (0, 1, 0)$, then

$$\vec{a} \cdot \vec{b} = (-1 \times 0 + 2 \times 1 + 5 \times 0) = 2$$

$$|\vec{b}| = \sqrt{0^2 + 1 + 0^2} = 1$$

Therefore the scalar projection is $\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = \frac{2}{1} = 2$,

The vector equation is $2 \times \frac{\vec{b}}{|\vec{b}|} = 2 \times \frac{(0, 1, 0)}{1} = 2$;

The same is also true when $\vec{a} = (-1, 2, 5)$ and $\vec{b} = (0, 0, 1)$ then

$$\vec{a} \cdot \vec{b} = (-1 \times 0 + 2 \times 0 + 5 \times 1) = 5$$

$$|\vec{b}| = \sqrt{0^2 + 0^2 + 1^2} = 1$$

Therefore the scalar projection is $\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = \frac{5}{1} = 5$,

The vector equation is $5 \times \frac{\vec{b}}{|\vec{b}|} = 5 \times \frac{(0, 0, 1)}{1} = 5$;

Without having to use formulae, a projection of $(-1, 2, 5)$ on $\vec{i}, \vec{j}$, or $\vec{k}$ is the same as a projection of $(-1, 0, 0)$ on $\vec{i}$, $(0, 2, 0)$ on $\vec{j}$, and $(0, 0, 5)$ on $\vec{k}$ which intuitively yields the same result.

6. a.
$$\vec{p} \cdot \vec{q} = (3 \times -4) + (6 \times 5) + (-22 \times -20) = -12 + 30 + 440 = 458$$

$$|\vec{q}| = \sqrt{(-4)^2 + 5^2 + (-20)^2} = \sqrt{16 + 25 + 400} = \sqrt{441} = 21$$

Therefore the scalar projection is $\frac{\vec{p} \cdot \vec{q}}{|\vec{q}|} = \frac{458}{21}$.

$$\text{The vector equation} = \frac{458}{21} \times \frac{\vec{q}}{|\vec{q}|} = \frac{458}{21}\frac{(-4, 5, -20)}{21} = \frac{458}{441}(-4, 5, 20).$$

b. Direction angles for $\vec{p}$ where $\vec{p} = (a, b, c)$ include α, β, and γ.

$$\cos\alpha = \frac{a}{\sqrt{a^2 + b^2 + c^2}} = \frac{3}{\sqrt{3^2 + 6^2 + (-22)^2}} = \frac{3}{\sqrt{9 + 36 + 484}} = \frac{3}{\sqrt{529}} = \frac{3}{23},$$

Therefore $\alpha = \cos^{-1}\left(\frac{3}{23}\right) \doteq 82.5°$;

$$\cos\beta = \frac{b}{\sqrt{a^2 + b^2 + c^2}} = \frac{6}{\sqrt{3^2 + 6^2 + (-22)^2}} = \frac{6}{\sqrt{9 + 36 + 484}} = \frac{6}{\sqrt{529}} = \frac{6}{23},$$

Therefore $\beta = \cos^{-1}\left(\frac{6}{23}\right) \doteq 74.9°$;

$$\cos\gamma = \frac{c}{\sqrt{a^2+b^2+c^2}}$$
$$= \frac{-22}{\sqrt{3^2+6^2+(-22)^2}}$$
$$= \frac{-22}{\sqrt{9+36+484}}$$
$$= \frac{-22}{\sqrt{529}}$$
$$= \frac{-22}{23}.$$

Therefore $\gamma = \cos^{-1}\left(\frac{-22}{23}\right)$

$\doteq 163.0°$

7. a. $\vec{x}\cdot\vec{y} = (1\times 1) + (1\times -1)$
$= 1 + (-1)$
$= 0$

$|\vec{y}| = \sqrt{1^2 + (-1)^2}$
$= \sqrt{2}$

The scalar projection is $\frac{\vec{x}\cdot\vec{y}}{|\vec{y}|} = \frac{0}{\sqrt{2}}$
$= 0;$

The vector projection is $0 \times \frac{\vec{y}}{|\vec{y}|} = \vec{0}$

b. $\vec{x}\cdot\vec{y} = (2\times 1) + (2\sqrt{3}\times 0)$
$= 2$

$|\vec{y}| = \sqrt{1^2+0^2}$
$= 1$

The scalar projection is $\frac{\vec{x}\cdot\vec{y}}{|\vec{y}|} = \frac{2}{1}$
$= 2;$

The vector projection is $2\times\frac{\vec{y}}{|\vec{y}|} = 2\times\frac{(1,0)}{1}$
$= 2\vec{i}$

c. $\vec{x}\cdot\vec{y} = (2\times -5) + (5\times 12)$
$= -10 + 60$
$= 50$

$|\vec{y}| = \sqrt{(-5)^2 + 12^2}$
$= \sqrt{25+144}$
$= \sqrt{169}$
$= 13$

The scalar projection is $\frac{\vec{x}\cdot\vec{y}}{|\vec{y}|} = \frac{50}{13}.$

The vector projection is $\frac{50}{13}\times\frac{\vec{y}}{|\vec{y}|} = \frac{50}{13}\times\frac{(-5,12)}{13}$
$= \frac{50}{169}(-5,12)$

8. a. The scalar projection of $\vec{a}$ on the x-axis $(X,0,0)$ is $\frac{\vec{a}\cdot(X,0,0)}{|(X,0,0)|}$

$$\frac{\vec{a}\cdot(X,0,0)}{|(X,0,0)|} = \frac{(-1\times X)+(2\times 0)+(4\times 0)}{\sqrt{X^2+0^2+0^2}}$$
$$= \frac{-X}{X}$$
$$= -1;$$

The vector projection of $\vec{a}$ on the x-axis is
$$-1\times\frac{(X,0,0)}{\sqrt{X^2+0^2+0^2}} = -1\times\frac{(X,0,0)}{X}$$
$$= -\vec{i};$$

The scalar projection of $\vec{a}$ on the y-axis $(0,Y,0)$ is
$$\frac{\vec{a}\cdot(0,Y,0)}{|(0,Y,0)|} = \frac{(-1\times 0)+(2\times Y)+(4\times 0)}{\sqrt{0^2+Y^2+0^2}}$$
$$= \frac{2Y}{Y}$$
$$= 2$$

The vector projection of $\vec{a}$ on the y-axis is
$$2\times\frac{(0,Y,0)}{\sqrt{0^2+Y^2+0^2}} = 2\times\frac{(0,Y,0)}{Y}$$
$$= 2\vec{j};$$

The scalar projection of $\vec{a}$ on the z-axis $(0,0,Z)$ is
$$\frac{\vec{a}\cdot(0,0,Z)}{|(0,0,Z)|} = \frac{(-1\times 0)+(2\times 0)+(4\times Z)}{\sqrt{0^2+0^2+Z^2}}$$
$$= \frac{4Z}{Z}$$
$$= 4;$$

The vector projection of $\vec{a}$ on the z-axis is
$$4\times\frac{(0,0,Z)}{\sqrt{0^2+0^2+Z^2}} = 4\times\frac{(0,0,Z)}{Z}$$
$$= 4\vec{k}.$$

b. The scalar projection of $m\vec{a}$ on the x-axis $(X,0,0)$ is
$$\frac{m\vec{a}\cdot(X,0,0)}{|(X,0,0)|} = \frac{(-m\times X)+(2m\times 0)}{\sqrt{X^2+0^2+0^2}} + \frac{(4m\times 0)}{\sqrt{X^2+0^2+0^2}}$$
$$= \frac{-mX}{X}$$
$$= -m$$

The vector projection of $m\vec{a}$ on the x-axis is
$$-m\times\frac{(X,0,0)}{\sqrt{X^2+0^2+0^2}} = -m\times\frac{(X,0,0)}{X}$$
$$= -m\vec{i};$$

The scalar projection of $m\vec{a}$ on the y-axis $(0, Y, 0)$ is

$$\frac{m\vec{a} \cdot (0, Y, 0)}{|(0, Y, 0)|} = \frac{(-m \times 0) + (2m \times Y)}{\sqrt{0^2 + Y^2 + 0^2}} + \frac{(4m \times 0)}{\sqrt{0^2 + y^2 + 0^2}}$$
$$= \frac{2mY}{Y}$$
$$= 2m;$$

The vector projection of $m\vec{a}$ on the y-axis is

$$2m \times \frac{(0, Y, 0)}{\sqrt{0^2 + Y^2 + 0^2}} = 2m \times \frac{(0, Y, 0)}{Y}$$
$$= 2m\vec{j};$$

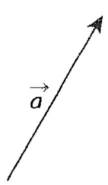
The scalar projection of $m\vec{a}$ on the z-axis $(0, 0, Z)$ is

$$\frac{m\vec{a} \cdot (0, 0, Z)}{|(0, 0, Z)|} = \frac{(-m \times 0) + (2m \times 0)}{\sqrt{0^2 + 0^2 + Z^2}} + \frac{(4m \times Z)}{\sqrt{0^2 + 0^2 + Z^2}}$$
$$= \frac{4mZ}{Z}$$
$$= 4m;$$

The vector projection of $m\vec{a}$ on the z-axis is

$$4m \times \frac{(0, 0, Z)}{\sqrt{0^2 + 0^2 + Z^2}} = 4m \times \frac{(0, 0, Z)}{Z}$$
$$= 4m\vec{k}.$$

9. a.



$\vec{a}$ projected onto itself will yield itself. The scalar projection will be the magnitude of itself.

b. Using the formula for the scalar projection

$$|\vec{a}|\cos\theta = |\vec{a}|\cos 0$$
$$= |\vec{a}|(1)$$
$$= |\vec{a}|.$$

The vector projection is the scalar projection multiplied by $\frac{\vec{a}}{|\vec{a}|}$, $|\vec{a}| \times \frac{\vec{a}}{|\vec{a}|} = \vec{a}$.

10. a.

b. $$\frac{(-\vec{a}) \cdot \vec{a}}{|\vec{a}|} = \frac{-|\vec{a}|^2}{|\vec{a}|}$$
$$= -|\vec{a}|$$

So the vector projection is $-|\vec{a}|\left(\frac{|\vec{a}|}{|\vec{a}|}\right) = -\vec{a}$.

11. a. $$\overrightarrow{AB} = \text{Point } B - \text{Point } A$$
$$= (-1, 3, 4) - (1, 2, 2)$$
$$= (-2, 1, 2)$$

The scalar projection of $\overrightarrow{AB}$ on the x-axis $(X, 0, 0)$ is

$$\frac{\vec{a} \cdot (X, 0, 0)}{|(X, 0, 0)|} = \frac{(-2 \times X) + (1 \times 0) + (2 \times 0)}{\sqrt{X^2 + 0^2 + 0^2}}$$
$$= \frac{-2X}{X}$$
$$= -2;$$

The vector projection of $\overrightarrow{AB}$ on the x-axis is

$$-2 \times \frac{(X, 0, 0)}{\sqrt{X^2 + 0^2 + 0^2}} = -2 \times \frac{(X, 0, 0)}{X}$$
$$= -2\vec{i};$$

The scalar projection of $\overrightarrow{AB}$ on the y-axis $(0, Y, 0)$ is

$$\frac{\vec{a} \cdot (0, Y, 0)}{|(0, Y, 0)|} = \frac{(-2 \times 0) + (1 \times Y) + (2 \times 0)}{\sqrt{0^2 + Y^2 + 0^2}}$$
$$= \frac{Y}{Y}$$
$$= 1;$$

The vector projection of $\overrightarrow{AB}$ on the y-axis is

$$1 \times \frac{(0, Y, 0)}{\sqrt{0^2 + Y^2 + 0^2}} = 1 \times \frac{(0, Y, 0)}{Y}$$
$$= \vec{j};$$

The scalar projection of $\overrightarrow{AB}$ on the z-axis $(0, 0, Z)$ is

$$\frac{\vec{a} \cdot (0, 0, Z)}{|(0, 0, Z)|} = \frac{(-2 \times 0) + (1 \times 0) + (2 \times Z)}{\sqrt{0^2 + 0^2 + Z^2}}$$
$$= \frac{2Z}{Z}$$
$$= 2;$$

The vector projection of $\overrightarrow{AB}$ on the z-axis is

$$2 \times \frac{(0, 0, Z)}{\sqrt{0^2 + 0^2 + Z^2}} = 2 \times \frac{(0, 0, Z)}{Z}$$
$$= 2\vec{k}$$

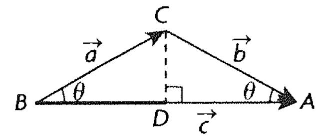
b. The angle made with the y-axis is β

$$\cos\beta = \frac{b}{\sqrt{a^2 + b^2 + c^2}}$$
$$= \frac{1}{\sqrt{(-2)^2 + 1^2 + 2^2}}$$
$$= \frac{1}{\sqrt{4 + 1 + 4}}$$
$$= \frac{1}{\sqrt{9}}$$
$$= \frac{1}{3}.$$

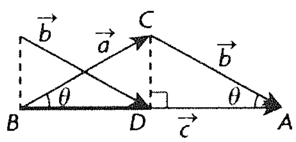
Therefore $\beta = \cos^{-1}\left(\frac{1}{3}\right)$

$\doteq 70.5°$

12. a. $|\overrightarrow{BD}|$



b. $|\overrightarrow{BD}|$



c. In an isosceles triangle, CD is a median and a right bisector of BA. Therefore $\vec{a}$ and $\vec{b}$ have the same magnitude projected on $\vec{c}$.

d. Yes, not only do they have the same magnitude, but they are in the same direction as well which makes them have equivalent vector projections.

13. a. Use the formula for the scalar projection of $\vec{a}$ on $\vec{b} = |\vec{a}|\cos\theta$

$= 10\cos 135°$

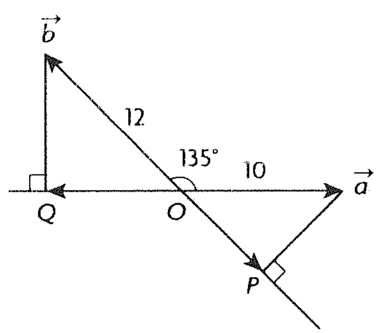
$= -7.07$

And the formula for the scalar projection of $\vec{b}$ on $\vec{a} = |\vec{b}|\cos\theta$

$= 12\cos 135°$

$= -8.49$

b.



$\overrightarrow{OQ}$ is the vector projection of $\vec{b}$ on $\vec{a}$

$\overrightarrow{OP}$ is the vector projection of $\vec{a}$ on $\vec{b}$

14. a. $\overrightarrow{AB}$ = Point B − Point A

$= (1, 3, 3) - (-2, 1, 4)$

$= (3, 2, -1)$

The scalar projection of $\overrightarrow{AB}$ on $\overrightarrow{OD}$ is

$$\frac{\overrightarrow{AB}\cdot\overrightarrow{OD}}{|\overrightarrow{OD}|} = \frac{(3 \times -1) + (2 \times 2) + (-1 \times 2)}{\sqrt{(-1)^2 + 2^2 + 2^2}}$$

$$= \frac{(-3) + 4 + (-2)}{\sqrt{1 + 4 + 4}}$$

$$= \frac{-1}{\sqrt{9}}$$

$$= -\frac{1}{3}$$

b. $\overrightarrow{BC}$ = Point C − Point B

$= (-6, 7, 5) - (1, 3, 3)$

$= (-7, 4, 2)$

The scalar projection of $\overrightarrow{BC}$ on $\overrightarrow{OD}$ is

$$\frac{\overrightarrow{BC}\cdot\overrightarrow{OD}}{|\overrightarrow{OD}|} = \frac{(-7 \times -1) + (4 \times 2) + (2 \times 2)}{\sqrt{(-1)^2 + 2^2 + 2^2}}$$

$$= \frac{7 + 8 + 4}{\sqrt{1 + 4 + 4}}$$

$$= \frac{19}{\sqrt{9}}$$

$$= \frac{19}{3}$$

$$\frac{\overrightarrow{AB}\cdot\overrightarrow{OD}}{|\overrightarrow{OD}|} + \frac{\overrightarrow{BC}\cdot\overrightarrow{OD}}{|\overrightarrow{OD}|} = -\frac{1}{3} + \frac{19}{3}$$

$$= \frac{18}{3}$$

$$= 6$$

$\overrightarrow{AC}$ = Point C − Point A

$= (-6, 7, 5) - (-2, 1, 4)$

$= (-4, 6, 1)$

The scalar projection of $\overrightarrow{AC}$ on $\overrightarrow{OD}$ is

$$\frac{\overrightarrow{AC}\cdot\overrightarrow{OD}}{|\overrightarrow{OD}|} = \frac{(-4 \times -1) + (6 \times 2) + (1 \times 2)}{\sqrt{(-1)^2 + 2^2 + 2^2}}$$

$$= \frac{4 + 12 + 2}{\sqrt{1 + 4 + 4}}$$

$$= \frac{18}{\sqrt{9}}$$

$$= \frac{18}{3}$$

$$= 6$$

c. Same lengths and both are in the direction of $\overrightarrow{OD}$. Add to get one vector.

15. a. $1 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma$

$$= \left(\frac{a}{\sqrt{a^2 + b^2 + c^2}}\right)^2 + \left(\frac{b}{\sqrt{a^2 + b^2 + c^2}}\right)^2 + \left(\frac{c}{\sqrt{a^2 + b^2 + c^2}}\right)^2$$

$$= \frac{a^2}{a^2 + b^2 + c^2} + \frac{b^2}{a^2 + b^2 + c^2} + \frac{c^2}{a^2 + b^2 + c^2}$$

$$= \frac{a^2 + b^2 + c^2}{a^2 + b^2 + c^2}$$
$$= 1$$

b. $\alpha = 90°, \beta = 30°, \gamma = 60°$

$\cos \alpha = \cos 90°$
$= 0,$
$x = 0$

$\cos \beta = \cos 30°$
$= \frac{\sqrt{3}}{2},$

y is a multiple of $\frac{\sqrt{3}}{2}$.

$\cos \gamma = \cos 60°$
$= \frac{1}{2},$

z is a multiple of $\frac{1}{2}$.

Answers include $\left(0, \frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, $\left(0, \sqrt{3}, 1\right)$, etc.

c. If two angles add to 90°, then all three will add to 180°.

16. a. $\alpha = \beta = \gamma$

$\cos \alpha = \cos \beta = \cos \gamma$

$\cos^2 \alpha = \cos^2 \beta = \cos^2 \gamma$

$1 = \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma$

$1 = 3 \cos^2 x$

$\frac{1}{3} = \cos^2 x$

$\sqrt{\frac{1}{3}} = \cos x$

$x = \cos^{-1} \sqrt{\frac{1}{3}}$

$x \doteq 54.7°$

b. For obtuse, use $\cos x = -\sqrt{\frac{1}{3}}$.

$x = \cos^{-1}\left(-\sqrt{\frac{1}{3}}\right)$

$x \doteq 125.3°$

17. $\cos^2 x + \sin^2 x = 1$

$\cos^2 x = 1 - \sin^2 x$

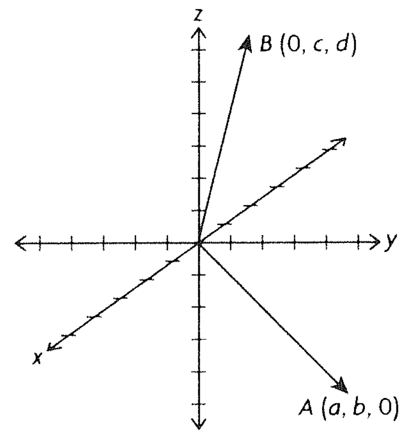
$1 = \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma$

$1 = (1 - \sin^2 \alpha) + (1 - \sin^2 \beta) + (1 - \sin^2 \gamma)$

$1 = 3 - (\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma)$

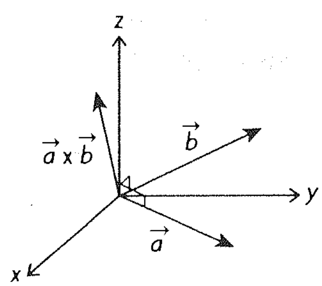
$\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = 2$

18. Answers may vary. For example:

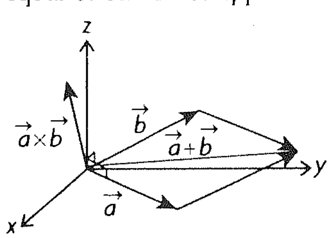


7.6 The Cross Product of Two Vectors, pp. 407–408

1. a.



$\vec{a} \times \vec{b}$ is perpendicular to $\vec{a}$. Thus, their dot product must equal 0. The same applies to the second case.



b. $\vec{a} + \vec{b}$ is still in the same plane formed by $\vec{a}$ and $\vec{b}$, thus $\vec{a} + \vec{b}$ is perpendicular to $\vec{a} \times \vec{b}$ making the dot product 0 again.

c. Once again, $\vec{a} - \vec{b}$ is still in the same plane formed by $\vec{a}$ and $\vec{b}$, thus $\vec{a} - \vec{b}$ is perpendicular to $\vec{a} \times \vec{b}$ making the dot product 0 again.

2. $\vec{a} \times \vec{b}$ produces a vector, not a scalar. Thus, the equality is meaningless.

3. a. It's possible because there is a vector crossed with a vector, then dotted with another vector, producing a scalar.

b. This is meaningless because $\vec{a} \cdot \vec{b}$ produces a scalar. This results in a scalar crossed with a vector, which is meaningless.

c. This is possible. $\vec{a} \times \vec{b}$ produces a vector, and $\vec{c} + \vec{d}$ also produces a vector. The result is a vector dotted with a vector producing a scalar.
d. This is possible. $\vec{a} \cdot \vec{b}$ produces a scalar, and $\vec{c} \times \vec{d}$ produces a vector. The product of a scalar and vector produces a vector.
e. This is possible. $\vec{a} \times \vec{b}$ produces a vector, and $\vec{c} \times \vec{d}$ produces a vector. The cross product of a vector and vector produces a vector.
f. This is possible. $\vec{a} \times \vec{b}$ produces a vector. When added to another vector, it produces another vector.

4. a. $(2, -3, 5) \times (0, -1, 4)$
$= (-3(4) - 5(-1), 5(0) - 2(4), 2(-1) - (-3)(0))$
$= (-7, -8, -2)$
$(2, -3, 5) \cdot (-7, -8, -2) = 0$
$(0, -1, 4) \cdot (-7, -8, -2) = 0$

b. $(2, -1, 3) \times (3, -1, 2)$
$= (-1(2) - 3(-1), 3(3) - 2(2), 2(-1) - (-1)(3))$
$= (1, 5, 1)$
$(2, -1, 3) \cdot (1, 5, 1) = 0$
$(3, -1, 2) \cdot (1, 5, 1) = 0$

c. $(5, -1, 1) \times (2, 4, 7)$
$= (-1(7) - 1(4), 1(2) - 5(7), 5(4) - (-1)(2))$
$= (-11, -33, 22)$
$(5, -1, 1) \cdot (-11, -33, 22) = 0$
$(2, 4, 7) \cdot (-11, -33, 22) = 0$

d. $(1, 2, 9) \times (-2, 3, 4)$
$= (2(4) - 9(3), 9(-2) - 1(4), 1(3) - 2(-2))$
$= (-19, -22, 7)$
$(1, 2, 9) \cdot (-19, -22, 7) = 0$
$(-2, 3, 4) \cdot (-19, -22, 7) = 0$

e. $(-2, 3, 3) \times (1, -1, 0)$
$= (3(0) - 3(-1), 3(1) - (-2)(0), -2(-1) - 3(1))$
$= (3, 3, -1)$
$(-2, 3, 3) \cdot (3, 3, -1) = 0$
$(1, -1, 0) \cdot (3, 3, -1) = 0$

f. $(5, 1, 6) \times (-1, 2, 4)$
$= (1(4) - 6(2), 6(-1) - 5(4), 5(2) - 1(-1))$
$= (-8, -26, 11)$
$(5, 1, 6) \cdot (-8, -26, 11) = 0$
$(-1, 2, 4) \cdot (-8, -26, 11) = 0$

5. $(-1, 3, 5) \times (0, a, 1)$
$= (3(1) - 5(a), 5(0) - (-1)(1), -1(a) - 3(0))$

If we look at the x component, we know that:
$3(1) - 5(a) = -2$
$-5(a) = -5$
$a = 1$

6. a. $\vec{a} \times \vec{b} = (1(1) - 1(5), 1(0) - 0(1), 0(5) - 0(1))$
$= (-4, 0, 0)$

b. Vectors of the form $(0, b, c)$ are in the yz-plane. Thus, the only vectors perpendicular to the yz-plane are those of the form $(a, 0, 0)$ because they are parallel to the x-axis.

7. a. $(1, 2, 1) \times (2, 4, 2)$
$= (2(2) - 1(4), 1(2) - 1(2), 1(4) - 2(2))$
$= (0, 0, 0)$

b. $(a, b, c) \times (ka, kb, kc)$
$= (b(kc) - c(kb), c(ka) - a(kc), a(kb) - b(ka))$
Using the commutative law of multiplication we can rearrange this:
$= (bck - bck, ack - ack, abk - abk)$
$= (0, 0, 0)$

8. a. $\vec{p} \times (\vec{q} + \vec{r}) = (1, -2, 4) \times [(1, 2, 7) + (-1, 1, 0)]$
$= (1, -2, 4) \times (1 - 1, 2 + 1, 7 + 0)$
$= (1, -2, 4) \times (0, 3, 7)$
$= (-2(7) - 4(3), 4(0) - 1(7), 1(3) + 2(0))$
$= (-26, -7, 3)$
$\vec{p} \times \vec{q} + \vec{p} \times \vec{r} = (-2(7) - 4(2), 4(1) - 1(7), 1(2) + 2(1)) + (-2(0) - 4(1), 4(-1) - 1(0), 1(1) + 2(-1))$
$= (-22, -3, 4) + (-4, -4, -1)$
$= (-26, -7, 3)$

b. $\vec{p} \times (\vec{q} + \vec{r}) = (4, 1, 2) \times [(3, 1, -1) + (0, 1, 2)]$
$= (4, 1, 2) \times (3, 1 + 1, -1 + 2)$
$= (4, 1, 2) \times (3, 2, 1)$
$= (1(1) - 2(2), 3(2) - 4(1), 4(2) - 1(3))$
$= (-3, 2, 5)$
$\vec{p} \times \vec{q} + \vec{p} \times \vec{r} = (1(-1) - 2(1), 2(3) - 4(-1), 4(1) - 1(3)) + (1(2) - 2(1), 2(0) - 4(2), 4(1) - 1(0))$
$= (-3, 10, 1) + (0, -8, 4)$
$= (-3, 2, 5)$

9. a. $\vec{i} \times \vec{j} = (1, 0, 0) \times (0, 1, 0)$
$= (0 - 0, 0 - 0, 1 - 0)$
$= (0, 0, 1)$
$= \vec{k}$

$$-\vec{j} \times \vec{i} = (0, -1, 0) \times (1, 0, 0)$$
$$= (0 - 0, 0 - 0, 0 - (-1))$$
$$= (0, 0, 1)$$
$$= \vec{k}$$

b. $\vec{j} \times \vec{k} = (0, 1, 0) \times (0, 0, 1)$
$$= (1 - 0, 0 - 0, 0 - 0)$$
$$= (1, 0, 0)$$
$$= \vec{i}$$
$$-\vec{k} \times \vec{j} = (0, 0, -1) \times (0, 1, 0)$$
$$= (0 - (-1), 0 - 0, 0 - 0)$$
$$= (1, 0, 0)$$
$$= \vec{i}$$

c. $\vec{k} \times \vec{i} = (0, 0, 1) \times (1, 0, 0)$
$$= (0 - 0, 1 - 0, 0 - 0)$$
$$= (0, 1, 0)$$
$$= \vec{j}$$
$$-\vec{i} \times \vec{k} = (-1, 0, 0) \times (0, 0, 1)$$
$$= (0 - 0, 0 - (-1), 0 - 0)$$
$$= (0, 1, 0)$$
$$= \vec{j}$$

10. $k(a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1) \cdot (a_1, a_2, a_3)$
$$= k(a_1a_2b_3 - a_1a_3b_2 + a_2a_3b_1 - a_2a_1b_3 + a_3a_1b_2 - a_3a_2b_1)$$
$$= k(0)$$
$$= 0$$
$\vec{a}$ is perpendicular to $k(\vec{a} \times \vec{b})$.

11. a. $\vec{a} \times \vec{b} = (2, 0, 0) \times (0, 3, 0)$
$$= (0 - 0, 0 - 0, 6 - 0)$$
$$= (0, 0, 6)$$
$$\vec{c} \times \vec{d} = (2, 3, 0) \times (4, 3, 0)$$
$$= (0 - 0, 0 - 0, 6 - 12)$$
$$= (0, 0, -6)$$

b. $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = (0, 0, 6) \times (0, 0, -6)$
(by part **a.**)
$$= (0 - 0, 0 - 0, 0 - 0)$$
$$= (0, 0, 0)$$

c. All the vectors are in the xy-plane. Thus, the cross product in part **b.** is between vectors parallel to the z-axis and so parallel to each other. The cross product of parallel vectors is $\vec{0}$.

12. Let $\vec{x} = (1, 0, 1)$
$$\vec{y} = (1, 1, 1)$$
$$\vec{z} = (1, 2, 3)$$
Then $\vec{x} \times \vec{y} = (0 - 1, 1 - 1, 1 - 0)$
$$= (-1, 0, 1)$$
$$(\vec{x} \times \vec{y}) \times \vec{z} = (0 - 2, 1 - (-3), -3 - 0)$$
$$= (-2, 4, -3)$$
$$\vec{y} \times \vec{z} = (3 - 2, 1 - 3, 2 - 1)$$
$$= (1, -2, 1)$$
$$\vec{x} \times (\vec{y} \times \vec{z}) = (0 + 2, 1 - 1, -2 - 0)$$
$$= (2, 0, -2)$$
Thus $(\vec{x} \times \vec{y}) \times \vec{z} \neq \vec{x} \times (\vec{y} \times \vec{z})$.

13. $(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b})$
By the distributive property of cross product:
$$= (\vec{a} - \vec{b}) \times \vec{a} + (\vec{a} - \vec{b}) \times \vec{b}$$
By the distributive property again:
$$= \vec{a} \times \vec{a} - \vec{b} \times \vec{a} + \vec{a} \times \vec{b} - \vec{b} \times \vec{b}$$
A vector crossed with itself equals $\vec{0}$, thus:
$$= -\vec{b} \times \vec{a} + \vec{a} \times \vec{b}$$
$$= \vec{a} \times \vec{b} - \vec{b} \times \vec{a}$$
$$= \vec{a} \times \vec{b} - (-\vec{a} \times \vec{b})$$
$$= 2\vec{a} \times \vec{b}$$

7.7 Applications of the Dot Product and Cross Product, pp. 414–415

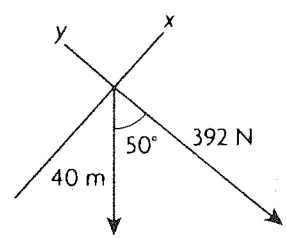
1. By pushing as far away from the hinge as possible, $|\vec{r}|$ is increased making the cross product bigger. By pushing at right angles, sine is its largest value, 1, making the cross product larger.

2. a. $\vec{a} \times \vec{b} = (1, 2, 1) \times (2, 4, 2)$
$$= (2(2) - 1(4), 1(2) - 1(2), 1(4) - 2(2))$$
$$= (0, 0, 0)$$
$$|\vec{a} \times \vec{b}| = 0$$

b. This makes sense because the vectors lie on the same line. Thus, the parallelogram would just be a line making its area 0.

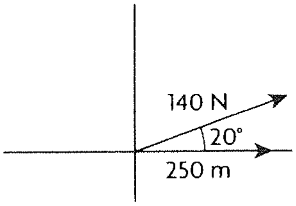
3. a. $\vec{f} \cdot \vec{s} = 3 \cdot 150 = 450$ J

b.



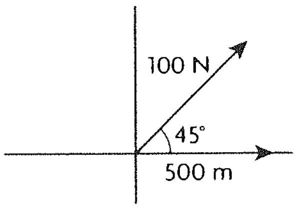
The axes are tilted to illustrate the force of gravity can be split up into components to find the part in the direction of the motion. Let x be the component of force going in the motion's direction.
$$\cos(50°) = \frac{x}{392}$$
$$x = (392)\cos(50°)$$
Now we have our force, so:
$(392)\cos 50°\ \text{N} \cdot 40\ \text{m} \doteq 10\,078.91$ J

c.



First find the x component of the force:
$(140)\cos(20°) = x$
Calculate work:
$140 \cos 20° \text{ N} \cdot 250 \text{ m} \doteq 32\,889.24 \text{ J}$

d.



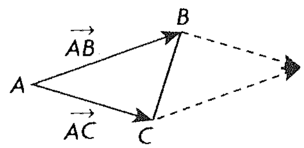
First calculate the x component of the force:
$x = (100)\cos(45°)$
Calculate work:
$100 \cos 45° \cdot 500 \text{ m} = 35\,355.34 \text{ J}$

4. a. $\vec{i} \times \vec{j} = \vec{k}$
The square formed by the 2 vectors has an area of 1. The 2 vectors are on the xy-plane, thus, the cross product must be $\vec{k}$ by the right hand rule.
b. $-\vec{i} \times \vec{j} = -\vec{k}$
Once again, the area is 1, making the possible vector have a magnitude of 1. Also, the 2 vectors are on the xy-plane again so the cross product must lie on the z axis. However, because of the right hand rule, the product must be $-\vec{k}$ this time.
c. $\vec{i} \times \vec{k} = -\vec{j}$
The square has an area of 1, so the magnitude of the vector produced must be 1. The 2 vectors are on the xz-plane. The new vector must be on the y axis making it $-\vec{j}$ because of the right hand rule.
d. $-\vec{i} \times \vec{k} = -\vec{j}$
The square has an area of 1. The 2 vectors are on the xz-plane. So the new vector must be $\vec{j}$ because of the right hand rule.

5. a.
$$\vec{a} \times \vec{b} = (1, 1, 0) \times (1, 0, 1) = (1 - 0, 0 - 1, 0 - 1) = (1, -1, -1)$$
$$|\vec{a} \times \vec{b}| = \sqrt{1 + 1 + 1} = \sqrt{3}$$
So the area of the parallelogram is $\sqrt{3}$ square units.
b.
$$\vec{a} \times \vec{b} = (1, -2, 3) \times (1, 2, 4) = (-8 - 6, 3 - 4, 2 + 2) = (-14, -1, 4)$$
$$|\vec{a} \times \vec{b}| = \sqrt{196 + 1 + 16} = \sqrt{213}$$
So the area of the parallelogram is $\sqrt{213}$ square units.

6.
$$\vec{p} \times \vec{q} = (a, 1, -1) \times (1, 1, 2) = (2 + 1, -2a - 1, a - 1) = (3, 2a + 1, a - 1)$$
$$|\vec{p} \times \vec{q}| = \sqrt{9 + (2a + 1)^2 + (a - 1)^2} = \sqrt{35}$$
$$9 + (2a + 1)^2 + (a - 1)^2 = 35$$
$$9 + 4a^2 + 4a + 1 + a^2 - 2a + 1 = 35$$
$$5a^2 + 2a - 24 = 0$$
$$a = \frac{-2 \pm \sqrt{2^2 - 4(5)(-24)}}{2(5)} = \frac{-2 \pm 22}{10} = 2, \frac{-12}{5}$$

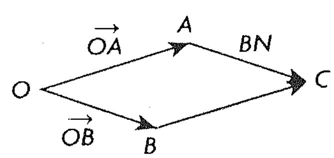
7. a.



As we see from the picture, the area of the triangle ABC is just half the area of the parallelogram determined by vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Thus, we use the magnitude of the cross product to calculate the area.
$$\overrightarrow{AB} = (1 + 2, 0 - 1, 1 - 3) = (3, -1, -2)$$
$$\overrightarrow{AC} = (2 + 2, 3 - 1, 2 - 3) = (4, 2, -1)$$
$$\overrightarrow{AB} \times \overrightarrow{AC} = (1 + 4, -3 + 8, 6 + 4) = (5, 5, 10)$$
$$|\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{25 + 25 + 100} = 5\sqrt{6}$$
Since triangle ABC is half the area of the parallelogram, its area is $\frac{5\sqrt{6}}{2}$ square units.
b. This is just a different way of describing the first triangle, thus the area is $\frac{5\sqrt{6}}{2}$ square units.
c. Any two sides of a triangle can be used to calculate its area.

8.
$$|\vec{r} \times \vec{f}| = (|\vec{r}|\sin(\theta))|\vec{f}| = (0.14)\sin(45°) \cdot 10 \doteq 0.99 \text{ J}$$

9.



We know that the area of a parallelogram is equal to its height multiplied with its base. Its height is BN and its base is $\overline{AC} = \overline{OB}$ as can be seen from the picture. We can calculate the area using the given vectors, then use the area to find BN.
$$\overrightarrow{OA} \times \overrightarrow{OB} = (8 - 4, 12 - 16, 4 - 6) = (4, -4, -2)$$
$$|\overrightarrow{OA} \times \overrightarrow{OB}| = \sqrt{16 + 16 + 4} = \sqrt{36} = 6$$

Now we need to calculate $|\overrightarrow{OB}|$ to know the length of the base.

$\overline{AC} = |\overrightarrow{OB}| = \sqrt{9 + 1 + 16} = \sqrt{26}$

Substituting these results into the equation for area:

$|\overrightarrow{OB}| \cdot BN = 6$

$\sqrt{26}\, BN = 6$

$BN = \dfrac{6}{\sqrt{26}}$ or about 1.18

10. a. $\vec{p} \times \vec{q} = (-6 - 3, 6 - 3, 1 + 4)$
$= (-9, 3, 5)$
$(\vec{p} \times \vec{q}) \times \vec{r} = (0 - 5, 5 + 0, -9 - 3)$
$= (-5, 5, -12)$

$a(1, -2, 3) + b(2, 1, 3) = (-5, 5, -12)$

Looking at x components:

$a + 2b = -5; a = -5 - 2b$

y components:

$-2a + b = 5$

Substitute in a:

$10 + 4b + b = 5$

$5b = -5$

$b = -1$

Substitute b back into the x components:

$a = -5 + 2; a = -3$

Check in z components:

$3a + 3b = -12$

$-9 - 3 = -12$

b. $\vec{p} \cdot \vec{r} = 1 - 2 + 0 = -1$

$\vec{q} \cdot \vec{r} = 2 + 1 + 0 = 3$

$(\vec{p} \cdot \vec{r})\vec{q} - (\vec{q} \cdot \vec{r})\vec{p} = -1(2, 1, 3) - 3(1, -2, 3)$
$= (2, -1, -3) - (3, -6, 9)$
$= (-2 - 3, -1 + 6, -3 - 9)$
$= (-5, 5, -12)$

Review Exercise, pp. 418–421

1. a. $\vec{a} \times \vec{b} = (2 - 0, -1 + 1, 0 + 2)$
$= (2, 0, 2)$

b. $\vec{b} \times \vec{c} = (0 - 4, -5 + 5, -4 - 0)$
$= (-4, 0, -4)$

c. 16

d. The cross products are parallel, so the original vectors are in the same plane.

2. a. $|\vec{a}| = \sqrt{2^2 + (-1)^2 + 2^2} = 3$

b. $|\vec{b}| = \sqrt{6^2 + 3^2 + (-2)^2} = 7$

c. $\vec{a} - \vec{b} = (2 - 6, -1 - 3, 2 + 2)$
$= (-4, -4, 4)$

$|\vec{a} - \vec{b}| = \sqrt{(-4)^2 + (-4)^2 + 4^2} = 4\sqrt{3}$

d. $\vec{a} + \vec{b} = (2 + 6, -1 + 3, 2 - 2)$
$= (8, 2, 0)$

$|\vec{a} + \vec{b}| = \sqrt{8^2 + 2^2 + 0^2} = 2\sqrt{17}$

e. $\vec{a} \cdot \vec{b} = 2(6) - 1(3) + 2(-2) = 5$

f. $\vec{a} - 2\vec{b} = (2 - 12, -1 - 6, 2 + 4)$
$= (-10, -7, 6)$

$\vec{a} \cdot (\vec{a} - 2\vec{b}) = 2(-10) - 1(-7) + 2(6) = -1$

3. a. If $a = 6$, then $\vec{y}$ will be twice $\vec{x}$, thus collinear.

b. $\vec{x} \times \vec{y} = (3, a, 9) \cdot (a, 12, 18) = 0$

$3a + 12a + 162 = 0$

$15a = -162$

$a = \dfrac{-54}{5}$

4. $\cos(\theta) = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$

$\vec{a} \cdot \vec{b} = 4(-3) + 5(6) + 20(22) = 458$

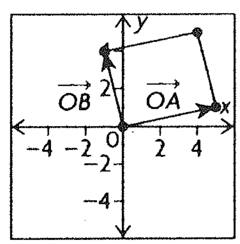
$|\vec{a}| = \sqrt{4^2 + 5^2 + 20^2} = 21$

$|\vec{b}| = \sqrt{(-3)^2 + 6^2 + 22^2} = 23$

$\theta = \cos^{-1}\left(\dfrac{458}{483}\right)$

$\theta \doteq 18.52°$

5. a.



b. We can use the dot product of the 2 diagonals to calculate the angle. The diagonals are the vectors $\overrightarrow{OA} + \overrightarrow{OB}$ and $\overrightarrow{OA} - \overrightarrow{OB}$.

$\overrightarrow{OA} + \overrightarrow{OB} = (5 - 1, 1 + 4) = (4, 5)$

$\overrightarrow{OA} - \overrightarrow{OB} = (5 + 1, 1 - 4) = (6, -3)$

$\cos(\theta) = \dfrac{(\overrightarrow{OA} + \overrightarrow{OB}) \cdot (\overrightarrow{OA} - \overrightarrow{OB})}{|\overrightarrow{OA} + \overrightarrow{OB}||\overrightarrow{OA} - \overrightarrow{OB}|}$

$(\overrightarrow{OA} + \overrightarrow{OB}) \cdot (\overrightarrow{OA} - \overrightarrow{OB}) = 4(6) + 5(-3) = 9$

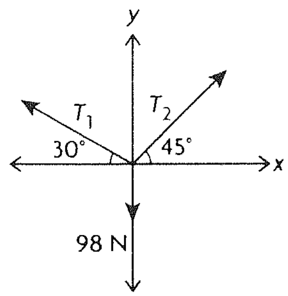
$|\overrightarrow{OA} + \overrightarrow{OB}| = \sqrt{4^2 + 5^2} = \sqrt{41}$

$|\overrightarrow{OA} - \overrightarrow{OB}| = \sqrt{6^2 + (-3)^2} = 3\sqrt{5}$

$\theta = \cos^{-1}\left(\dfrac{9}{3\sqrt{205}}\right)$

$\theta \doteq 77.9°$

6.



The vertical components of the tensions must equal the downward force:

$T_1 \sin(30°) + T_2 \sin(45°) = 98$ N

$$\frac{1}{2}T_1 + \frac{1}{\sqrt{2}}T_2 = 98$$

$$T_1 = 196 - \sqrt{2}T_2$$

The horizontal components:

$T_1 \cos(30°) + T_2 \cos(45°) = 0$ N

$$\frac{\sqrt{3}}{2}T_1 - \frac{1}{\sqrt{2}}T_2 = 0$$

Substitute in T_1:

$$98\sqrt{3} - \frac{\sqrt{6}}{2}T_2 = -98\sqrt{3}$$

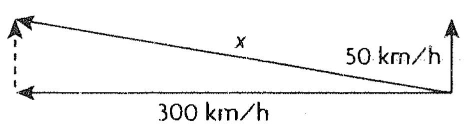
$$\frac{-\sqrt{6} - \sqrt{2}}{2}T_2 = -98\sqrt{3}$$

$$T_2 \doteq 87.86N$$

Substitute this back in to get T_1:

$T_1 \doteq 71.74N$

7.

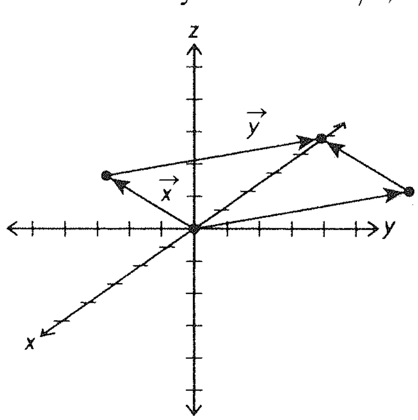


$x = \sqrt{50^2 + 300^2} \doteq 304.14$

$\tan^{-1}\left(\frac{50}{300}\right) \doteq 9.46°$

The resultant velocity is 304.14 km/h, W 9.46° N.

8. a.



b. $\vec{x} \times \vec{y} = (-15 - 35, -5 - 15, 21 - 3)$

$= (-50, -20, 18)$

$|\vec{x} \times \vec{y}| = \sqrt{50^2 + 20^2 + 18^2} = \sqrt{3224} \doteq 56.78$

9. $(0, 3, -5) \times (2, 3, 1)$

$= (3 + 15, -10 - 0, 0 - 6) = (18, -10, -6)$

The cross product is perpendicular to the given vectors, but its magnitude is $\sqrt{18^2 + (-10)^2 + (-6)^2}$, or $2\sqrt{115}$. A unit vector perpendicular to the given vectors is

$$\left(\frac{9}{\sqrt{115}}, -\frac{5}{\sqrt{115}}, -\frac{3}{\sqrt{115}}\right).$$

10. a. $\cos(\alpha) = \dfrac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|}$

$\overrightarrow{AB} = (0, -3, 4) - (2, 3, 7) = (-2, -6, -3)$

$\overrightarrow{AC} = (5, 2, -4) - (2, 3, 7) = (3, -1, -11)$

$\overrightarrow{AB} \cdot \overrightarrow{AC} = -2(3) - 6(-1) - 3(-11) = 33$

$|\overrightarrow{AB}| = \sqrt{(-2)^2 + (-6)^2 + (-3)^2} = 7$

$|\overrightarrow{AC}| = \sqrt{3^2 + (-1)^2 + (-11)^2} = \sqrt{131}$

$$\alpha = \cos^{-1}\frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|}$$

$$= \cos^{-1}\frac{33}{7\sqrt{131}}$$

$$\doteq 65.68°$$

$$\beta = \cos^{-1}\frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{|\overrightarrow{BA}||\overrightarrow{BC}|}$$

$\overrightarrow{BA} = -\overrightarrow{AB} = (2, 6, 3)$

$\overrightarrow{BC} = (5 - 0, 2 + 3, -4 - 4.) = (5, 5, -8)$

$\overrightarrow{BA} \cdot \overrightarrow{BC} = 2(5) + 6(5) + 3(-8) = 16$

$|\overrightarrow{BA}| = \sqrt{2^2 + 6^2 + 3^2} = 7$

$|\overrightarrow{BC}| = \sqrt{5^2 + 5^2 + (-8)^2} = \sqrt{144}$

$$\beta = \cos^{-1}\frac{16}{7\sqrt{114}}$$

$$\doteq 77.64°$$

$$\gamma = 180 - \alpha - \beta \doteq 36.68°$$

So $\beta \doteq 77.64°$ is the largest angle.

b. The area is half the magnitude of the cross product of $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$$\frac{1}{2}\overrightarrow{AB} \times \overrightarrow{AC} = \frac{1}{2}|(63, -31, 20)| \doteq 36.50$$

11. The triangle formed by the two strings and the ceiling is similar to a 3-4-5 right triangle, with the 30 cm and 40 cm strings as legs. So the angle adjacent to the 30 cm leg satisfies

$$\cos\theta = \frac{3}{5}$$

The angle adjacent to the 40 cm leg satisfies

$$\cos\phi = \frac{4}{5}$$

Also,

$$\sin\theta = \frac{4}{5} \text{ and } \sin\phi = \frac{3}{5}.$$

Let T_1 be the tension in the 30 cm string, and T_2 be the tension in the 40 cm string. Then

$$T_1 \cos\theta - T_2 \cos\phi = 0$$

$$T_1\frac{3}{5} - T_2\frac{4}{5} = 0$$

$$T_1 = \frac{4}{3}T_2$$

Also,

$$T_1 \sin\theta + T_2 \sin\phi = (10)(9.8) = 98$$

$$T_1\frac{4}{5} - T_2\frac{3}{5} = 98$$

$$\left(\frac{4}{3}T_2\right)\frac{4}{5} + T_2\frac{3}{5} = 98$$

$$\frac{5}{3}T_2 = 98$$

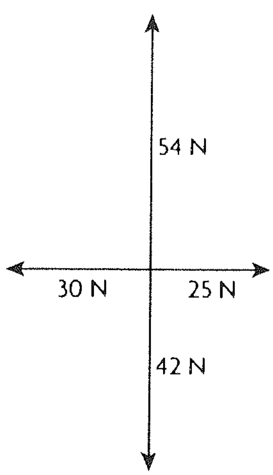
$$T_2 = 58.8 \text{ N}$$

$$T_1 = \frac{4}{3}(58.8)$$

$$= 78.4 \text{ N}$$

So the tension in the 30 cm string is 78.4 N and the tension in the 40 cm string is 58.8 N.

12. a.



b. The east- and west-pulling forces result in a force of 5 N west. The north- and south-pulling forces result in a force of 12 N north. The 5 N west and 12 N north forces result in a force pulling in the north-westerly direction with a force of

$$\sqrt{5^2 + 12^2} = 13 \text{ N},$$

by using the Pythagorean theorem. To find the exact direction of this force, use the definition of sine.

If θ is the angle west of north, then

$$\sin\theta = \frac{5}{13}$$

$$\theta \doteq 22.6°$$

So the resultant is 13 N in a direction N22.6°W. The equilibrant is 13 N in a direction S22.6°E.

13. a. Let D be the origin, then:

$A = (2, 0, 0)$, $B = (2, 4, 0)$, $C = (0, 4, 0)$,
$D = (0, 0, 0)$, $E = (2, 0, 3)$, $F = (2, 4, 3)$,
$G = (0, 4, 3)$ $H = (0, 0, 3)$

b. $\overrightarrow{AF} = (0, 4, 3)$

$\overrightarrow{AC} = (-2, 4, 0)$

$$\overrightarrow{AF} \cdot \overrightarrow{AC} = 0 + 16 + 0 = 16$$

$$|\overrightarrow{AF}| = \sqrt{0^2 + 4^2 + 3^2} = 5$$

$$|\overrightarrow{AC}| = \sqrt{(-2)^2 + 4^2 + 0^2} = 2\sqrt{5}$$

$$\cos(\theta) = \frac{\overrightarrow{AF} \cdot \overrightarrow{AC}}{|\overrightarrow{AF}||\overrightarrow{AC}|}$$

$$\theta = \cos^{-1}\left(\frac{16}{10\sqrt{5}}\right)$$

$$\theta \doteq 44.31°$$

c. Scalar projection $= |\overrightarrow{AF}|\cos(\theta)$

By part **b.**:

$= (5)\cos(44.31°)$

$\doteq 3.58$

14. $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos(\theta) = \cos(\theta)$

$\cos(\theta) = -\frac{1}{2}$ (cosine law)

$(2\vec{a} - 5\vec{b}) \cdot (\vec{b} + 3\vec{a})$

$= -13\vec{a} \cdot \vec{b} + 6\vec{a} \cdot \vec{a} - 5\vec{b} \cdot \vec{b}$

$= -13\vec{a} \cdot \vec{b} + 1$

$= -13\cos(\theta) + 1$

$= 7.5$

15. a. The angle to the bank, θ, will satisfy

$$\sin(90° - \theta) = \frac{2}{3}$$

$$90° - \theta \doteq 41.8°$$

$$\theta \doteq 48.2°$$

b. By the Pythagorean theorem, Kayla's net swimming speed will be

$\sqrt{3^2 - 2^2} = \sqrt{5}$ km/h.

So since distance = rate × time, it will take her

$$t = \frac{0.3}{\sqrt{5}}$$

$\doteq 0.13$ h

$\doteq 8$ min 3 sec

to swim across.

c. Such a situation would have resulted in a right triangle where one of the legs is longer than the hypotenuse, which is impossible.

16. a. The diagonals are $\overrightarrow{OA} + \overrightarrow{OB}$ and $\overrightarrow{OA} - \overrightarrow{OB}$.

$$\overrightarrow{OA} + \overrightarrow{OB} = (3 - 6, 2 + 6, -6 - 2)$$
$$= (-3, 8, -8)$$
$$\overrightarrow{OA} - \overrightarrow{OB} = (3 + 6, 2 - 6, -6 + 2)$$
$$= (9, -4, -4)$$

b. $\overrightarrow{OA} \cdot \overrightarrow{OB} = 3(-6) + 2(6) - 6(-2) = 6$

$$|\overrightarrow{OA}| = \sqrt{3^2 + 2^2 + (-6)^2} = 7$$
$$|\overrightarrow{OB}| = \sqrt{(-6)^2 + 6^2 + (-2)^2} = 2\sqrt{19}$$
$$\cos(\theta) = \frac{\overrightarrow{OA} \cdot \overrightarrow{OB}}{|\overrightarrow{OA}||\overrightarrow{OB}|}$$
$$\theta = \cos^{-1} = \left(\frac{6}{14\sqrt{19}}\right)$$
$$\doteq 84.36°$$

17. a. The z value is double, so if $a = 4$ and $b = -4$, the vector $\vec{q}$ will be collinear.

b. If $\vec{p}$ and $\vec{q}$ are perpendicular, then their dot product will equal 0.

$\vec{p} \cdot \vec{q} = 2a - 2b - 18 = 0$

c. Let $a = 9$, and $b = 0$, then we have a vector perpendicular to $\vec{p}$. Now it must be divided by its magnitude to make it a unit vector:

$|\vec{p}| = \sqrt{81 + 0 + 324} = 9\sqrt{5}$

So the unit vector is:

$$\left(\frac{1}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}}\right)$$

18. a. $\vec{m} \cdot \vec{n} = 2\sqrt{3} - 2\sqrt{3} + 3 = 3$

$$|\vec{m}| = \sqrt{3 + 4 + 9} = 4$$
$$|\vec{n}| = \sqrt{4 + 3 + 1} = 2\sqrt{2}$$
$$\cos(\theta) = \frac{\vec{m} \cdot \vec{n}}{|\vec{m}||\vec{n}|}$$
$$\theta = \cos^{-1}\left(\frac{3}{8\sqrt{2}}\right)$$
$$\doteq 74.62°$$

b. Scalar projection $= |\vec{n}|\cos(\theta)$

$$= 2\sqrt{2}\cos(74.62°)$$
$$\doteq 0.75$$

c. Scalar projection multiplied with the unit vector in the direction of $\vec{m}$:

$$= (0.75)\frac{\vec{m}}{|\vec{m}|}$$
$$= (0.75)\frac{(\sqrt{3}, -2, -3)}{4}$$
$$= (0.1875)(\sqrt{3}, -2, -3)$$

d. $\vec{m} \cdot \vec{k} = -3$

$$\theta = \cos^{-1}\left(\frac{-3}{4}\right)$$
$$\doteq 138.59°$$

19. a. If the dot product is 0, then the vectors are perpendicular:

$(1, 0, 0) \cdot (0, 0, -1) = 0 + 0 + 0 = 0$

$(1, 0, 0) \cdot (0, 1, 0) = 0 + 0 + 0 = 0$

$(0, 0, -1) \cdot (0, 1, 0) = 0 + 0 + 0 = 0$ special

b. $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) \cdot \left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$= -\frac{1}{\sqrt{6}} + \frac{1}{\sqrt{6}} + 0$

$= 0$

$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) \cdot (0, 0, -1) = 0 + 0 + 0 = 0$

$\left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \cdot (0, 0, -1)$

$= 0 + 0 + -\frac{1}{\sqrt{3}} = -\frac{1}{\sqrt{3}}$ not special

20. a. $\vec{p} \times \vec{q}$

$= (-2(1) - 1(-1), 1(2) - 1(1), 1(-1) + 2(2))$

$= (-1, 1, 3)$

b. $\vec{p} - \vec{q} = (-1, -1, 0)$

$\vec{p} + \vec{q} = (3, -3, 2)$

$(\vec{p} - \vec{q}) \times (\vec{p} + \vec{q}) = (-2 - 0, 0 + 2, 3 - (-3))$

$= (\quad 2, 2, 6)$

c. $\vec{p} \times \vec{r} = (4 - 1, 0 + 2, 1 - 0)$

$= (3, 2, 1)$

$(\vec{p} \times \vec{r}) \cdot \vec{r} = 0 + 2 - 2 = 0$

d. $\vec{p} \times \vec{q} = (-2 + 1, 2 - 1, -1 + 4)$

$= (-1, 1, 3)$

21. Since the angle between the two vectors is 60°, the angle formed when they are placed head-to-tail is 120°. So the resultant, along with these two vectors, forms an isosceles triangle with top angle 120° and two equal angles 30°. By the cosine law, the two equal forces satisfy

$$20^2 = 2F^2 - 2F^2\cos 120°$$
$$F^2 = \frac{400}{3}$$
$$F = \frac{20}{\sqrt{3}}$$
$$\doteq 11.55\ N$$

22. $\vec{a} \times \vec{b} = (2 - 0, -5 - 3, 0 - 10)$

$= (2, -8, -10)$

23. First we need to determine the dot product of $\vec{x}$ and $\vec{y}$:

$\vec{x} \cdot \vec{y} = |\vec{x}||\vec{y}|\cos\theta$

$= (10)\cos(60°)$

$= 5$

$(\vec{x} - 2\vec{y}) \cdot (\vec{x} + 3\vec{y})$

By the distributive property:

$= \vec{x} \cdot \vec{x} + 3\vec{x} \cdot \vec{y} - 2\vec{x} \cdot \vec{y} - 6\vec{y} \cdot \vec{y}$

$= 4 + 15 - 10 - 150$

$= -141$

24. $|(2, 2, 1)| = \sqrt{2^2 + 2^2 + 1^2} = 3$

Since the magnitude of the scalar projection is 4, the scalar projection itself has value 4 or -4.

If it is 4, we get

$\frac{(1, m, 0) \cdot (2, 2, 1)}{3} = 4$

$2 + 2m = 12$

$m = 5$

If it is -4, we get

$\frac{(1, m, 0) \cdot (2, 2, 1)}{3} = -4$

$2 + 2m = -12$

$m = -7$

So the two possible values for m are 5 and -7.

25. $\vec{a} \cdot \vec{j} = -3$

$|\vec{a}| = \sqrt{144 + 9 + 16} = 13$

$\theta = \cos^{-1}\left(\frac{-3}{13}\right)$

$\doteq 103.34°$

26. a. $C = (3, 0, 5)$, $F = (0, 4, 0)$

b. $\overrightarrow{CF} = (0, 4, 0) - (3, 0, 5) = (-3, 4, -5)$

c. $|\overrightarrow{CF}| = \sqrt{9 + 16 + 25} = 5\sqrt{2}$

$\overrightarrow{OP} = (3, 4, 5)$

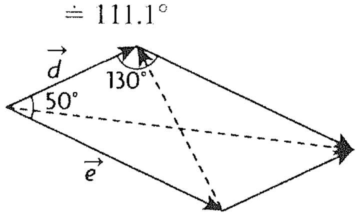
$|\overrightarrow{OP}| = \sqrt{9 + 16 + 25} = 5\sqrt{2}$

$\overrightarrow{CF} \cdot \overrightarrow{OP} = -9 + 16 - 25 = -18$

$\theta = \cos^{-1}\left(\frac{-18}{50}\right)$

$\doteq 111.1°$

27.



a. Using properties of parallelograms, we know that the other angle is 130° (Angles must add up to 360°, opposite angles are congruent).

Using the cosine law,

$|\vec{d} + \vec{e}|^2 = 3^2 + 5^2 - 2(3)(5)\cos 130°$

$|\vec{d} + \vec{e}| \doteq 7.30$

b. Using the cosine law,

$|\vec{d} - \vec{e}|^2 = 3^2 + 5^2 - 2(3)(5)\cos 50°$

$|\vec{d} - \vec{e}| \doteq 3.84$

c. $\vec{e} - \vec{d}$ is the vector in the opposite direction of $\vec{d} - \vec{e}$, but with the same magnitude. So:

$|\vec{e} - \vec{d}| = |\vec{d} - \vec{e}| \doteq 3.84$

28. a. Scalar: $\frac{(\vec{i} + \vec{j}) \cdot (\vec{i})}{|\vec{i}|} = 1$

Vector: $1\left(\frac{\vec{i}}{|\vec{i}|}\right) = \vec{i}$

b. Scalar: $\frac{(\vec{i} + \vec{j}) \cdot (\vec{j})}{|\vec{j}|} = 1$

Vector: $1\left(\frac{\vec{j}}{|\vec{j}|}\right) = \vec{j}$

c. Scalar: $\frac{(\vec{i} + \vec{j}) \cdot (\vec{k} + \vec{j})}{|\vec{k} + \vec{j}|} = \frac{1}{\sqrt{2}}$

Vector: $\frac{1}{\sqrt{2}} \cdot \frac{(\vec{k} + \vec{j})}{|\vec{k} + \vec{j}|} = \frac{1}{2}(\vec{k} + \vec{j})$

29. a. If its magnitude is 1, it's a unit vector:

$|\vec{a}| = \sqrt{\frac{1}{4} + \frac{1}{9} + \frac{1}{36}} \neq 1$ not a unit vector

$|\vec{b}| = \sqrt{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}} = 1$, unit vector

$|\vec{c}| = \sqrt{\frac{1}{4} + \frac{1}{2} + \frac{1}{4}} = 1$, unit vector

$|\vec{d}| = \sqrt{1 + 1 + 1} \neq 1$, not a unit vector

b. $\vec{a}$ is. When dotted with $\vec{d}$, it equals 0.

30. $25 \cdot \sin(30°) \cdot 0.6 = 7.50$ J

31. a. $\vec{a} \cdot \vec{b} = 6 - 5 - 1 = 0$

b. $\vec{a}$ with the x-axis:

$|\vec{a}| = \sqrt{4 + 25 + 1} = \sqrt{30}$

$\cos(\alpha) = \frac{2}{\sqrt{30}}$

$\vec{a}$ with the y-axis:

$\cos(\beta) = \frac{5}{\sqrt{30}}$

$\vec{a}$ with the z-axis:

$\cos(\gamma) = \frac{-1}{\sqrt{30}}$

$|\vec{b}| = \sqrt{9 + 1 + 1} = \sqrt{11}$

$\vec{b}$ with the x-axis:

$\cos(\alpha) = \frac{3}{\sqrt{11}}$

$\vec{b}$ with the y-axis:

$$\cos(\beta) = \frac{-1}{\sqrt{11}}$$

$\vec{b}$ with the z-axis:

$$\cos(\gamma) = \frac{1}{\sqrt{11}}$$

c. $\vec{m}_1 \cdot \vec{m}_2 = \frac{6}{\sqrt{330}} - \frac{5}{\sqrt{330}} - \frac{1}{\sqrt{330}} = 0$

32. Need to show that the magnitudes of the diagonals are equal to show that it is a rectangle.

$|3\vec{i} + 3\vec{j} + 10\vec{k}| = \sqrt{9 + 9 + 100} = \sqrt{118}$

$|-\vec{i} + 9\vec{j} - 6\vec{k}| = \sqrt{1 + 81 + 36} = \sqrt{118}$

33. a. Direction cosine for x-axis:

$$\cos(30^\circ) = \frac{\sqrt{3}}{2}$$

We know the identity

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$

Since $\alpha = 30\gamma$, and $\beta = \gamma$, we get

$$2\cos^2\beta = 1 - \frac{3}{4}$$

$$\cos\beta = \cos\gamma = \pm\frac{1}{2\sqrt{2}}$$

$$\cos\alpha = \frac{\sqrt{3}}{2}$$

So there are two possibilities, depending upon whether $\beta = \gamma$ is acute or obtuse.

b. If γ is acute, then

$$\cos\gamma = \frac{1}{2\sqrt{2}}$$

$$\gamma \doteq 69.3^\circ$$

If Á is obtuse, then

$$\cos\gamma = \frac{1}{2\sqrt{2}}$$

$$\gamma \doteq 110.7^\circ$$

34. $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos(\theta) = \frac{1}{2}$

$(\vec{a} - 3\vec{b}) \cdot (m\vec{a} + \vec{b}) = 0$

$m\vec{a} \cdot \vec{a} + \vec{a} \cdot \vec{b} - 3m\vec{a} \cdot \vec{b} - 3\vec{b} \cdot \vec{b} = 0$

$m + \frac{1}{2} - \frac{3}{2}m - \frac{6}{2} = 0$

$-\frac{1}{2}m = \frac{5}{2}$

$m = -5$

35. $\vec{a} \cdot \vec{b} = 0 - 20 + 12 = -8$

$\vec{a} + \vec{b} = (-1, -1, -8)$

$|\vec{a} + \vec{b}| = \sqrt{1 + 1 + 64} = \sqrt{66}$

$\vec{a} - \vec{b} = (1, 9, -4)$

$|\vec{a} - \vec{b}| = \sqrt{1 + 81 + 16} = \sqrt{98}$

$\frac{1}{4}|\vec{a} + \vec{b}|^2 - \frac{1}{4}|\vec{a} - \vec{b}|^2 = \frac{66}{4} - \frac{98}{4} = -8$

36. $\vec{c} = \vec{b} - \vec{a}$

$|\vec{c}|^2 = |\vec{b} - \vec{a}|^2$

$= (\vec{b} - \vec{a}) \cdot (\vec{b} - \vec{a})$

$= \vec{b} \cdot \vec{b} - \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{a} - \vec{a} \cdot \vec{b}$

$= |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a} \cdot \vec{b}$

$= |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta$

37. $\vec{AB} = (2, 0, 4)$

$|\vec{AB}| = \sqrt{4 + 0 + 16} = 2\sqrt{5}$

$|\vec{AC}| = (1, 0, 2)$

$|\vec{AC}| = \sqrt{1 + 0 + 4} = \sqrt{5}$

$\vec{BC} = (-1, 0, -2)$

$|\vec{BC}| = \sqrt{1 + 0 + 4} = \sqrt{5}$

$$\cos A = \frac{\vec{AB} \cdot \vec{AC}}{|\vec{AB}||\vec{AC}|} = \frac{10}{10} = 1$$

But this means that angle $A = 0^\circ$, so that this triangle is degenerate. For completeness, though, notice that $\vec{BC} = -\vec{AC}$ and $\vec{AB} = 2\vec{AC}$. This means that point C sits at the midpoint of the line segment joining A and B. So angle $C = 180^\circ$ and angle $B = 0^\circ$. So

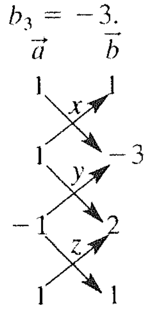
$\cos B = 1;$

$\cos C = -1.$

The area of triangle ABC is, of course, 0.

Chapter 7 Test, p. 422

1. a. We use the diagram to calculate $\vec{a} \times \vec{b}$, noting $a_1 = -1, a_2 = 1, a_3 = 1$ and $b_1 = 2, b_2 = 1, b_3 = -3.$



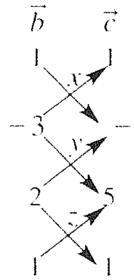
$x = 1(-3) - 1(1) = -4$

$y = 1(2) - (-1)(-3) = -1$

$z = -1(1) - 1(2) = -3$

So, $\vec{a} \times \vec{b} = (-4, -1, -3)$

b. We use the diagram again:



$x = 1(-7) - (-3)(1) = -4$

$y = -3(5) - (2)(-7) = -1$

$z = 2(1) - 1(5) = -3$

So, $\vec{b} \times \vec{c} = (-4, -1, -3)$

c. $\vec{a} \cdot (\vec{b} \times \vec{c}) = (-1, 1, 1) \cdot (-4, -1, -3)$
$= (-1)(-4) + (1)(-1) + (1)(-3)$
$= 0$

d. We could use the diagram method again, or, we note that for any vectors $\vec{x}, \vec{y}, \vec{x} \times \vec{y} = -(\vec{y} \times \vec{x})$, so letting $\vec{y} = \vec{x}$, we have $\vec{x} \times \vec{x} = 0$ from the last equation. Since $\vec{a} \times \vec{b} = \vec{b} \times \vec{c}$ from the first two parts of the problem, $(\vec{a} \times \vec{b}) \times (\vec{b} \times \vec{c}) = 0$.

2. a. To find the scalar and vector projections of $\vec{a}$ on $\vec{b}$, we need to calculate $\vec{a} \cdot \vec{b}$ and $|\vec{b}| = \sqrt{\vec{b} \cdot \vec{b}}$

$\vec{a} \cdot \vec{b} = (1, -1, 1) \cdot (2, -1, -2)$
$= (1)(2) + (-1)(-1) + (1)(-2)$
$= 1$

$|\vec{b}| = \sqrt{2^2 + (-1)^2 + (-2)^2}$
$= 3$

So, $|\vec{b}| = 3$

The scalar projection of $\vec{a}$ on $\vec{b}$ is $\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = \frac{1}{3}$, and

the vector projection of $\vec{a}$ on $\vec{b}$ is $\left(\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2}\right)\vec{b} = \frac{1}{9}(2, -1, -2)$.

b. We find the direction cosines for $\vec{b}$:

$\cos(\alpha) = \frac{b_1}{|\vec{b}|} = \frac{2}{3}$

$\alpha \doteq 48.2°$.

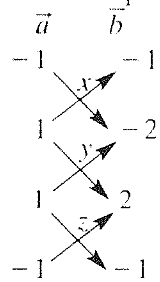
$\cos(\beta) = \frac{b_2}{|\vec{b}|} = \frac{-1}{3}$

$\beta \doteq 109.5°$.

$\cos(\gamma) = \frac{b_3}{|\vec{b}|} = \frac{-2}{3}$

$\gamma \doteq 131.8°$.

c. The area of the parallelogram is the magnitude of the cross product.



$x = (-1)(-2) - 1(-1) = 3$

$y = 1(2) - (1)(-2) = 4$

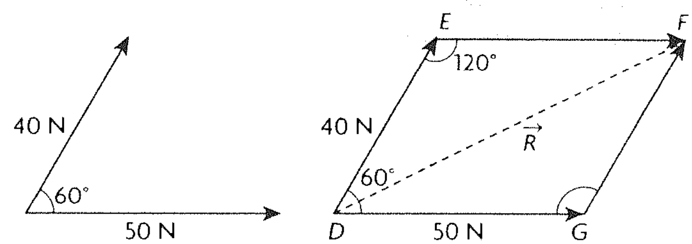
$z = (1)(-1) - (-1)(2) = 1$

So, $\vec{a} \times \vec{b} = (3, 4, 1)$ and thus,

$|\vec{a} \times \vec{b}| = \sqrt{3^2 + 4^2 + 1^2}$
$= \sqrt{26}$

So the area of the parallelogram formed by $\vec{a}$ and $\vec{b}$ is $\sqrt{26}$ or 5.10 square units.

3. We first draw a diagram documenting the situation:



In triangle DEF, we use the cosine law:

$|\vec{R}| = \sqrt{40^2 + 50^2 - 2(40)(50)\cos(120°)}$

$|\vec{R}| \doteq 78.10$

We now use the sine law to find $\angle EDF$:

$\frac{\sin \angle EDF}{|\vec{EF}|} = \frac{\sin \angle DEF}{|\vec{R}|}$

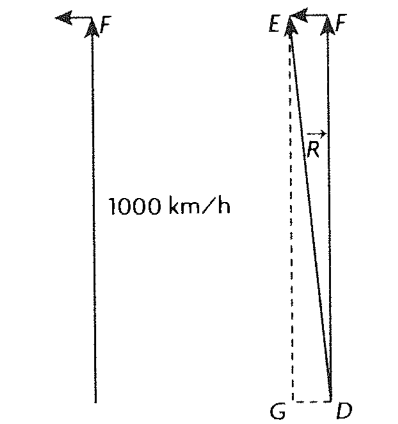
$\frac{\sin \angle EDF}{50} \doteq \frac{\sin 120°}{78.10}$

$\sin \angle EDF \doteq 0.5544$

$\angle EDF \doteq 33.7°$

The equilibrant force is equal in magnitude and opposite in direction to the resultant force, so both forces have a magnitude of 78.10 N. The resultant makes an angle 33.7° to the 40 N force and 26.3° to the 50 N force. The equilibrant makes an angle 146.3° to the 40 N force and 153.7° to the 50 N force.

4. We find the resultant velocity of the airplane.



Position diagram Vector diagram

Since the airplane's velocity is perpendicular to the wind, the resultant's magnitude is given by the Pythagorean theorem:

$|\vec{R}| = \sqrt{1000^2 + 100^2}$

$|\vec{R}| \doteq 1004.99$

The angle is determined using the tangent ratio:

$\tan \angle EDF = \dfrac{100}{1000}$

$\angle EDF \doteq 5.7°$

Thus, the resultant velocity is 1004.99 km/h, N 5.7° W (or W 84.3° N).

5. a. The canoeist will travel 200 m across the stream, so the total time he will paddle is:

$t = \dfrac{d}{r_{canoeist}}$

$t = \dfrac{200 \text{ m}}{2.5 \text{ m/s}}$

$t = 80 \text{ s}$

The current is flowing 1.2 m/s downstream, so the distance that the canoeist travels downstream is:

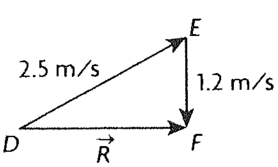
$d = r_{current} \times t$

$d = (1.2 \text{ m/s})(80 \text{ s})$

$d = 96 \text{ m}$

So, the canoeist will drift 96 m south.

b. In order to arrive directly across stream, the canoeist must take into account the change in his velocity caused by the current. That is, he must initially paddle upstream in a direction such that the resultant velocity is directed straight across the stream. The resultant velocity:



Since the resultant velocity is perpendicular to the current, the direction in which the canoeist should head is determined by the sine ratio.

$\sin \angle EDF = \dfrac{1.2}{2.5}$

$\angle EDF \doteq 28.7°$

The canoeist should head 28.7° upstream.

6. The area of the triangle is exactly:

$A_{\Delta ABC} = \dfrac{1}{2}|\overrightarrow{AB} \times \overrightarrow{BC}|$

$\overrightarrow{AB} = (2, 1, 3) - (-1, 3, 5)$
$= (3, -2, -2)$

$\overrightarrow{BC} = (-1, 1, 4) - (2, 1, 3)$
$= (-3, 0, 1)$

$\overrightarrow{AB}$ $\overrightarrow{BC}$

−2 0

$x = (-2)(1) - (-2)(0) = -2$

−2 1

$y = (-2)(-3) - (3)(1) = 3$

3 −3

$z = (3)(0) - (-2)(-3) = -6$

−2 0

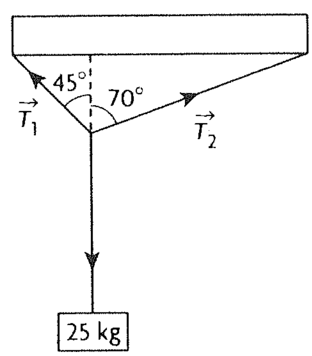
So, $\overrightarrow{AB} \times \overrightarrow{BC} = (-2, 3, -6)$ and

$|\overrightarrow{AB} \times \overrightarrow{BC}| = \sqrt{(-2)^2 + 3^2 + (-6)^2}$
$= \sqrt{49}$
$= 7$

So, $A_{\Delta ABC} = \dfrac{1}{2}|\overrightarrow{AB} \times \overrightarrow{BC}| = \dfrac{7}{2}$.

The area of the triangle is 3.50 square units.

7.



The system is in equilibrium (i.e. it is not moving), so we know that the horizontal components of $\vec{T_1}$ and $\vec{T_2}$ are equal:

$|\vec{T_1}|\sin(45°) = |\vec{T_2}|\sin(70°)$

$|\vec{T_2}| = \dfrac{\sin(45°)}{\sin(70°)}|\vec{T_1}|$

Also, the vertical component of $\vec{T}_1 + \vec{T}_2$ must equal the gravitational force on the block:

$|\vec{T}_1|\cos 45° + |\vec{T}_2|\cos 70° = (25\text{ kg})(9.8\text{ m/s}^2)$

Substituting in for $\vec{T}_2$, we find that:

$$|\vec{T}_1|\cos 45° + |\vec{T}_1|\frac{\sin 45°}{\sin 70°}\cos 70° = (25\text{ kg})(9.8\text{ m/s}^2)$$

$$|\vec{T}_1|\left(\cos 45° + \frac{\sin 45°}{\sin 70°}\cos 70°\right) = 245\text{ N}$$

$$|\vec{T}_1|(0.9645) \doteq 245\text{ N}$$

$$|\vec{T}_1| \doteq 254.0\text{ N}$$

So, we can now find

$$|\vec{T}_2| = \frac{\sin(45°)}{\sin(70°)}|\vec{T}_1|$$

$$|\vec{T}_2| \doteq \frac{\sin(45°)}{\sin(70°)}(254.0\text{ N})$$

$$|\vec{T}_2| \doteq 191.1\text{ N}$$

The direction of the tensions are indicated in the diagram.

8. a. We explicitly calculate both sides of the equation. The left side is:

$$\begin{aligned}\vec{x}\cdot\vec{y} &= (3, 3, 1)\cdot(-1, 2, -3)\\ &= (3)(-1) + (3)(2) + (1)(-3)\\ &= 0\end{aligned}$$

We perform a few computations before computing the right side:

$$\begin{aligned}\vec{x}+\vec{y} &= (3, 3, 1) + (-1, 2, -3)\\ &= (2, 5, -2)\end{aligned}$$

$$\begin{aligned}|\vec{x}+\vec{y}|^2 &= (\vec{x}+\vec{y})\cdot(\vec{x}+\vec{y})\\ &= 2^2 + 5^2 + (-2)^2\\ &= 33\end{aligned}$$

$$\begin{aligned}\vec{x}-\vec{y} &= (3, 3, 1) - (-1, 2, -3)\\ &= (4, 1, 4)\end{aligned}$$

$$\begin{aligned}|\vec{x}-\vec{y}|^2 &= (\vec{x}-\vec{y})\cdot(\vec{x}-\vec{y})\\ &= 4^2 + 1^2 + 4^2\\ &= 33\end{aligned}$$

Thus, the right side is

$$\begin{aligned}\frac{1}{4}|\vec{x}+\vec{y}|^2 - \frac{1}{4}|\vec{x}-\vec{y}|^2 &= \frac{1}{4}(33) - \frac{1}{4}(33)\\ &= 0\end{aligned}$$

So, the equation holds for these vectors.

b. We now verify that the formula holds in general. We will compute the right side of the equation, but we first perform some intermediary computations:

$$\begin{aligned}|\vec{x}+\vec{y}|^2 &= (\vec{x}+\vec{y})\cdot(\vec{x}+\vec{y})\\ &= (\vec{x}\cdot\vec{x}) + (\vec{x}\cdot\vec{y}) + (\vec{y}\cdot\vec{x}) + (\vec{y}\cdot\vec{y})\\ &= (\vec{x}\cdot\vec{x}) + 2(\vec{x}\cdot\vec{y}) + (\vec{y}\cdot\vec{y})\end{aligned}$$

$$\begin{aligned}|\vec{x}-\vec{y}|^2 &= (\vec{x}-\vec{y})\cdot(\vec{x}-\vec{y})\\ &= (\vec{x}\cdot\vec{x}) + (\vec{x}\cdot -\vec{y}) + (-\vec{y}\cdot\vec{x})\\ &\quad + (-\vec{y}\cdot -\vec{y})\\ &= (\vec{x}\cdot\vec{x}) - 2(\vec{x}\cdot\vec{y}) + (\vec{y}\cdot\vec{y})\end{aligned}$$

So, the right side of the equation is:

$$\begin{aligned}\frac{1}{4}|\vec{x}+\vec{y}|^2 - \frac{1}{4}|\vec{x}-\vec{y}|^2 &= \frac{1}{4}(4(\vec{x}\cdot\vec{y}))\\ &= \vec{x}\cdot\vec{y}\end{aligned}$$

Thus, the equation holds for arbitrary vectors.

CHAPTER 8

Equations of Lines and Planes

Review of Prerequisite Skills, pp. 424–425

1. a. $(3, -2, 1) - (1, 7, -5)$
$= (3 - 1, -2 - 7, 1 - (-5))$
$= (2, -9, 6)$
b. $5(2, -3, -4) + 3(1, 1, -7)$
$= (5 \times 2, 5 \times (-3), 5 \times (-4))$
$+ (3 \times 1, 3 \times 1, 3 \times (-7))$
$= (10, -15, -20) + (3, 3, -21)$
$= (13, -12, -41)$
2. a. The points A, B, and C are collinear if and only if the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are collinear.
$\overrightarrow{AB} = (4, 2) - (1, -3)$
$= (3, 5)$
$\overrightarrow{AC} = (-8, -18) - (1, -3)$
$= (-9, -15)$
$= -3(3, 5)$
So $\overrightarrow{AC} = -3\overrightarrow{AB}$, and so $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are collinear.
b. The points J, K, and L are collinear if and only if the vectors $\overrightarrow{JK}$ and $\overrightarrow{JL}$ are collinear.
$\overrightarrow{JK} = (4, 5) - (-4, 3)$
$= (8, 2)$
$\overrightarrow{JL} = (0, 4) - (-4, 3)$
$= (4, 1)$
$= \frac{1}{2}(8, 2)$
So $\overrightarrow{JK} = \frac{1}{2}\overrightarrow{JL}$, and so $\overrightarrow{JK}$ and $\overrightarrow{JL}$ are collinear.
c. The points A, B, and C are collinear if and only if the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are collinear.
$\overrightarrow{AB} = (4, 7, 0) - (1, 2, 1)$
$= (3, 5, -1)$
$\overrightarrow{AC} = (7, 12, -1) - (1, 2, 1)$
$= (6, 10, -2)$
$= 2(3, 5, -1)$
So $\overrightarrow{AC} = 2\overrightarrow{AB}$, and so $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are collinear.
d. The points R, S, and T are collinear if and only if the vectors $\overrightarrow{RS}$ and $\overrightarrow{RT}$ are collinear.
$\overrightarrow{RS} = (4, 1, 3) - (1, 2, -3)$
$= (3, -1, 6)$
$\overrightarrow{RT} = (2, 4, 0) - (1, 2, -3)$
$= (1, 2, 3)$
Since the ratios of the components are not equal, $\overrightarrow{RS}$ and $\overrightarrow{RT}$ are not collinear. So R, S, and T do not lie on the same line.
3. ABC is a right triangle if its sides, $|\overrightarrow{AB}|$, $|\overrightarrow{AC}|$, and $|\overrightarrow{BC}|$, satisfy the Pythagorean theorem.
$\overrightarrow{AB} = (2, 5, 3) - (1, 6, -2)$
$= (1, -1, 5)$
So $|\overrightarrow{AB}| = \sqrt{1^2 + (-1)^2 + 5^2}$
$= \sqrt{27}$.
$\overrightarrow{AC} = (5, 3, 2) - (1, 6, -2)$
$= (4, -3, 4)$
So $|\overrightarrow{AC}| = \sqrt{4^2 + (-3)^2 + 4^2}$
$= \sqrt{41}$.
$\overrightarrow{BC} = (5, 3, 2) - (2, 5, 3)$
$= (3, -2, -1)$
So $|\overrightarrow{BC}| = \sqrt{3^2 + (-2)^2 + (-1)^2}$
$= \sqrt{14}$.
Since $|\overrightarrow{AB}|^2 + |\overrightarrow{BC}|^2 = |\overrightarrow{AC}|^2$, ABC is a right triangle.
4. The vectors $\vec{u}$ and $\vec{v}$ are perpendicular if $\vec{u} \cdot \vec{v} = 0$.
$\vec{u} \cdot \vec{v} = (t, -1, 3) \cdot (2, t, -6)$
$= 2t + (-1)t + 3(-6)$
$= t - 18$
So if $t = 18$, then $\vec{u} \cdot \vec{v} = 0$.
5. a. A vector, (t_1, t_2), is perpendicular to $\vec{a}$ if $\vec{a} \cdot (t_1, t_2) = 0$ and t_1 and t_2 are not both zero.
$\vec{a} \cdot (t_1, t_2) = (1, -3) \cdot (t_1, t_2)$
$= 1(t_1) - 3(t_2)$
So if $t_1 = 3$ and $t_2 = 1$, then $\vec{a} \cdot (t_1, t_2) = 0$. So $(3, 1)$ is perpendicular to $\vec{a}$.
b. A vector, (t_1, t_2), is perpendicular to $\vec{b}$ if $\vec{b} \cdot (t_1, t_2) = 0$ and t_1 and t_2 are not both zero.
$\vec{b} \cdot (t_1, t_2) = (6, -5) \cdot (t_1, t_2)$
$= 6(t_1) - 5(t_2)$
So if $t_1 = 5$ and $t_2 = 6$, then $\vec{a} \cdot (t_1, t_2) = 0$. So $(5, 6)$ is perpendicular to $\vec{b}$.
c. A vector, (t_1, t_2, t_3), is perpendicular to $\vec{c}$ if $\vec{c} \cdot (t_1, t_2, t_3) = 0$ and t_1, t_2, t_3 are not all zero.
$\vec{c} \cdot (t_1, t_2, t_3) = (-7, -4, 0) \cdot (t_1, t_2, t_3)$
$= -7(t_1) - 4(t_2) + 0(t_3)$

So if $t_1 = -4$, $t_2 = 7$, and $t_3 = 0$, then $\vec{c} \cdot (t_1, t_2, t_3) = 0$. So $(-4, 7, 0)$ is perpendicular to $\vec{c}$.

6. The area of a parallelogram formed by two vectors is determined by the magnitude of the cross product of the vectors.

$(4, 10, 9) \times (3, 1, -2)$
$= ((10)(-2) - (9)(1), (9)(3) - (4)(2),$
$(4)(1) - (10)(3))$
$= (-29, 35, -26)$

A = area of parallelogram
$= |(-29, 35, -26)|$
$= \sqrt{(-29)^2 + 35^2 + (-26)^2}$
$= \sqrt{2802}$

7. a. $\vec{a} \times \vec{b} = ((1)(2) - (-4)(-5), (-4)(3) - (2)(-2), (2)(-5) - (1)(3))$
$= (-22, -8, -13)$

$\vec{a} \cdot (-22, -8, -13) = (2)(-22) + (1)(-8) + (-4)(-13)$
$= -44 - 8 + 52$
$= 0$

$\vec{b} \cdot (-22, -8, -13) = (3)(-22) + (-5)(-8) + (-2)(-13)$
$= -66 + 40 + 26$
$= 0$

So $(-22, -8, -13)$ is a vector perpendicular to both vectors.

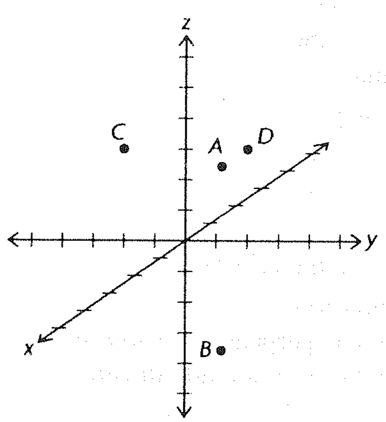
b. $\vec{a} \times \vec{b} = ((-2)(0) - (0)(-1), (0)(-2) - (-1)(0), (-1)(-1) - (-2)(-2))$
$= (0, 0, -3)$

$\vec{a} \cdot (0, 0, -3) = (-1)(0) + (-2)(0) + (0)(-3)$
$= 0$

$\vec{b} \cdot (-22, -8, -13)$
$= (-2)(0) + (-1)(0) + (0)(-3)$
$= 0$

So $(0, 0, -3)$ is a vector perpendicular to both vectors.

8.



9. a. $\vec{p} = (-3, 5) - (4, 8)$
$= (-7, -3)$

b. $\vec{p} = (3, 8) - (-7, -6)$
$= (10, 14)$

c. $\vec{p} = (3, -6, 9) - (1, 2, 4)$
$= (2, -8, 5)$

d. $\vec{p} = (0, 5, 0) - (4, 0, -4)$
$= (-4, 5, 4)$

10. a. $\vec{p} = (4, 8) - (-3, 5)$
$= (7, 3)$

b. $\vec{p} = (-7, -6) - (3, 8)$
$= (-10, -14)$

c. $\vec{p} = (1, 2, 4) - (3, -6, 9)$
$= (-2, 8, -5)$

d. $\vec{p} = (4, 0, -4) - (0, 5, 0)$
$= (4, -5, -4)$

11. a. The y-intercept occurs when $x = 0$.
$y = -2x - 5$
$= -2(0) - 5$
$= -5$

So the y-intercept is -5. The slope is equal to -2.

b. $4x - 8y = 8$
$4(0) - 8y = 8$
$y = \frac{8}{-8}$
$= -1$

So the y-intercept is -1. The slope is equal to $\frac{4}{8} = \frac{1}{2}$.

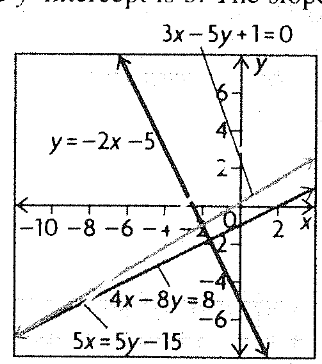
c. $3x - 5y + 1 = 0$
$3(0) - 5y + 1 = 0$
So $-5y = -1$

So the y-intercept is $\frac{-1}{-5} = \frac{1}{5}$. The slope is equal to $\frac{3}{5}$.

d. $5x = 5y - 15$
$5(0) = 5y - 15$
So $5y = 15$

So the y–intercept is 3. The slope is equal to $\frac{5}{5} = 1$.

a. –d.



12. Any positive scalar multiple of a vector is a collinear vector in the same direction. Answers may vary. For example:

a. $2(4, 7) = (8, 14)$

b. $3(-5, 4, 3) = (-15, 12, 9)$

c. $\frac{1}{2}(2\vec{i} + 6\vec{j} - 4\vec{k}) = \vec{i} + 3\vec{j} - 2\vec{k}$

d. $4(-5\vec{i} + 8\vec{j} + 2\vec{k}) = -20\vec{i} + 32\vec{j} + 8\vec{k}$

13. To simplify $\vec{v}$ can be written in algebraic notation. So $\vec{v} = (4, -2, 1)$

a. $\vec{u} \cdot \vec{v} = (4)(4) + (-9)(-2) + (-1)(1)$
$= 16 + 18 - 1$
$= 33$

b. $-\vec{v} = -1(4, -2, 1)$
$= (-4, 2, -1)$

So $-\vec{v} \cdot \vec{u} = (-4)(4) + (2)(-9) + (-1)(-1)$
$= -16 - 18 + 1$
$= -33$

c. $\vec{u} + \vec{v} = (4, -9, -1) + (4, -2, 1)$
$= (8, -11, 0)$

$\vec{u} - \vec{v} = (4, -9, -1) - (4, -2, 1)$
$= (0, -7, -2)$

$(\vec{u} + \vec{v}) \cdot (\vec{u} - \vec{v}) = (8)(0) + (-11)(-7) + (0)(-2)$
$= 77$

d. $\vec{u} \times \vec{v} = ((-9)(1) - (-1)(-2), (-1)(4) - (4)(1), (4)(-2) - (-9)(4))$
$= (-11, -8, 28)$

e. $\vec{v} \times \vec{u}$ is merely the negative of $\vec{u} \times \vec{v}$.
So $\vec{v} \times \vec{u} = -(-11, -8, 28)$
$= (11, 8, -28)$

f. $2\vec{u} + \vec{v} = 2(4, -9, -1) + (4, -2, 1)$
$= (8, -18, -2) + (4, -2, 1)$
$= (12, -20, -1)$

$\vec{u} - 2\vec{v} = (4, -9, -1) - 2(4, -2, 1)$
$= (4, -9, -1) - (8, -4, 2)$
$= (-4, -5, -3)$

$(2\vec{u} + \vec{v}) \times (\vec{u} - 2\vec{v}) = ((-20)(-3) - (-1)(-5), (-1)(-4) - (12)(-3), (12)(-5) - (-20)(-4))$
$= (55, 40, -140)$

14. The dot product of two vectors yields a real number, while the cross product of two vectors gives another vector.

8.1 Vector and Parametric Equations of a Line in R^2, pp. 433–434

1. Direction vectors for a line are unique only up to scalar multiplication. So since each of the given vectors is just a scalar multiple of $\left(\frac{1}{3}, \frac{1}{6}\right)$ each is an acceptable direction vectors for the line.

2. a. Simply find x and y coordinates for three values of t. Three possible values are $t = -1$, $t = 0$, and $t = 1$. At $t = -1$, $x = 1 + 3(-1) = -2$ and $y = 5 - 2(-1) = 7$. At $t = 0$, $x = 1 + 3(0) = 1$ and $y = 5 - 2(0) = 5$. At $t = 1$, $x = 1 + 3(1) = 4$ and $y = 5 - 2(1) = 3$. So $(-2, 7)$, $(1, 5)$, and $(4, 3)$ are three points on the line.

b. Find the t value when the y-coordinate is 15. So solve $15 = 5 - 2t$ for t.
$-2t = 10$
$t = -5$
If $t = -5$, the $x = 1 + 3(-5) = -14$. So $P(-14, 15)$ is a point on the line.

3. Answers may vary. For example:

a. $(3, 4)$ is a point on the line and $(2, 1)$ is a direction vector for the line.

b. $(1, 3)$ is a point on the line and $(2, -7)$ is a direction vector for the line.

c. $(4, 1)$ is a point on the line and $(0, 2)$ is a direction vector for the line.

d. $(0, 6)$ is a point on the line and $(-5, 0)$ is a direction vector for the line.

4. Answers may vary. For example: One possible line has $A(2, 1)$ as its origin point and $\overrightarrow{AB}$ as its direction vector, while another has $B(-3, 5)$ as its origin point and $\overrightarrow{BA}$ as its direction vector.
$\overrightarrow{AB} = (-3, 5) - (2, 1) = (-5, 4)$
So the first case is $\vec{r} = (2, 1) + t(-5, 4), t \in \mathbf{R}$.
$\overrightarrow{BA} = (2, 1) - (-3, 5) = (5, -4)$
The second case is $\vec{q} = (-3, 5) + s(5, -4), s \in \mathbf{R}$.

5. a. Find the t value when the y-coordinate is 18. So solve $18 = 4 + 2t$ for t.
$2t = 14$
$t = 7$
If $t = 7$, the $x = -2 - 7 = -9$. So $R(-9, 18)$ is a point on the line.

b. Answers may vary. For example: A directional vector for the line is $(-1, 2)$. Since $R(-9, 18)$ is a point on the line, a possible vector equation is $\vec{r} = (-9, 18) + t(-1, 2), t \in \mathbf{R}$.

c. Answers may vary. For example: We may take $t = 0$ to find another point on the line. So $x = -2 - 0 = -2$ and $y = 4 + 2(0) = 4$. Hence $(-2, 4)$ is a point on the line. So another vector equation is $\vec{r} = (-2, 4) + t(-1, 2), t \in \mathbf{R}$.

6. Answers may vary. For example:

a. Three different s values will yield three different points on the line. If $s = -1$, then

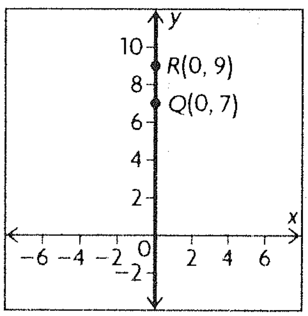
$s(3, 4) = (-3, -4)$. If $s = 0$, then $s(3, 4) = (0, 0)$ and if $s = 1$, then $s(3, 4) = (3, 4)$. Hence $(-3, -4)$, $(0, 0)$, and $(3, 4)$ are three points on the line.

b. $\vec{r} = t(1, 1)$, $t \in \mathbf{R}$ is a line that passes through the origin different from the line in part **a.**

c. If $t = -3$, then $(9, 12) + t(3, 4) = (0, 0)$. So $\vec{r} = (9, 12) + t(3, 4)$, $t \in \mathbf{R}$, is a line that passes through the origin with a direction vector of $(3, 4)$. Hence this describes the same line as part **a.**

7. One can multiply a direction vector by a constant to keep the same line, but multiplying the point yields a different line.

8. a.



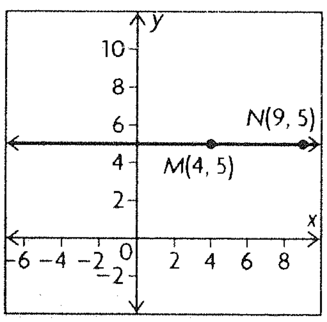
b. $\overrightarrow{QR}$ is a possible direction vector for this line and $Q(0, 7)$ is a point on the line.

$\overrightarrow{QR} = (0, 9) - (0, 7) = (0, 2)$

So a vector equation for the line is

$\vec{r} = (0, 7) + t(0, 2)$, $t \in \mathbf{R}$. The corresponding parametric equation is $x = 0$, $y = 7 + 2t$, $t \in \mathbf{R}$.

9. a.



b. $\overrightarrow{MN}$ is a possible direction vector for this line and $M(4, 5)$ is a point on the line.

$\overrightarrow{MN} = (9, 5) - (4, 5) = (5, 0)$

So a vector equation for the line is

$\vec{r} = (4, 5) + t(5, 0)$, $t \in \mathbf{R}$. The corresponding parametric equation is $x = 4 + 5t$, $y = 5$, $t \in \mathbf{R}$.

10. a. A line perpendicular to L would have a direction vector that is perpendicular to the direction vector of L. (t_1, t_2) is perpendicular to $(3, 5)$ if $(3, 5) \cdot (t_1, t_2) = 0$ and t_1 and t_2 are not both zero.

$(3, 5) \cdot (t_1, t_2) = 3(t_1) + 5(t_2)$

So if $t_1 = 5$ and $t_2 = -3$, then $(3, 5) \cdot (t_1, t_2) = 0$. So an equation for a line with $(5, -3)$ as a direction vector and $P(2, 0)$ as a line is $\vec{r} = (2, 0) + t(5, -3)$, $t \in \mathbf{R}$.

b. The line intersects the y-axis when the x coordinate is zero. The x coordinate is zero, when $2 + 5t = 0$, or $t = -0.4$. The y coordinate at this point is $0 + 3t$ or $y = -1.2$. So the line intersects the y-axis at the point $(0, -1.2)$.

11. The line crosses the x-axis, when $y = 0$, so $8 + s = 0$, or $s = -8$. So the x coordinate at this point is $x = -10 - 2(-8) = 6$. The line crosses the y-axis, when $x = 0$, so $-10 - 2s = 0$, or $s = -5$. So the y coordinate at this point is $y = 8 + (-5) = 3$. So the triangle formed by the origin, $A(6, 0)$, and $B(0, 3)$ is a right triangle with a base of six units and a height of three units. So the area is $\frac{1}{2}(3)(6) = 9$.

12. First all the relevant vectors are found.

$$\overrightarrow{AB} = ((1, 2) + 1(-2, 3)) - ((1, 2) + 0(-2, 3)) = (-2, 3)$$

$$\overrightarrow{AC} = ((1, 2) + 2(-2, 3)) - ((1, 2) + 0(-2, 3)) = (-4, 6)$$

$$\overrightarrow{AD} = ((1, 2) + 3(-2, 3)) - ((1, 2) + 0(-2, 3)) = (-6, 9)$$

a. $\overrightarrow{AC} = (-4, 6) = 2(-2, 3) = 2\overrightarrow{AB}$

b. $\overrightarrow{AD} = (-6, 9) = 3(-2, 3) = 3\overrightarrow{AB}$

c. $\overrightarrow{AC} = (-4, 6) = \frac{2}{3}(-6, 9) = \frac{2}{3}\overrightarrow{AD}$

13. a. Find the t values such that x and y coordinates satisfy $x^2 + y^2 = 169$ or similarly $x^2 + y^2 - 169 = 0$.

$$\begin{aligned} x^2 + y^2 - 169 &= (2 + t)^2 + (9 + t)^2 - 169 \\ &= 4 + 4t + t^2 + 81 + 18t \\ &\quad + t^2 - 169 \\ &= t^2 + 11t - 42 \\ &= (t - 3)(t + 14) \end{aligned}$$

So $x^2 + y^2 - 169 = 0$, when $t = 3$ or $t = -14$. Let A be the point where $t = 3$. So x coordinate of A is $2 + 3 = 5$, and the y coordinate is $9 + 3 = 12$. Let B be the point where $t = -14$. So x coordinate of B is $2 - 14 = -12$, and the y coordinate is $9 - 14 = -5$. So A is $(5, 12)$ and B is $(-12, -5)$

b. $\overrightarrow{AB} = (-12, -5) - (5, 12) = (-17, -17)$ and hence the length of $AB = |\overrightarrow{AB}|$

$= \sqrt{(-17)^2 + (-17)^2}$

$= \sqrt{578}$, or about 24.04

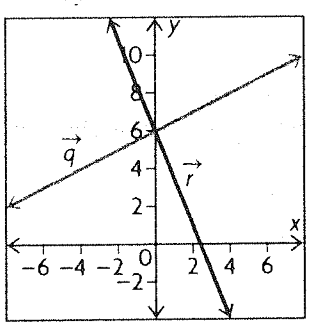
14. In the parametric form, the second equation becomes $x = 1 + 6t$, $y = 6 + 4t$, $t \in \mathbf{R}$. If t is solved for in this equation, we obtain $t = \frac{x - 1}{6}$ and $t = \frac{y - 6}{4}$.

Setting these two expressions equal to each other, the line is described by $\frac{x - 1}{6} = \frac{y - 6}{4}$, or by simplifying, $y - 6 = \frac{2}{3}x - \frac{2}{3}$. So the second equation describes a line with a slope of $\frac{2}{3}$. If y is solved for in the first expression, we see that $y = \frac{2}{3}x + 5$. $(1, 6)$ is on the second line but not the first. Hence both equations are lines with slope of $\frac{2}{3}$ with no point in common and must be parallel.

8.2 Cartesian Equation of a Line, pp. 443–444

1. a. $\vec{m} = (6, -5)$ is a direction vector parallel to the line.
b. For a vector perpendicular to the line, a suitable $\vec{n}$ has to be found, such that $\vec{m} \cdot \vec{n} = 0$. $\vec{n} = (5, 6)$ is a such a vector.
c. If $x = 0$, then $y = 9$, so $(0, 9)$ is a point on the given line.
d. A direction vector was found in part **a.**, so a vector equation for a parallel line passing through $A(7, 9)$ is $\vec{r} = (7, 9) + t(6, -5)$, $t \in \mathbf{R}$. The corresponding parametric equation is $x = 7 + 6t$, $y = 9 - 5t$, $t \in \mathbf{R}$.
e. A direction vector was found in part **b.**, so a vector equation for a perpendicular line passing through $B(-2, 1)$ is $\vec{r} = (-2, 1) + t(5, 6)$, $t \in \mathbf{R}$. The corresponding parametric equation is $x = -2 + 5t$, $y = 1 + 6t$, $t \in \mathbf{R}$.

2. a.-b.



c. Switching the components of the direction vector with the coordinates of the point on the line produces a different line.

3. a. A direction vector parallel to the line is $(8, 7)$, and if $x = 0$, then $y = -6$. So $(0, -6)$ is a point on the line. So a vector equation for the line is $\vec{r} = (0, -6) + t(8, 7)$, $t \in \mathbf{R}$. The corresponding parametric equation is $x = 8t$, $y = -6 + 7t$, $t \in \mathbf{R}$.
b. A direction vector parallel to the line is $(2, 3)$, and if $x = 0$, then $y = 5$. So $(0, 5)$ is a point on the line. So a vector equation for the line is $\vec{r} = (0, 5) + t(2, 3)$, $t \in \mathbf{R}$. The corresponding parametric equation is $x = 2t$, $y = 5 + 3t$, $t \in \mathbf{R}$.
c. The equation $y = -1$ describes a horizontal line in the xy-plane, so a direction vector parallel to this line is $(1, 0)$. Also $(0, -1)$ is a point on this line, so a vector equation for the line is $\vec{r} = (0, -1) + t(1, 0)$, $t \in \mathbf{R}$, which gives a parametric equation of $x = t$, $y = -1$, $t \in \mathbf{R}$.
d. The equation $x = 4$ describes a vertical line in the xy-plane, so a direction vector parallel to this line is $(0, 1)$. Also $(4, 0)$ is a point on this line, so a vector equation for the line is $\vec{r} = (4, 0) + t(0, 1)$, $t \in \mathbf{R}$, which gives a parametric equation of $x = 4$, $y = t$, $t \in \mathbf{R}$.

4. If the two lines have direction vectors that are collinear and share a point in common, then the two lines are coincident. In this example, both have $(3, 2)$ as a parallel direction vector and both have $(-4, 0)$ as a point on the line. Hence the two lines are coincident.

5. a. The normal vectors for the lines are $(2, -3)$ and $(4, -6)$, which are collinear. Since in two dimensions, any two direction vector perpendicular to $(2, -3)$ are collinear, the lines have collinear direction vectors. Hence the lines are parallel.
b. The lines will be coincident if they share a common point. $(0, 2)$ is a point in the first line. So the lines are coincident if and only if $4(0) - 6(2) + k = 0$, or equivalently $k = 12$. So only if $k = 12$, are the lines coincident.

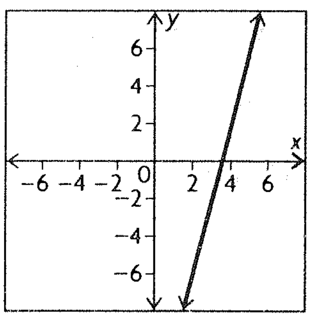
6. Since the normal vector is $(4, 5)$, the Cartesian equation of the line is $4x + 5y + k = 0$, for some constant k. Since $A(-1, 5)$ is a point on the graph,
$4(-1) + 5(5) + k = 0$.
So $k = 4 - 25 = -21$.
So the equation of the line is $4x + 5y - 21 = 0$.

7. So the slope of this line is equal to $\frac{4 - 5}{-2 - (-3)} = -1$. Hence the equation for the line satisfies $\frac{y - 5}{x - (-3)} = -1$, or by multiplying both sides by

$x - (-3)$, $y - 5 = -1(x + 3)$. Moving everything the left hand side yields $y - 5 + x + 3 = 0$, or $x + y - 2 = 0$, which is the equation in Cartesian form.

8. So the directional vector of the line is collinear with the normal vector $(2, -4)$, and so has slope equal to -2. Furthermore $P(7, 2)$ is a point on the line. Hence the equation for the line satisfies $\frac{y - 2}{x - 7} = -2$, or by multiplying both sides by $x - 7$, $y - 2 = -2(x - 7)$. Moving everything to the left side yields $y - 2 + 2x - 14 = 0$, or $2x + y - 16 = 0$, which is the equation in Cartesian form.

9. a.



b. First solve for t in both coordinates. So $t = 3 - x$ and $t = \frac{y + 2}{-4}$. Then set these two sides equal to each other to obtain $3 - x = \frac{y + 2}{-4}$, or simply $-4(3 - x) = y + 2$. So $-12 + 4x = y + 2$ or $4x - y - 14 = 0$.

10. The acute angle of the intersection between two vectors $\vec{a}$ and $\vec{b}$ is found by taking the inverse cosine of the absolute value of $\frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$.

a. $(2, -5) \cdot (-4, -1) = -3$, $|(2, -5)| = \sqrt{29}$, and $|(-4, -1)| = \sqrt{17}$. So the acute angle is $\cos^{-1}\left(\frac{3}{\sqrt{29}\sqrt{17}}\right) \doteq 82°$.

b. $(-5, 4) \cdot (1, -6) = -29$, $|(-5, 4)| = \sqrt{41}$, and $|(1, -6)| = \sqrt{37}$. So the acute angle is $\cos^{-1}\left(\frac{29}{\sqrt{41}\sqrt{37}}\right) \doteq 42°$.

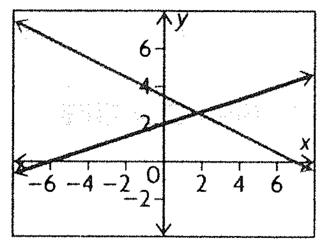
c. The direction vector for the first line is $(2, 1)$ and a direction vector for the second is $(4, -3)$. $(2, 1) \cdot (4, -3) = 5$, $|(2, 1)| = \sqrt{5}$, and $|(4, -3)| = \sqrt{25}$. So the acute angle is $\cos^{-1}\left(\frac{5}{\sqrt{25}\sqrt{5}}\right) \doteq 63°$.

d. A direction vector for the second line is $(2, 1)$. $(2, 4) \cdot (2, 1) = 8$, $|(2, 4)| = \sqrt{20}$, and $|(2, 1)| = \sqrt{5}$. So the acute angle is $\cos^{-1}\left(\frac{8}{\sqrt{20}\sqrt{5}}\right) \doteq 37°$.

e. $(2, -5) \cdot (-4, 1) = -13$, $|(2, -5)| = \sqrt{29}$, and $|(-4, 1)| = \sqrt{17}$. So the acute angle is $\cos^{-1}\left(\frac{13}{\sqrt{29}\sqrt{17}}\right) \doteq 54°$.

f. $x = 3$ has a direction vector of $(0, 1)$ and the direction vector for the second line is $(2, 1)$. $(0, 1) \cdot (2, 1) = 1$, $|(0, 1)| = \sqrt{1}$, and $|(2, 1)| = \sqrt{5}$. So the acute angle is $\cos^{-1}\left(\frac{1}{\sqrt{1}\sqrt{5}}\right) \doteq 63°$.

11. a.



b. The normal vectors are $(1, -3)$ and $(1, 2)$. $(1, -3) \cdot (1, 2) = -5$, $|(1, -3)| = \sqrt{10}$, and $|(1, 2)| = \sqrt{5}$. So the acute angle is $\cos^{-1}\left(\frac{5}{\sqrt{10}\sqrt{5}}\right) = 45°$ and the obtuse angle is $180° - 45° = 135°$.

12. a. Let the coordinates of C be (x, y). They must satisfy the equation $(x, y) - (-6, 6) + t(3, -4)$. Rewrite this equation in Cartesian form. The slope is $m = -\frac{4}{3}$. The equation is of the form $y = -\frac{4}{3}x + b$. Substitute $(-6, 6)$ into the equation to solve for b.

$$6 = -\frac{4}{3}(-6) + b$$
$$6 = 8 + b$$
$$-2 = b$$

The equation of the line is $y = -\frac{4}{3}x - 2$.
If C is the vertex of the right triangle, $\overline{CA}$ and $\overline{CB}$ must be perpendicular, meaning that their dot product must be 0.

$\overrightarrow{CA} = (-3 - x, 2 - y)$

$\overrightarrow{CB} = (8 - x, 4 - y)$

$\overrightarrow{CA} \cdot \overrightarrow{CB} = (-3 - x)(8 - x) + (2 - y)(4 - y)$

$(-3 - x)(8 - x) + (2 - y)(4 - y) = 0$

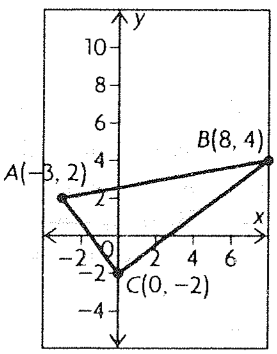
So $-24 - 5x + x^2 + 8 - 6y + y^2 = 0$.

Substitute $-\frac{4}{3}x - 2$ for y.

$$-24 - 5x + x^2 + 8 - 6y + y^2 = 0$$
$$-16 - 5x + x^2 - 6\left(-\frac{4}{3}x - 2\right) + \left(-\frac{4}{3}x - 2\right)^2 = 0$$
$$-16 - 5x + x^2 + 8x + 12 + \frac{16}{9}x^2 + \frac{16}{3}x + 4 = 0$$
$$\frac{25}{9}x^2 + 3x + \frac{16}{3}x = 0$$
$$25x^2 + 75x = 0$$
$$25x(x + 3) = 0$$

So $x = 0$ or $x = -3$.
When $x = 0$, $y = -2$.
When $x = -3$, $y = 2$. But then C would have the same coordinates as A. This would not produce a right triangle. So the coordinates of C are $(0, -2)$.

b.



c. $\overrightarrow{CA} = (-3 - 0, 2 - (-2))$
$= (-3, 4)$
$\overrightarrow{CB} = (8 - 0, 4 - (-2))$
$= (8, 6)$
$\overrightarrow{CA} \cdot \overrightarrow{CB} = (-3)(8) + (4)(6)$
$= -24 + 24$
$= 0$

Since the dot product of the vectors is 0, the vectors are perpendicular, and $\angle ACB = 90°$.

13. The sum of the interior angles of a quadrilateral is 360°. The normals make 90° angles with their respective lines at A and C. The angle of the quadrilateral at B is $180° - \theta$. Let x represent the measure of the interior angle of the quadrilateral at O.

$$90° + 90° + 180° - \theta + x = 360°$$
$$360° - \theta + x = 360°$$
$$x = \theta$$

Therefore, the angle between the normals is the same as the angle between the lines.

14. The normal vector for the first line is $(1, -1)$ and $(1, k)$ for the second. $(1, -1) \cdot (1, k) = 1 - k$, $|(1, -1)| = \sqrt{2}$, and $|(1, k)| = \sqrt{1 + k^2}$.
So $\cos(60°) = \frac{1 - k}{\sqrt{2}\sqrt{1 + k^2}}$ and $\cos(60°) = 0.5$. We obtain after squaring both sides that $\frac{1 - 2k + k^2}{2(1 + k^2)} = \frac{1}{4}$.
So $2 - 4k + 2k^2 = 1 + k^2$ or simply $k^2 - 4k + 1 = 0$. Solving by the quadratic equation gives $k = 2 \pm \sqrt{3}$.

8.3 Vector, Parametric, and Symmetric Equations of a Line in R^3, pp. 449–450

1. a. A point on this line is $(-3, 1, 8)$.
b. A point on this line is $(1, -1, 3)$.
c. A point on this line is $(-2, 1, 3)$.
d. A point on this line is $(-2, -3, 1)$.
e. A point on this line is $(3, -2, -1)$.
f. A point on this line is $\left(\frac{1}{3}, -\frac{3}{4}, \frac{2}{5}\right)$.
2. a. A direction vector is $(-1, 1, 9)$.
b. A direction vector is $(2, 1, -1)$.
c. A direction vector is $(3, -4, -1)$.
d. A direction vector is $(-1, 0, 2)$.
e. A direction vector is $(0, 0, 2)$.
f. A direction vector is $\left(\frac{1}{2}, -\frac{1}{4}, \frac{1}{2}\right)$, which if multiplied by the least common denominator, 4, yields a vector of $(2, -1, 2)$.
3. a. $\overrightarrow{AB} = (3, -3, 5) - (-1, 2, 4) = (4, -5, 1)$ is a direction vector, as well as $\overrightarrow{BA} = (-1, 2, 4) - (3, -3, 5) = (-4, 5, -1)$. So $\vec{r} = (-1, 2, 4) + t(4, -5, 1), t \in \mathbf{R}$, is one possible vector equation $\vec{q} = (3, -3, 5) + s(-4, 5, -1)$, $s \in \mathbf{R}$ is another.
b. The parametric equation corresponding with the first vector equation is $x = -1 + 4t$, $y = 2 - 5t$, $z = 4 + t$, $t \in \mathbf{R}$. The second parametric equation is $x = 3 - 4s$, $y = -3 + 5s$, $z = 5 - s$, $s \in \mathbf{R}$.
4. a. $\overrightarrow{AB} = (2, 5, -4) - (-1, 5, -4) = (3, 0, 0)$. So $(3, 0, 0)$ is a direction vector for the equation, and so $(1, 0, 0)$ may be used as the direction vector. Hence $\vec{r} = (-1, 5, 4) + t(1, 0, 0), t \in \mathbf{R}$, is a vector equation for a line containing the points $A(A-1, 5, -4)$ and $B(2, 5, -4)$.
b. The corresponding parametric equation is $x = -1 + t$, $y = 5$, $z = -4$, $t \in \mathbf{R}$.
c. Since two of the coordinates in the direction vector are zero, a symmetric equation cannot exist.

5. a. So $\vec{r} = (-1, 2, 1) + t(3, -2, 1), t \in \mathbf{R}$, is a vector equation for the line and the corresponding parametric equation is $x = -1 + 3t$, $y = 2 - 2t$, $z = 1 + t$, $t \in \mathbf{R}$. So the symmetric equation is $\frac{x+1}{3} = \frac{y-2}{-2} = \frac{z-1}{1}$.
b. $\overrightarrow{AB} = (-1, 2, 1) - (-1, 1, 0) = (0, 1, 1)$ is a direction vector for the line. So $\vec{r} = (-1, 1, 0) + t(0, 1, 1), t \in \mathbf{R}$ is a vector equation for the line and the corresponding parametric equation is $x = -1$, $y = 1 + t$, $z = t$, $t \in \mathbf{R}$. So the symmetric equation is $\frac{y-1}{1} = \frac{z}{1}$, $x = -1$.
c. $\overrightarrow{MN} = (-2, 4, 7) - (-2, -2, 1) = (0, -6, -6)$ is a direction vector for the line. Since $(0, -6, -6) = -6(0, 1, 1)$, $(0, 1, 1)$ is also a direction vector for this line. So $\vec{r} = (-2, 3, 0) + t(0, 1, 1), t \in \mathbf{R}$ is a vector equation for the line and the corresponding parametric equation is $x = -2$, $y = 3 + t$, $z = t$, $t \in \mathbf{R}$. So the symmetric equation is $\frac{y-3}{1} = \frac{z}{1}$, $x = -2$.

d. $\overrightarrow{DE} = (-1, 1, 0) - (-1, 0, 0) = (0, 1, 0)$ is a direction vector for the line. So $\vec{r} = (-1, 0, 0) + t(0, 1, 0), t \in \mathbf{R}$ is a vector equation for the line and the corresponding parametric equation is $x = -1$, $y = t$, $z = 0$, $t \in \mathbf{R}$. Since two of the coordinates in the direction vector are zero, there is no symmetric equation for this line.
e. $\overrightarrow{XO} = (-4, 3, 0) - (0, 0, 0) = (-4, 3, 0)$ is a direction vector for the line. So $\vec{r} = t(-4, 3, 0)$, $t \in \mathbf{R}$ is a vector equation for the line and the corresponding parametric equation is $x = -4t$, $y = 3t$, $z = 0$, $t \in \mathbf{R}$. So the symmetric equation is $\frac{x}{-4} = \frac{y}{3}$, $z = 0$.
f. The direction vector for the z-axis is $(0, 0, 1)$, so a line parallel to the z-axis has $(0, 0, 1)$ as a direction vector. So $\vec{r} = (1, 2, 4) + t(0, 0, 1), t \in \mathbf{R}$ is a vector equation for the line and the corresponding parametric equation is $x = 1$, $y = 2$, $z = 4 + t$, $t \in \mathbf{R}$. Since two of the coordinates in the direction vector are zero, there is no symmetric equation for this line.

6. a. So the first line is given by $\frac{x+6}{1} = \frac{y-10}{-1} = \frac{z-7}{1}(=t)$. If x, y, and z are solved for in terms of t, the corresponding parametric equations is $x = -6 + t$, $y = 10 - t$, $z = 7 + t$, $t \in \mathbf{R}$. So the first line has a direction vector of $(1, -1, 1)$. The second line is given by $\frac{x+7}{1} = \frac{y-11}{-1}(=s)$, $z = 5$. If x and y are solved for in terms of s, $x = -7 + s$, and $y = 11 - s$ are obtained. So the parametric equation for the second line is $x = -7 + s$, $y = 11 - s$, $z = 5$, $s \in \mathbf{R}$, and so has a direction vector of $(1, -1, 0)$.
b. $(1, -1, 1) \cdot (1, -1, 0)$
$= 1(1) - 1(-1) + 0(1) = 2$. $|(1, -1, 0)| = \sqrt{2}$ and $|(1, -1, 1)| = \sqrt{3}$. So the angle between the two lines is $\cos^{-1}\left(\frac{2}{\sqrt{2}\sqrt{3}}\right) \doteq 35.3°$.
7. The directional vector of the first line is $(8, 2, -2) = -2(-4, -1, 1)$. So $(-4, -1, 1)$ is a directional vector for the first line as well. Since $(-4, -1, 1)$ is also the directional vector of the second line, the lines are the same if the lines share a point. $(1, 1, 3)$ is a point on the second line. Since $1 = \frac{1+7}{8} = \frac{1+1}{2} = \frac{3-5}{-2}$, $(1, 1, 3)$ is a point on the first line as well. Hence the lines are the same.
8. a. The line that passes through $(0, 0, 3)$ with a directional vector of $(-3, 1, -6)$ is given by the parametric equation is $x = -3t$, $y = t$, $z = 3 - 6t$, $t \in \mathbf{R}$. So the y coordinate is equal to -2 only when $t = -2$. At $t = -2$, $x = -3(-2) = 6$ and $z = 3 - 6(-2) = 15$. So $A(6, -2, 15)$ is a point on the line. So the y coordinate is equal to 5 only when $t = 5$. At $t = 5$, $x = -3(5) = -15$ and $z = 3 - 6(5) = -27$. So $B(-15, 5, -27)$ is a point on the line.
b. Since the point A occurs when $t = -2$, and point B occurs when $t = 5$, the line segment connecting the two points is precisely all the t values between -2 and 5. So the equation is $x = -3t$, $y = t$, $z = 3 - 6t$, $-2 \le t \le 5$.
9. The direction vector for the first line is $(k, 2, k - 1)$ and for the second line is $(-2, 0, 1)$. The lines are perpendicular precisely when $(k, 2, k - 1) \cdot (-2, 0, 1) = 0$. So $(k, 2, k - 1) \cdot (-2, 0, 1)$
$= -2(k) + 0(2) + 1(k - 1) = -k - 1$. So if $k = -1$, then $(k, 2, k - 1) \cdot (-2, 0, 1) = 0$, and the lines are perpendicular.
10. a. Three different points occur at three different values of t. At $t = -1$, the corresponding point on the line is $(4, -2, 5) - (-4, -6, 8) = (8, 4, -3)$. At $t = 1$, the corresponding point on the line is

$(4, -2, 5) + (-4, -6, 8) = (0, -8, 13)$. The point at the origin is $(4, -2, 5)$.
b. Three different points occur at three different values of s. At $s = -1$, the corresponding point on the line is when $x = -4 + 5(-1) = -9$, $y = 2 - (-1) = 3$, and $z = 9 - 6(-1) = 15$. At $s = 1$, the corresponding point on the line is when $x = -4 + 5(1) = 1$, $y = 2 - (1) = 1$, and $z = 9 - 6(1) = 3$. So $(-9, 3, 15)$ and $(1, 1, 3)$ are two points on the line. The point at the origin is $(-4, 2, 9)$.
c. $\frac{x+1}{3} = \frac{y-2}{-1} = \frac{z}{4}$ is actually equal to $\frac{x+1}{3} = \frac{y-2}{-1} = \frac{z}{4}(=t)$, for any $t \in \mathbf{R}$. So we can pick different t values to obtain different points on the lines. At $t = -1$, the corresponding point on the line is found by solving for x, y, and z, in the equation $\frac{x+1}{3} = \frac{y-2}{-1} = \frac{z}{4}(=-1)$. So
$x = (-1)3 - 1 = -4$, $y = (-1)(-1) + 2 = 3$, and $z = (-1)4 = -4$. So $(-4, 3, -4)$ is a point on the line. At $t = 1$ and solving for x, y, and z, in the equation $\frac{x+1}{3} = \frac{y-2}{-1} = \frac{z}{4}(=1)$, yields
$x = (1)3 - 1 = 2$, $y = (1)(-1) + 2 = 1$, and $z = (1)4 = 4$. So $(2, 1, 4)$ is a point on the line. Also the point at the origin is $(-1, 2, 0)$.
d. $x = -4$, $\frac{y-2}{3} = \frac{z-3}{5}$ is actually equal to $x = -4$, $\frac{y-2}{3} = \frac{z-3}{5}(=t)$, for any $t \in \mathbf{R}$. So we can pick different t values to obtain different points on the lines. At $t = -1$, the corresponding point on the line is found by solving for x, y, and z, in the equation $x = -4$, $\frac{y-2}{3} = \frac{z-3}{5}(=-1)$. So
$x = -4$, $y = (-1)(3) + 2 = -1$, and $z = (-1)5 + 3 = -2$. So $(-4, -1, -2)$ is a point on the line. At $t = 1$ and solving for x, y, and z, in the equation $x = -4$, $\frac{y-2}{3} = \frac{z-3}{5}(=1)$, yields
$x = -4$, $y = (1)(3) + 2 = 5$, and $z = (1)5 + 3 = 8$. So $(-4, 5, 8)$ is a point on the line. Also the point at the origin is $(-4, 2, 3)$.
11. For part **a.** the corresponding parametric equation is $x = 4 - 4t$, $y = -2 - 6t$, $z = 5 + 8t$, $t \in \mathbf{R}$. The corresponding symmetric equation is $\frac{x-4}{-4} = \frac{y+2}{-6} = \frac{z-5}{8}$.
For part **b.** the corresponding vector equation is $\vec{r} = (-4, 2, 9) + s(5, -1, -6)$, $s \in \mathbf{R}$. The corresponding symmetric equation is $\frac{x+4}{5} = \frac{y-2}{-1} = \frac{z-9}{-6}$.
For part **c.** the point at the origin is $(-1, 2, 0)$ and the direction vector is $(3, -1, 4)$. So the corresponding vector equation is $\vec{r} = (-1, 2, 0) + t(3, -1, 4)$ $t \in \mathbf{R}$, and parametric equation $x = -1 + 3t$, $y = 2 - t$, $z = 4t$, $t \in \mathbf{R}$.
For part **d.** the point at the origin is $(-4, 2, 3)$ and a direction vector is $(0, 3, 5)$. So the corresponding vector equation is $\vec{r} = (-4, 2, 3) + t(0, 3, 5)$, $t \in \mathbf{R}$, and parametric equation $x = -4$, $y = 2 + 3t$, $z = 3 + 5t$, $t \in \mathbf{R}$.
12. The direction vector of the first line is $(-4, -7, 3)$ and the direction vector of the second line is $(3, 2, 4)$. The cross product of these two vectors gives a vector that is perpendicular to both direction vectors.
$(-4, -7, 3) \times (3, 2, 4)$
$= ((-7)4 - (3)2, (3)3 - (-4)4, (-4)2 - (-7)3)$
$= (-28 - 6, 9 + 16, -8 + 21)$
$= (-34, 25, 13)$
So a line with a direction vector of $(-34, 25, 13)$ is perpendicular to the two initial lines. A parametric equation of such a line passing through the point $(2, -5, 0)$ is $x = 2 - 34t$, $y = -5 + 25t$, $z = 13t$, $t \in \mathbf{R}$.
13. Since $x = 10 + 2s$, $y = 5 + s$, and $z = 2$, if $x^2 + y^2 + z^2 = 9$, then $(10 + 2s)^2 + (5 + s)^2 + (2)^2 = 9$ or equivalently $(10 + 2s)^2 + (5 + s)^2 + (2)^2 - 9 = 0$.
$(10 + 2s)^2 + (5 + s)^2 + (2)^2 - 9$
$= 5s^2 + 50s + 120$
$= 5(s + 6)(s + 4)$.
So if $x^2 + y^2 + z^2 = 9$, then $s = -6$ or $s = -4$. Also if $s = -6$ or $s = -4$, then $x^2 + y^2 + z^2 = 9$. So the only two points occur at $s = -6$ and $s = -4$. At $s = -6$, $x = 10 + 2(-6) = -2$, $y = 5 + (-6) = -1$, and $z = 2$, or $(-2, -1, 2)$. At $s = -4$, $x = 10 + 2(-4) = 2$, $y = 5 + (-4) = 1$, and $z = 2$, or $(2, 1, 2)$.
14. Let $P_1(4 + 2t, 4 + t, -3 - t)$ and $P_2(-2 + 3s, -7 + 2s, 2 - 3s)$ be two such points for some real numbers s and t. So $\overline{P_1P_2}$ is perpendicular to the lines L_1 and L_2, and so since

the direction vectors for the lines are $(2, 1, -1)$ and $(3, 2, -3)$, respectively, $\overline{P_1P_2} \cdot (2, 1, -1) = 0$ and $\overline{P_1P_2} \cdot (3, 2, -3) = 0$.

$\overline{P_1P_2} = (-2 + 3s, -7 + 2s, 2 - 3s) - (4 + 2t, 4 + t, -3 - t)$
$= (-6 + 3s - 2t, -11 + 2s - t, 5 - 3s + t)$

So $\overline{P_1P_2} \cdot (2, 1, -1) = 2(-6 + 3s - 2t) + 1(-11 + 2s - t) + (-1)(5 - 3s + t) = -28 + 11s - 6t = 0$.

So $\overline{P_1P_2} \cdot (3, 2, -3) = 3(-6 + 3s - 2t) + 2(-11 + 2s - t) + (-3)(5 - 3s + t) = -55 + 22s - 11t = 0$

So $(-2)[\overline{P_1P_2} \cdot (2, 1, -1)] + [\overline{P_1P_2} \cdot (3, 2, -3)] = -2(0) + 0 = 0$.

Yet $(-2)[\overline{P_1P_2} \cdot (2, 1, -1)] + [\overline{P_1P_2} \cdot (3, 2, -3)] = (-2)(-28 + 11s - 6t) + (-55 + 22s - 11t) = 1 + t$. So $1 + t = 0$, or $t = -1$.

Since $-28 + 11s - 6t = 0$, $-28 + 11s - 6(-1) = 0$, or $11s = 22$. So $s = 2$.

At $t = -1$, $x = 4 + 2(-1) = 2$, $y = 4 + (-1) = 3$, and $z = -3 - (-1) = -2$. At $s = 2$, $x = -2 + 3(2) = 4$, $y = -7 + 2(2) = -3$, and $z = 2 - 3(2) = -4$. So $P_1(2, 3, -2)$ and $P_2(4, -3, -4)$ are the points that work.

15. The direction vector for the first line is $(2, 1, 0)$ and the direction vector for the second line is $(3, 2, 1)$. $(2, 1, 0) \cdot (3, 2, 1) = 2(3) + 1(2) + 0(1) = 8$. $|(2, 1, 0)| = \sqrt{5}$ and $|(3, 2, 1)| = \sqrt{14}$. So the angle between the two lines is $\cos^{-1}\left(\frac{8}{\sqrt{5}\sqrt{14}}\right) \doteq 17°$.

Chapter 8 Mid-Chapter Review, pp. 451–452

1. a. Any three different t values yield three different points. At $t = -1$, $x = 2(-1) - 5 = -7$, $y = 3(-1) + 1 = -2$. At $t = 0$, $x = 2(0) - 5 = -5$, $y = 3(0) + 1 = 1$, and at $t = 1$, $x = 2(1) - 5 = -3$, $y = 3(1) + 1 = 4$. So $(-7, -2)$, $(-5, 1)$, and $(-3, 4)$ are three points on the line.

b. Pick any three s values. At $s = -1$, $(2, 3) + (-1)(3, -2) = (-1, 5)$. At $s = 0$, $(2, 3) + (0)(3, -2) = (2, 3)$, and at $s = 1$, $(2, 3) + (1)(3, -2) = (5, 1)$. So $(-1, 5)$, $(2, 3)$, and $(5, 1)$ are three points on the line.

c. Pick three different x values and solve for y to obtain the three points. At $x = -1$, $3(-1) + 5y - 8 = 0$, or $5y = 11$. So $y = \frac{11}{5}$, when $x = -1$. Similarly at $x = 0$, $3(0) + 5y - 8 = 0$, or $y = \frac{8}{5}$. At $x = 1$, $3(1) + 5y - 8 = 0$, or $y = \frac{5}{5} = 1$. So three points on the line are $\left(-1, \frac{11}{5}\right)$, $\left(0, \frac{8}{5}\right)$, and $(1, 1)$.

d. $\frac{x - 1}{3} = \frac{y + 2}{2} = \frac{z - 5}{1}$ is actually equal to $\frac{x - 1}{3} = \frac{y + 2}{2} = \frac{z - 5}{1} (= t)$, for any $t \in \mathbf{R}$. So we can pick different t values to obtain different points on the lines. At $t = -1$, the corresponding point on the line is found by solving for x, y, and z, in the equation $\frac{x - 1}{3} = \frac{y + 2}{2} = \frac{z - 5}{1} (= -1)$. So $x = (-1)3 + 1 = -2$, $y = (-1)2 - 2 = -4$, and $z = (-1)1 + 5 = 4$. So $(-2, -4, 4)$ is a point on the line. At $t = 1$ and solving for x, y, and z, in the equation $\frac{x - 1}{3} = \frac{y + 2}{2} = \frac{z - 5}{1} (= 1)$, yields $x = (1)3 + 1 = 4$, $y = (1)2 - 2 = 0$, and $z = (1)1 + 5 = 6$. So $(4, 0, 6)$ is another point on the line. Also the point at the origin is $(1, -2, 5)$.

2. a. The x-intercept occurs when $y = 0$, so solve for the t values when $y = 0$, to find the point. At $y = 0$, $1 + 5t = 0$, so $t = \frac{-1}{5}$. So $x = 3 - 3\left(\frac{-1}{5}\right) = \left(\frac{18}{5}\right)$. So the x-intercept is at $\left(\frac{18}{5}, 0\right)$. The y-intercept occurs when $x = 0$, or $3 - 3t = 0$. So at the y-intercept, $t = 1$. So $y = 1 + 5(1) = 6$. So the y-intercept is at $(0, 6)$.

b. The x-intercept occurs when $y = 0$, so solve for the s values when $y = 0$, to find the point. At $y = 0$, $3 - 2s = 0$, so $s = \frac{3}{2}$. So $x = -6 + 2\left(\frac{3}{2}\right) = -\frac{14}{3}$. So the x-intercept is at $\left(-\frac{14}{3}, 0\right)$. The y-intercept occurs when $x = 0$, or $-6 + 2s = 0$. So at the y-intercept, $t = 3$. So $y = 3 - 2(3) = -3$. So the y-intercept is at $(0, -3)$.

3. The direction vector for the first line is $(-4, 7)$ and the direction vector for the second is $(2, 1)$. $(-4, 7) \cdot (2, 1) = -1$, $|(-4, 7)| = \sqrt{65}$, and $|(2, 1)| = \sqrt{5}$. So the angle between the lines is $\cos^{-1}\left(\frac{-1}{\sqrt{5}\sqrt{65}}\right) \doteq 93.2°$. The acute angle between the lines is approximately $180° - 93.2° = 86.8°$.

4. The direction vector for the x-axis is $(1, 0)$ and the direction vector for the y-axis is $(0, 1)$. The direction vector of the line is $(4, -5)$. $(4, -5) \cdot (1, 0) = 4$, $|(4, -5)| = \sqrt{41}$, and $|(1, 0)| = \sqrt{1} = 1$. So the angle the line makes with the x-axis is $\cos^{-1}\left(\frac{4}{1\sqrt{41}}\right) \doteq 51°$. $(4, -5) \cdot (0, 1) = -5$,

$|(4, -5)| = \sqrt{41}$, and $|(0, 1)| = \sqrt{1} = 1$. So the angle the line makes with the y-axis is $\cos^{-1}\left(\frac{-5}{1\sqrt{41}}\right) \doteq 141°$. The acute angle between them is approximately $180° - 141° = 39°$.

5. Since the perpendicular line has $(5, -7)$ as a direction vector, $(5, -7)$ is a normal vector for the desired line. So a Cartesian equation for this line is $5x - 7y + C = 0$, for some constant C. C is found by knowing that $(4, -3)$ is a point on the line. So $5(4) - 7(-3) + C = 0$ or $41 + C = 0$. Hence $C = -41$, and the Cartesian equation is $5x - 7y - 41 = 0$.

6. Parallel lines have collinear direction vectors. Since the direction vector for the first line is $(3, -4, 4)$, it may also be the direction vector for the desired line. The symmetric equation for this line having $(0, 0, 2)$ as its origin point is $\frac{x}{3} = \frac{y}{-4} = \frac{z-2}{4}$.

7. $\overrightarrow{KL} = (3, -5, 6) - (2, 4, 5) = (1, -9, 1)$. Since parallel lines have collinear direction vectors, $(1, -9, 1)$ may be taken to be the direction vector for the parallel line. So the parametric equation with $(1, 2, 5)$ as its origin point is $x = 1 + t$, $y = 2 - 9t$, $z = 5 + t$, $t \in \mathbf{R}$.

8. The direction vector for this line is $(2, -8, 7)$. The direction angles are found by finding the angles this vector makes with the coordinate axes. The direction vectors for the x-axis, y-axis, and z-axis are $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, respectively.
$|(2, -8, 7)| = \sqrt{2^2 + (-8)^2 + 7^2} = \sqrt{117}$, and $|(1, 0, 0)| = |(0, 1, 0)| = |(0, 0, 1)| = \sqrt{1} = 1$.
$(2, -8, 7) \cdot (1, 0, 0) = 2$, so the angle the line makes with the x-axis is $\cos^{-1}\left(\frac{2}{1\sqrt{117}}\right) \doteq 79.3°$.
$(2, -8, 7) \cdot (0, 1, 0) = -8$, so the angle the line makes with the y-axis is $\cos^{-1}\left(\frac{-8}{1\sqrt{117}}\right) \doteq 137.7°$.
$(2, -8, 7) \cdot (0, 0, 1) = 7$, so the angle the line makes with the z-axis is $\cos^{-1}\left(\frac{7}{1\sqrt{117}}\right) \doteq 49.7°$.
So the direction angles are approximately $79.3°$, $137.7°$, and $49.7°$.

9. If (a, b, c) is a unit vector with direction vectors $60°$, $90°$, and $30°$, then
$\cos(60°) = (a, b, c) \cdot (1, 0, 0)$. Yet $\cos(60°) = \frac{1}{2}$ and $(a, b, c) \cdot (1, 0, 0) = a$. So $a = \frac{1}{2}$. Similarly $\cos(90°) = (a, b, c) \cdot (0, 1, 0)$, so $0 = b$. Also $\cos(30°) = (a, b, c) \cdot (0, 0, 1)$, so $\frac{\sqrt{3}}{2} = c$. So $\left(\frac{1}{2}, 0, \frac{\sqrt{3}}{2}\right)$ is a direction vector for the line, as well as $(1, 0, \sqrt{3})$.
So the symmetric equation of the line with this direction vector and $P(3, -4, 6)$ as an origin point is $\frac{x-3}{1} = \frac{z-6}{\sqrt{3}}$, $y = -4$.

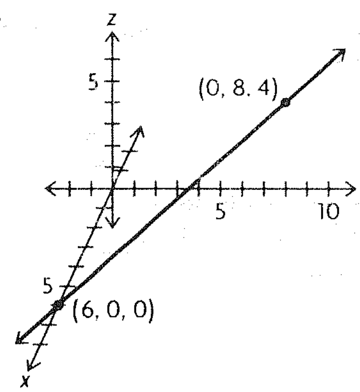
10. The direction vectors for the x-axis, y-axis, and z-axis are $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, respectively. The origin is a point on each of the axes, so it may be taken as the origin point for each equation. So a parametric equation for the x-axis is $x = t$, $y = 0$, $z = 0$, $t \in \mathbf{R}$. Similarly a parametric equation for the y-axis is $x = 0$, $y = t$, $z = 0$, $t \in \mathbf{R}$, and a parametric equation for the z-axis is $x = 0$, $y = 0$, $z = t$, $t \in \mathbf{R}$.

11. a. The direction vector for the first line is $(k + 1, 3k + 1, k - 3)$ and the direction vector for the second line is $(-3, -10, -5)$. The lines are parallel if and only if the direction vectors are collinear. The vectors are collinear only when one is a multiple of the other, which happens only when the ratio between the coordinates is constant. So the direction vectors are parallel if and only if $\frac{k+1}{-3} = \frac{3k+1}{-10} = \frac{k-3}{-5}$. If $\frac{k+1}{-3} = \frac{3k+1}{-10}$, then $-10(k + 1) = -3(3k + 1)$ or $-10k - 10 = -9k - 3$. So $k = -7$. If $k = -7$, then $\frac{k+1}{-3} = \frac{3k+1}{-10} = 2$, and since $\frac{-7-3}{-5} = 2$ as well, the ratios are a constant. So the lines are parallel if $k = -7$. The lines are perpendicular if and only if the dot product of the direction vectors is zero.
$(k + 1, 3k + 1, k - 3) \cdot (-3, -10, -5) =$
$(k + 1)(-3) + (3k + 1)(-10) + (k - 3)(-5)$
$= -38k + 2$.
So the dot product is zero when $-38k + 2 = 0$, or simply $k = \frac{1}{19}$. So if $k = \frac{1}{19}$, then the lines are perpendicular.

12. The x-intercept occurs when $y = 0$, so solve for the x value when $y = 0$, to find the point. At $y = 0$, $\frac{x-6}{3} = \frac{0+8}{-2} = -4$, so $x = (-4)3 + 6 = -6$. So the x-intercept is at $(-6, 0)$. The y-intercept occurs when $x = 0$. So at the y-intercept, $\frac{y+8}{-2} = \frac{0-6}{3} = -2$, so $y = (-2)(-2) - 8 = -4$. So the y-intercept is at $(0, -4)$. So the triangle with the origin has a base of 6 units and a height of 4 units.
Hence the hypotenuse has a length of $\sqrt{4^2 + 6^2} = \sqrt{52}$. So the perimeter is equal to $4 + 6 + \sqrt{52} \doteq 17.2$ units. The area of the triangle is $\frac{1}{2} \times 4 \times 6 = 12$.

13. a. Solving the Cartesian equation for y yields $y = \frac{-3}{4}x + 6$. So the direction of the line is $(4, -3)$

and the y-intercept, $(0, 6)$, may be the origin point of the line. So a vector equation is $\vec{r} = (0, 6) + t(4, -3), t \in \mathbf{R}$.
b. The corresponding parametric equation for the vector equation in part **a.** is $x = 4t$, $y = 6 - 3t, t \in \mathbf{R}$.
c. A direction vector for the x-axis is $(1, 0)$. $(4, -3) \cdot (1, 0) = 4$, $|(4, -3)| = \sqrt{25} = 5$, and $|(1, 0)| = 1$. So the angle between the line and the x-axis is $\cos^{-1}\left(\frac{4}{5}\right) \doteq 36.9°$.
d. The normal vector for the line is $(3, 4)$, which is a vector perpendicular to the line. So a line with the origin as its origin point with a direction vector of $(3, 4)$ is $\vec{r} = t(3, 4), t \in \mathbf{R}$.
14. $\frac{4 - 6}{8 - (-4)} = \frac{-2}{12} = -\frac{1}{6}$, is the slope of the line connecting $A(-4, 6)$ and $B(8, 4)$. Since $A(-4, 6)$ is a point, the scalar equation can be found from $\frac{y - 6}{x - (-4)} = -\frac{1}{6}$. So $6(y - 6) = -1(x + 4)$, which gives $x + 6y - 32 = 0$ as the scalar equation. $\vec{AB} = (8, 4) - (-4, 6) = (12, -2)$ is a direction vector for the line and we may take $A(-4, 6)$ to be the origin point for the line. So a vector equation for the line is $\vec{r} = (-4, 6) + t(12, -2), t \in \mathbf{R}$. The corresponding parametric equation is $x = -4 + 12t$, $y = 6 - 2t, t \in \mathbf{R}$.
15. The direction vector for the given line is $(2, -4)$. So a vector (t_1, t_2), is normal to $(2, -4)$ if $(2, -4) \cdot (t_1, t_2) = 0$ and t_1 and t_2 are not both zero.
$(2, -4) \cdot (t_1, t_2) = 2(t_1) - 4(t_2)$
So if $t_1 = 2$ and $t_2 = 1$, then $(2, -4) \cdot (t_1, t_2) = 0$.
So $(2, 1)$ is normal to the line. Since $|(2, 1)| = \sqrt{5}$, $\frac{1}{\sqrt{5}}(2, 1) = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$ is a unit vector normal to the given line.
16. a. Since the slope is $-\frac{2}{3}$, a direction vector for the line is $(3, -2)$. A parametric equation with an origin point of $(-5, 10)$ is $x = -5 + 3t$, $y = 10 - 2t, t \in \mathbf{R}$.
b. The direction vector for the given line is $(2, -2)$. A vector, (t_1, t_2), perpendicular to $(2, -2)$ satisfies $(2, -2) \cdot (t_1, t_2) = 0$ and t_1 and t_2 are not both zero. $(2, -2) \cdot (t_1, t_2) = 2(t_1) - 2(t_2)$
So if $t_1 = 1$ and $t_2 = 1$, then $(2, -4) \cdot (t_1, t_2) = 0$.
So $(1, 1)$ is a direction vector for a line perpendicular to the given line. A parametric equation for this perpendicular line with an origin point of $(1, -1)$ is $x = 1 + t$, $y = -1 + t, t \in \mathbf{R}$.
c. A direction vector for the line is $(0, 10) - (0, 7) = (0, 3)$. So we may use $(0, 1)$ as a direction vector. Also since the line connecting $(0, 10)$ and $(0, 7)$ has the origin as a point, the origin may be used as the origin point for the parametric equation. Hence a parametric equation for the line is $x = 0$, $y = t, t \in \mathbf{R}$.
17. a. The coordinate planes are $x = 0$, $y = 0$, and $z = 0$. If the x coordinate of the line is zero, then $12 - 3t = 0$, so $t = 4$. So the line intersects the yz-plane when $t = 4$. Since $(12, -8, -4) + 4(-3, 4, 2) = (0, 8, 4)$, the line intersects the yz-plane at $(0, 8, 4)$. Similarly if the y coordinate of the line is zero, $-8 + 4t = 0$, so $t = 2$. Since $(12, -8, -4) + 2(-3, 4, 2) = (6, 0, 0)$, the line intersects the xz-plane at $(6, 0, 0)$. If the z coordinate of the line is zero, t is also 2. Hence the line intersects the xy-plane at $(6, 0, 0)$.
b. Since any line intersecting a coordinate axis intersects two coordinate planes at the same point, the only possible points for intersection with an axis are $(0, 8, 4)$ and $(6, 0, 0)$. $(0, 8, 4)$ does not lie on a coordinate axis, but $(6, 0, 0)$ lies on the x-axis. So the line intersects the x-axis at $(6, 0, 0)$.
c.



18. a. Choose P_0 to be the origin point for the equations. So the vector equation is $\vec{r} = (1, -2, 8) + t(-5, -2, 1), t \in \mathbf{R}$. The corresponding parametric equation is $x = 1 - 5t$, $y = -2 - 2t$, $z = 8 + t, t \in \mathbf{R}$, and the symmetric equation is $\frac{x - 1}{-5} = \frac{y + 2}{-2} = \frac{z - 8}{1}$.
b. Choose P_0 to be the origin point for the equations. So the vector equation is $\vec{r} = (3, 6, 9) + (2, 4, 6)$, $t \in \mathbf{R}$. The corresponding parametric equation is $x = 3 + 2t$, $y = 6 + 4t$, $z = 9 + 6t, t \in \mathbf{R}$, and the symmetric equation is $\frac{x - 3}{2} = \frac{y - 6}{4} = \frac{z - 9}{6}$.
c. Choose P_0 to be the origin point for the equations. So the vector equation is $\vec{r} = (0, 0, 6) + t(-1, 5, 1)$, $t \in \mathbf{R}$. The corresponding parametric equation is

$x = -t, y = 5t, z = 6 + t, t \in \mathbf{R}$, and the symmetric equation is $\frac{x}{-1} = \frac{y}{5} = \frac{z-6}{1}$.

d. Choose P_0 to be the origin point for the equations. So the vector equation is $\vec{r} = (2, 0, 0) + t(0, 0, -2)$, $t \in \mathbf{R}$. The corresponding parametric equation is $x = 2, y = 0, z = -2t, t \in \mathbf{R}$. Since the direction vector has two zero coordinates, there is no symmetric equation for this line.

19. A line parallel to the line connecting the points $(-4, 5, 6)$ and $(6, -5, 4)$ has a direction vector of $(6, -5, 4) - (-4, 5, 6) = (10, -10, -2)$. Since collinear vectors of $(10, -10, -2)$ are also direction vectors for the line, $(5, -5, -1)$ is a direction vector. So the vector equation for a line with a direction vector of $(5, -5, -1)$ passing through the origin is $\vec{r} = t(5, -5, -1), t \in \mathbf{R}$.

20. The midpoint between $(2, 6, 10)$ and $(-4, 4, -8)$ is precisely $\frac{1}{2}[(2, 6, 10) + (-4, 4, -8)] = (-1, 5, 1)$. The line connecting the midpoint and the given point has a direction vector of $(0, -8, 1) - (-1, 5, 1) = (1, -13, 0)$. So the parametric equations of the line through the desired points is $x = t, y = -8 - 13t, z = 1, t \in \mathbf{R}$.

21. The direction vector for the first line is $(1, 3, -5)$, and the direction vector for the second line is $(-3, -9, 15) = -3(1, 3, -5)$. So the direction vectors are collinear. The direction vectors are collinear if and only if the lines are parallel, so the equations describe parallel lines.

22. Since $\frac{7-4}{3} = \frac{-1+2}{1} = \frac{8-6}{2} = 1$, the point $(7, -1, 8)$ lies on the line.

8.4 Vector and Parametric Equations of a Plane, pp. 459–460

1. a. plane; This is a vector equation of a plane in R^3.
b. line; This is a vector equation of a line in R^3.
c. line; This is a parametric equation for a line in R^3.
d. plane; This is a parametric equation of a plane in R^3 using $(0, 0, 0)$ as $\vec{r}_0$.

2. a. The first direction vector can be expressed with integers as follows:
$$\left(\frac{1}{3}, -2, \frac{3}{4}\right) \times 12 = (4, -24, 9).$$
b. The second direction vector can be reduced as follows:
$$(6, -12, 30) \times \frac{1}{6} = (1, -2, 5)$$
c. The resulting equation of the plane using the two new direction vectors is:
$\vec{r} = (2, 1, 3) + s(4, -24, 9) + t(1, -2, 5), t, s \in \mathbf{R}$

3. a. By inspection, if we choose $n = m = 0$, we get the point $(0, 0, -1)$.
b. Collecting the vector components of the n, and m, multiples we can rewrite the equation of the plane in vector form as:
$\vec{r} = (0, 0, -1) + m(2, -3, -3) + n(0, 5, -2)$; $m, n \in \mathbf{R}$
Thus our direction vectors are:
$(2, -3, -3)$ and $(0, 5, -2)$
c. $\vec{r} = (0, 0, -1) + m(2, -3, -3) + n(0, 5, -2)$; $m, n \in \mathbf{R}$
Letting $m = -1$ and $n = -4$ we get:
$$\vec{r} = (0, 0, -1) + (-1)(2, -3, -3) + (-4)(0, 5, -2)$$
$$= (0, 0, -1) + (-2, 3, 3) + (0, -20, 8)$$
$$= (-2, -17, 10)$$
d. Letting $\vec{r} = A(0, 15, 17)$
$$A(0, 15, -7) = (0, 0, -1) + m(2, -3, -3) + n(0, 5, -2)$$
We get the following parametric equations:
$0 = 0 + 2m + 0n$;
$0 = m$.
$15 = 0 + (-3)m + 5n$
$15 = 5n$
$3 = n$.
$-7 = -1 - 3m - 2n$; for $m = 0$ and $n = 3$ we get:
$-7 = -1 - 3(0) - 2(3)$
$-7 = -7$
So our solution is $m = 0$ and $n = 3$.
e. For the point $B(0, 15, -8)$ the first two parametric equations are the same; yielding $m = 0$ and $n = 3$, however the third equation would then give:
$-8 = -1 - 3m - 2n$
$-8 = -1 - 3(0) - 2(3)$
$-8 = -7$ which is not true. So there can be no solution.

4. a. $P(-2, 3, 1), Q(-2, 3, 2), R(1, 0, 1)$
$$\overrightarrow{PQ} = Q - P = (-2 - (-2), 3 - 3, 2 - 1) = (0, 0, 1)$$
$$\overrightarrow{PR} = R - P = (1 - (-2), 0 - 3, 1 - 1) = (3, -3, 0)$$
$$\vec{r} = (-2, 3, 1) + t(0, 0, 1) + s(3, -3, 0)$$
b. $\overrightarrow{QR} = R - Q = (1 - (-2), 0 - 3, 1 - 2) = (3, -3, -1)$

Using $\overrightarrow{PQ}$ as the other direction vector:
$\vec{r} = (-2, 3, -2) + t(0, 0, 1) + s(3, -3, -1)$, $t, s \in \mathbf{R}$
Using $\overrightarrow{PR}$ as the other direction vector:
$\vec{r} = (1, 0, 1) + t(3, -3, 0) + s(3, -3, -1), t, s \in \mathbf{R}$
5. a. $\vec{r} = (1, 0, -1) + s(2, 3, -4) + t(4, 6, -8)$, $t, s \in \mathbf{R}$, does not represent a plane because the direction vectors are the same. We can rewrite the second direction vector as:
$(2)(2, 3, -4)$
And so we can rewrite the equation as:
$\vec{r} = (1, 0, -1) + s(2, 3, -4) + 2t(2, 3, -4)$
$= (1, 0, -1) + (s + 2t)(2, 3, -4)$
$= (1, 0, -1) + n(2, 3, 4), n \in \mathbf{R}$
This is an equation of a line in R^3.
6. a. The plane with direction vectors $\vec{a} = (4, 1, 0)$ and $\vec{b} = (3, 4, -1)$, that passes through the point $A(-1, 2, 7)$ has a vector equation of:
$\vec{r} = (-1, 2, 7) + t(4, 1, 0) + s(3, 4, -1), t, s \in \mathbf{R}$
The parametric equations are then:
$x = -1 + 4t + 3s$
$y = 2 + t + 4s$
$z = 7 - s; t, s \in \mathbf{R}$
b. $\overrightarrow{AB} = (0, 1, 0) - (1, 0, 0) = (-1, 1, 0)$
$\overrightarrow{AC} = (0, 0, 1) - (1, 0, 0) = (-1, 0, 1)$
Using A(1, 0, 0) as our point with $\overrightarrow{AB}$ and $\overrightarrow{AC}$ as our direction vectors, our vector equation is:
$\vec{r} = (1, 0, 0) + t(-1, 1, 0) + s(-1, 0, 1), t, s \in \mathbf{R}$
And thus our parametric equations are:
$x = 1 - t - s$
$y = t$
$z = s, t, s \in \mathbf{R}$
c. $\overrightarrow{AB} = B - A = (3, 4, -6)$ using this and $\vec{a} = (7, 1, 2)$ as our direction vectors and $A(1, 1, 0)$ as our point, the vector equation is:
$\vec{r} = (1, 1, 0) + t(3, 4, -6) + s(7, 1, 2), t, s \in \mathbf{R}$
The parametric equations are:
$x = 1 + 3t + 7s$
$y = 1 + 4t + s$
$z = -6t + 2s, t, s \in \mathbf{R}$
7. a. $(5, 3, 2) = (2, 0, 1) + s(4, 2, -1) + t(-1, 1, 2)$
This gives the parametric equations:
$5 = 2 + 4s - t \Rightarrow t = -3 + 4s.$
$3 = 2s + t$. Substituting for t gives:
$3 = 2s + (-3 + 4s)$
$6 = 6s$
$1 = s.$
$t = -3 + 4(1) = 1.$
$2 = 1 - s + 2t$
$2 = 1 - 1 + 2(1)$
$2 = 2$; which is true so $s = 1$ and $t = 1$.
b. $(0, 5, -4) = (2, 0, 1) + s(4, 2, -1) + t(-1, 1, 2)$
Gives the following parametric equations:
$0 = 2 + 4s - t \Rightarrow t = 2 + 4s.$
$5 = 2s + t$
$5 = 2s + (2 + 4s)$
$3 = 6s$
$\frac{1}{2} = s.$
$t = 2 + 4\left(\frac{1}{2}\right)$
$t = 2 + 2 = 4.$
The third equation then says:
$-4 = 1 - s + 2t$
$-4 = 1 - \frac{1}{2} + 2(4)$
$-4 = \frac{17}{2}$, which is a false statement. So the point $A(0, 5, -4)$ is not on the plane.
8. a. Using the direction vectors $\vec{a} = (-1, 1, 2)$, $\vec{b} = (2, 1, -3)$ and the point $A(-3, 5, 6)$, two equations of intersecting lines on the plane in vector form are:
$\vec{l} = (-3, 5, 6) + s(-1, 1, 2); s \in \mathbf{R}$
$\vec{p} = (-3, 5, 6) + t(2, 1, -3); t \in \mathbf{R}$
b. When $s = 0$ and $t = 0$ it is easily seen that these two lines both have the point $(-3, 5, 6)$ in common.
9. $\vec{r} = (4, 1, 6) + s(11, -1, 3) + t(-7, 2, -2)$ has parametric equations:
$x = 4 + 11s - 7t$
$y = 1 - s + 2t$
$z = 6 + 3s - 2t$
The plane crosses the z-axis when both x and y equal 0.
$0 = 1 - s + 2t \Rightarrow s = 1 + 2t$
$0 = 4 + 11s - 7t$
$0 = 4 + 11(1 + 2t) - 7t$
$0 = 15 + 15t$
$t = -1.$
$s = 1 + 2(-1) = -1$. And so the z-coordinate is:
$z = 6 + 3(-1) - 2(-1) = 5$. The plane crosses the z-axis at the point (0, 0, 5)
10. Using the point $Q(2, 1, 3)$ on the line and the point $P(-1, 2, 1)$, we get another direction vector:
$\vec{a} = Q - P = (3, -1, 2)$. The equation of the plane having the given properties is then:
$\vec{r} = (2, 1, 3) + s(4, 1, 5) + t(3, -1, 2), t, s \in \mathbf{R}$

11. Using the point $A(-2, 2, 3)$ and the point $(0, 0, 0)$ on the line we get another direction vector of: $\vec{a} = (-2, 2, 3)$. So the equation of the plane with the given properties is:
$\vec{r} = m(2, -1, 7) + n(-2, 2, 3), m, n \in \mathbf{R}$.

12. a. The xy-plane in R^3 has no z-coordinate so two sets of direction vectors are: $(1, 0, 0)$, $(0, 1, 0)$ and $(1, 1, 0)$, $(-1, 1, 0)$.

b. A vector equation for the xy-plane in R^3 is:
$\vec{r} = s(1, 0, 0) + t(0, 1, 0), t, s \in \mathbf{R}$.
The parametric equations are:
$x = s$
$y = t$
$z = 0, t, s \in \mathbf{R}$

13. a. We can use the direction vectors $\overrightarrow{OA} = (-1, 2, 5)$ and $\overrightarrow{OC} = (3, -1, 7)$ and the origin to write the vector equation of the plane:
$\vec{r} = s(-1, 2, 5) + t(3, -1, 7), t, s \in \mathbf{R}$

b. Using $\overrightarrow{PQ} = Q - P = (-1, 2, 5)$ and $\overrightarrow{PR} = R - P = (3, -1, 7)$ as direction vectors, the vector equation of the plane is:
$\vec{r} = (-2, 2, 3) + s(-1, 2, 5) + t(3, -1, 7), t, s \in \mathbf{R}$

c. The two planes in parts **a.** and **b.** are parallel since they have the same direction vectors.

14. We simply need to show that the direction vectors can be expressed as a linear combination of the other two:

$$(-4, 7, 1) - (-3, 2, 4) = (-1, 5, -3)$$
$$\frac{27}{13}(-3, 2, 4) - \frac{17}{13}(-4, 7, 1) = (-1, -5, 7).$$

15. The plane
$\vec{r} = (1, 2, 3) + m(1, 2, 5) + n(1, -1, 3)$ has parametric equations:
$x = 1 + m + n$
$y = 2 + 2m - n$
$z = 3 + 5m + 3n$
Solving for the y-intercept:
$0 = 1 + m + n \Rightarrow n = -1 - m$
$0 = 3 + 5m + 3n$
$0 = 3 + 5m + 3(-1 - m)$
$0 = 4m$
$0 = m; n = -1$
$y = 2 + 2(0) - (-1) = 3$
Solving for the z-intercept:
$n = -1 - m$
$0 = 2 + 2m - (-1 - m)$
$0 = 3 + 3m$
$-1 = m; n = 0$
$z = 3 + 5(-1) + 3(0) = -2.$

The direction vector between the two points is then:
$(0, 3, 0) - (0, 0, -2) = (0, 3, 2)$.
And the equation of the line between them is:
$\vec{r} = (0, 3, 0) + t(0, 3, 2), t \in \mathbf{R}$

16. The fact that the plane
$\vec{r} = \overrightarrow{OP_0} + s\vec{a} + t\vec{b}$ contains both of the given lines is easily seen when letting $s = 0$ and $t = 0$ respectively.

8.5 The Cartesian Equation of a Plane, pp. 468–469

1. a. $\vec{n} = (A, B, C) = (1, -7, -18)$

b. In the Cartesian equation:
$Ax + By + Cz + D = 0$. If $D = 0$ the plane passes through the origin.

c. Three coordinates: $(0, 0, 0)$, $(11, -1, 1)$, $(11, -1, 1)$,

2. a. $\vec{n} = (A, B, C) = (2, -5, 0)$

b. In the Cartesian equation: $D = 0$. So the plane passes through the origin.

c. Three coordinates: $(0, 0, 0)$, $(5, 2, 0)$, $(5, 2, 1)$

3. a. $\vec{n} = (A, B, C) = (1, 0, 0)$

b. In the Cartesian equation: $D = 0$. So the plane passes through the origin.

c. Three coordinates: $(0, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$

4. a. $\vec{n} = (15, 75, -105)$ which is equivalent to $\vec{n} = (1, 5, -7)$. The Cartesian equation is:
$x + 5y - 7z + D = 0$. Since the plane passes through the origin $D = 0$. So the equation is:
$x + 5y - 7z = 0$.

b. $\vec{n} = (-\frac{1}{2}, \frac{3}{4}, \frac{7}{16})$ is equivalent to
$\vec{n} = (-8, 12, 7)$, so the Cartesian equation is:
$-8x + 12y + 7z + D = 0$, and since the plane passes through the origin $D = 0$. $-8x + 12y + 7z = 0$

5. *Method 1:* Let $A(x, y, z)$ be a point on the plane. Then $\overrightarrow{PA} = (x + 3, y - 3, z - 5)$ is a vector on the plane.

$$\vec{n} \cdot \overrightarrow{PA} = 0$$
$$(x + 3) + 7(y - 3) + 5(z - 5) = 0$$
$$x + 7y + 5z - 43 = 0$$

Method 2: $\vec{n} = (1, 7, 5)$ so the Cartesian equation is: $x + 7y + 5z + D = 0$.
We know the point $(-3, 3, 5)$ is on the plane and must satisfy the equation, so:
$$(-3) + 7(3) + 5(5) + D = 0$$
$$43 + D = 0$$
$$D = -43.$$
This also gives the equation:
$$x + 7y + 5z - 43 = 0.$$

6. a. $\overrightarrow{PQ} = (3 - (-1), 1 - 2, 4 - 1)$
$= (4, -1, 3)$
$\overrightarrow{QR} = (-2 - 3, 3 - 1, 5 - 4) = (-5, 2, 1)$
$\overrightarrow{PQ} \times \overrightarrow{QR} = ((-1)(1) - (3)(2), (3)(-5) - (4)(1), (4)(2) - (-1)(-5))$
$= (-7, -19, 3)$
$= -1(7, 19, -3).$
Using $\vec{n} = (7, 19, -3)$ the Cartesian equation is:
$7x + 19y - 3z + D = 0.$
Using the point $R(-2, 3, 5)$ on the plane to solve for D:
$7(-2) + 19(3) - 3(5) + D = 0$
$-14 + 57 - 15 + D = 0$
$28 + D = 0$
$D = -28.$
$7x + 19y - 3z - 28 = 0$
b. $\overrightarrow{QP} = (-1 - 3, 2 - 1, 1 - 4)$
$= (-4, 1, -3)$
$\overrightarrow{PR} = (-2 - (-1), 3 - 2, 5 - 1)$
$= (-1, 1, 4)$
$\overrightarrow{QP} \times \overrightarrow{PR} = ((1)(4) - (-3)(1), (-3)(-1) - (-4)(4), (-4)(1) - (1)(-1))$
$= (7, 19, -3).$
Using $\vec{n} = (7, 19, -3)$ the Cartesian equation is:
$7x + 19y - 3z + D = 0.$
Using the point $P(-1, 2, 1)$ on the plane to solve for D:
$7(-1) + 19(2) - (3)(1) + D = 0$
$-7 + 38 - 3 + D = 0$
$28 + D = 0$
$D = -28.$
$7x + 19y - 3z - 28 = 0$
c. There is only one simplified Cartesian equation that satisfies the given information, so the equations must be the same.
7. $\overrightarrow{AB} = (5, 1, 4).$
$\overrightarrow{AC} = (3, -2, -1).$
$\overrightarrow{AB} \times \overrightarrow{AC} = ((1)(-1) - (4)(-2), (4)(3) - (5)(-1), (5)(-2) - (1)(3))$
$= (7, 17, -13)$
Using $\vec{n} = (7, 17, -13)$ the Cartesian equation is:
$7x + 17y - 13z + D = 0.$
Using the point $(1, 1, 0)$ on the plane to solve for D:
$7(1) + 17(1) - 13(0) + D = 0$
$24 + D = 0$
$D = -24.$
$7x + 17y - 13z - 24 = 0$

8. The point $Q(2, 0, 1)$ is on the line and thus also on the plane and we can get another direction vector from:
$\overrightarrow{PQ} = (1, -3, 1)$. Using $\vec{a} = (-4, 5, 5)$ as the other direction vector we can find the normal vector:
$\vec{n} = \overrightarrow{PQ} \times \vec{a} = ((-3)(5) - (1)(5), (1)(-4) - (1)(5), (1)(5) - (-3)(-4))$
$= (-20, -9, -7) = -1(20, 9, 7).$
Our Cartesian equation is thus:
$20x + 9y + 7z + D = 0.$
Using the point $(1, 3, 0)$ to determine D:
$20(1) + 9(3) + 7(0) + D = 0.$
$47 + D = 0$
$D = -47.$
$20x + 9y + 7z - 47 = 0$
9. a. $2x + 2y - z - 1 = 0$
$\vec{n} = (2, 2, -1)$
$|\vec{n}| = \sqrt{4 + 4 + 1}$
$= 3$
So the unit normal vector is:
$\frac{\vec{n}}{|\vec{n}|} = \left(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right)$
b. $4x - 3y + z - 3 = 0$
$\vec{n} = (4, -3, 1)$
$|\vec{n}| = \sqrt{16 + 9 + 1}$
$= \sqrt{26}$
So the unit normal vector is:
$\frac{\vec{n}}{|\vec{n}|} = \left(\frac{4}{\sqrt{26}}, -\frac{3}{\sqrt{26}}, \frac{1}{\sqrt{26}}\right)$
c. $3x - 4y + 12z - 1 = 0$
$\vec{n} = (3, -4, 12)$
$|\vec{n}| = \sqrt{9 + 16 + 144}$
$= \sqrt{169}$
$= 13$
So the unit normal vector is:
$\frac{\vec{n}}{|\vec{n}|} = \left(\frac{3}{13}, -\frac{4}{13}, \frac{12}{13}\right)$
10. We know the point $P(1, 1, 5)$ is on the plane, and can obtain another direction vector from:
$\overrightarrow{AP} = (1, 1, -6)$. Let $\vec{a} = (2, 1, 3)$ be our other direction vector.
$\vec{n} = \overrightarrow{AP} \times \vec{a} = ((1)(3) - (-6)(1), (-6)(2) - (1)(3), (1)(1) - (1)(2))$
$= (21, -15, -1)$
The Cartesian equation is then:
$21x - 15y - z + D = 0.$

Using the point $(1, 1, 5)$ to solve for D:
$$21(1) - 15(1) - (5) + D = 0$$
$$1 + D = 0$$
$$D = 11$$
$$21x - 15y - z + 11 = 0.$$
11. Since the normal vector is perpendicular to the plane, we can use the direction vector of the line as our normal vector:
$\vec{n} = (3, -2, 0) - (1, 2, 1) = (2, -4, -1)$.
The Cartesian equation is then:
$2x - 4y - z + D = 0$.
We need the point $(-1, 1, 0)$ to be on the plane so:
$$2(-1) - 4(1) - (0) + D = 0.$$
$$-6 + D = 0$$
$D = 6$. And the Cartesian equation of the plane satisfying the given conditions is:
$2x - 4y - z + 6 = 0$.
12. a. To determine the angle between two planes, first determine their normal vectors. This is easily done if the equations given are in Cartesian form. Once the normal vectors are known, $\vec{n_1}$ and $\vec{n_2}$, then the angle between the two planes can be determined from the formula:
$$\cos(\theta) = \frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}||\vec{n_2}|}.$$
b. $\vec{n_1} = (1, 0, -1)$, $\vec{n_2} = (2, 1, -1)$.
$$\vec{n_1} \cdot \vec{n_2} = 2 + 0 + 1 = 3$$
$$\vec{n_1} \cdot \vec{n_2} = \sqrt{2} \cdot \sqrt{6} = \sqrt{12}.$$
$$\cos(\theta) = \frac{3}{\sqrt{12}} = \frac{\sqrt{3}}{2}$$
$$\theta = \frac{\pi}{6} = 30°$$
13. a. $\vec{n_1} = (1, 2, -3)$, $\vec{n_2} = (1, 2, 0)$
$$\vec{n_1} \cdot \vec{n_2} = 1 + 4 = 5$$
$$|\vec{n_1}||\vec{n_2}| = \sqrt{14} \cdot \sqrt{5} = \sqrt{70}$$
$$\cos(\theta) = \frac{5}{\sqrt{70}}.$$
$$\theta = \cos^{-1}\left(\frac{5}{\sqrt{70}}\right) = 53.3°$$
b. The parametric equations for the line are:
$x = 3 - 2t$
$y = -1 + 3t$
$z = -4 + t$,
which give the following vector equation:
$\vec{r} = (3, -1, -4) + t(-2, 3, 1)$. Since the line and normal vector are both perpendicular to the plane we may take:
$\vec{n} = (2, -3, -1)$.
The Cartesian equation for the plane is then:
$2x - 3y - z + D = 0$.
Using the point $P(1, 2, 1)$ to solve for D:
$$2(1) - (3)(2) - (1)(1) + D = 0$$
$$-5 + D = 0$$
$D = 5$. And the Cartesian equation becomes:
$2x - 3y - z + 5 = 0$.
14. a. $\vec{n_1} = (4, k, -2)$ and $\vec{n_2} = (2, 4, -1)$.
When $k = 8$, $\vec{n_1}$ is equivalent to: $\vec{n_1} = 2(2, 4, -1)$, so the planes are parallel when $k = 8$.
b. When the planes are perpendicular
$$\vec{n_1} \cdot \vec{n_2} = 0.$$
$$\vec{n_1} \cdot \vec{n_2} = 8 + 4k + 2 = 0$$
$$10 + 4k = 0$$
$$k = -\frac{10}{4} = -\frac{5}{2}$$
c. No the planes cannot ever be coincident. If they were then they would also be parallel, so $k = 8$, and we would have the two equations:
$4x + 8y - 2z + 1 = 0$.
$2x + 4y - z + 4 = 0 \Rightarrow 4x + 8y - 2z + 8 = 0$.
Here all of the coefficients are equal except for the D values, which means that they don't coincide.
15. Since the plane passes through the points $(1, 4, 5)$ and $(3, 2, 1)$ it contains the line and the direction vector between them. The direction vector is:
$\vec{r} = (2, -2, -4)$.
The normal vector, $\vec{n_1}$, must be perpendicular to the direction vector and to the normal vector, $\vec{n_2} = (2, -1, 1)$, of the other plane, so:
$$\vec{n_1} = \vec{r} \times \vec{n_2} = ((-2)(1) - (-4)(-1), (-4)(2) - (2)(1), (2)(-1) - (-2)(2))$$
$$= (-6, -10, 2) = -2(3, 5, -1)$$
Take $\vec{n_1} = (3, 5, -1)$ and the Cartesian equation of the plane is:
$3x + 5y - z + D = 0$
Use the point $(1, 4, 5)$ to determine D:
$$3(1) + 5(4) - 5 + D = 0$$
$$18 + D = 0$$
$$D = -18.$$
$$3x + 5y - z - 18 = 0$$

16. Let $\vec{n_1} = (A, B, C)$, be the normal vector of the unknown plane, and $\vec{n_2} = (1, 2, 0)$ be the normal vector to the perpendicular plane. $\vec{n_1} \cdot \vec{n_2} = 0$ so we get:

$A + 2B = 0.$

$A = -2B$

We also know that the z-axis has the direction vector $\vec{r} = (0, 0, 1)$. So:

$$\cos(30°) = \frac{\sqrt{3}}{2} = \frac{\vec{n_1} \cdot \vec{r}}{|\vec{n_1}||\vec{r}|} = \frac{C}{\sqrt{A^2 + B^2 + C^2}}.$$

The other constraint which we can choose is the length of $\vec{n_1}$. Since this is arbitrary (multiplication by any scalar will give an equivalent normal vector) choose $|\vec{n_1}| = 2$. We have:

$$\frac{\sqrt{3}}{2} = \frac{C}{2} \Rightarrow C = \sqrt{3}.$$

$A^2 + B^2 + C^2 = 4$

$4B^2 + B^2 + 3 = 4$

$$B^2 = \frac{1}{5}$$

$$B = \frac{1}{\sqrt{5}}; A = -\frac{2}{\sqrt{5}}.$$

The equation of the plane is then:

$$-\frac{2}{\sqrt{5}}x + \frac{1}{\sqrt{5}}y + \sqrt{3}z = 0.$$

17. The point equidistant from $(-1, 2, 4)$ and $(3, 1, -4)$ is the point $\frac{1}{2}((-1, 2, 4) + (3, 1, -4)) = (1, \frac{3}{2}, 0)$. If every point in the plane is equidistant from these two point than the normal to the plane must point in the same direction as the line connecting them:

$\vec{n} = (3, 1, -4) - (-1, 2, 4) = (4, -1, -8).$

The equation of the plane is thus:

$4x - y - 8z + D = 0.$

Using the point $(1, \frac{3}{2}, 0)$ to solve for D:

$$4(1) - \frac{3}{2} - 0 + D = 0$$

$$\frac{5}{2} + D = 0 \Rightarrow D = -\frac{5}{2}$$

We now have the equation of the plane:

$$4x - y - 8z - \frac{5}{2} = 0.$$

Or equivalently:

$8x - 2y - 16z - 5 = 0.$

8.6 Sketching Planes in R^3, pp. 476–477

1. a. A plane parallel to the yz-axis but two units away, in the negative x direction.

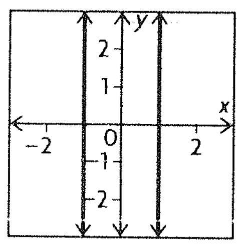
b. A plane parallel to the xz-axis but three units away, in the positive y direction.

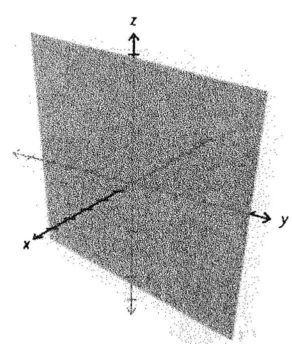
c. A plane parallel to the xy-axis but 4 units away, in the positive z direction.

2. The point of intersection of the three planes in problem 1 must lie in every plane. Therefore the point of intersection is: $(-2, 3, 4)$

3. The point $P(5, -3, -3)$ must lie on the plane π_1: $x = 5$, since the point has an x-coordinate of 5, and doesn't have a y-coordinate of 6.

4. In R^2, $x^2 - 1 = 0$ represents two lines, $x = -1$ and $x = 1$. In R^3, $x^2 - 1 = 0$ represents two planes with the same equations.





5. a. i. x-intercept is when $y = z = 0$.

$2x = 18$

$x = 9$

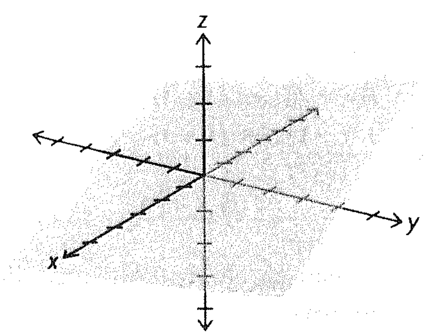
Similarly the y-intercept is:

$3y = 18$

$y = 6$

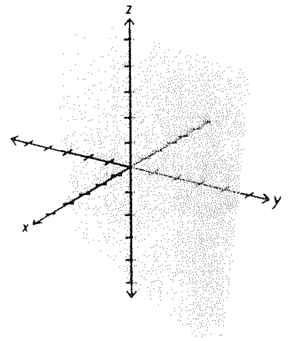
Since x and y cannot both be zero at the same time there is no z-intercept. The plane is parallel to the z-axis.

ii. x-intercept:
$3x = 120$
$x = 40$
y-intercept:
$-4y = 120$
$y = -30$
z-intercept:
$5z = 120$
$z = 24$
iii. There is no x-intercept since y and z cannot both be simultaneously zero.
y-intercept:
$13y = 39$
$y = 3$
z-intercept:
$-z = 39$
$z = -39$
b. i. Since the plane is parallel to the z-axis one directional vector is: $(0, 0, 1)$. The other lies along the line $2x + 3y = 18$, so $(3, -2, 0)$.
ii. We can find directional vectors by taking the difference between two points, namely the intercepts we found in **a.**: $(40, 0, 0) - (0, -30, 0) = (40, 30, 0)$ or equivalently $(4, 3, 0)$.
$(40, 0, 0) - (0, 0, 24) = (40, 0, -24)$ or equivalently $(5, 0, -3)$.
iii. Since the plane is parallel to the x-axis $(1, 0, 0)$ is one directional vector.
$(0, 3, 0) - (0, 0, -39) = (0, 3, 39)$. Or equivalently $(0, 1, 13)$.
6. a. i. π: $2x - y + 5z = 0$. Three points satisfying this equation are: $(0, 0, 0)$, $(1, 2, 0)$, $(0, 5, 1)$.
ii. The line where this plane intersects the xy-plane is simply the line when $z = 0$:
$2x - y = 0$.
b.

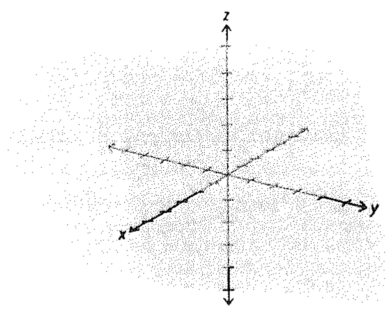


7. $xyz = 0$ has the solutions: $x = 0$, $y = 0$, $z = 0$. So the three planes are the yz-plane, xz-plane, and the xy-plane.

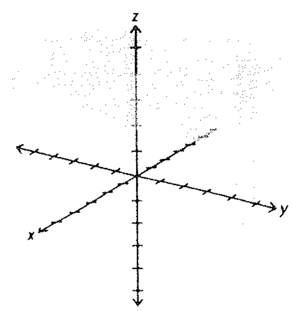
8. a.



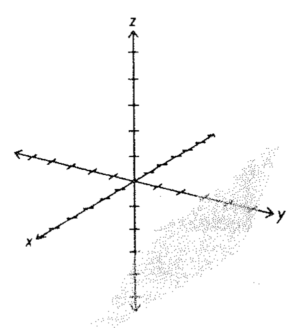
b.



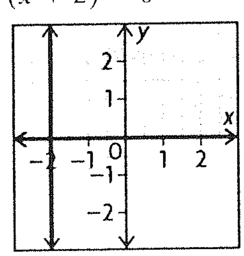
c.



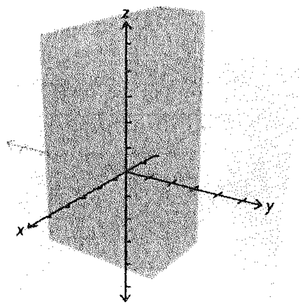
d.



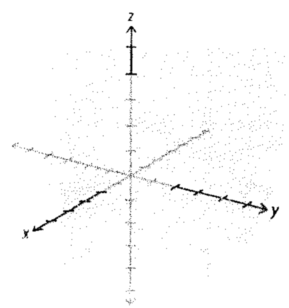
9. a. $xy + 2y = 0$
$y(x + 2) = 0$
b.



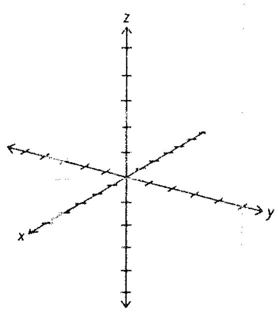
c.



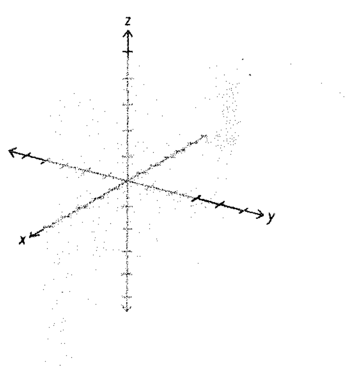
10. a.



b.



c.



11. a. The plane with x-, y-, z- intercepts of 3, 4, and 6, respectively is $\frac{x}{3} + \frac{y}{4} + \frac{z}{6} = 1$.

b. The plane with x- and z-intercepts of 5 and -7, respectively, and which is parallel to the y-axis is $\frac{x}{5} - \frac{z}{7} = 1$.

c. No x- or y-intercepts but with a z-intercept of 8 has the equation $\frac{z}{8} = 1$.

Review Exercise, pp. 480–483

1. Answers may vary. For example:
$A(1, 2, -1)$, $B(2, 1, 1)$, $C(3, 1, 4)$

$\overrightarrow{AB} = (1, -1, 2) = \vec{a}$

$\overrightarrow{BC} = (1, 0, 3) = \vec{b}$

$\vec{r} = \vec{r_0} + s\vec{a} + t\vec{b}$

$\vec{r} = (1, 2, -1) + s(1, -1, 2) + t(1, 0, 3), s, t \in \mathbf{R}$

$x = 1 + s + t$

$y = 2 - s$

$z = -1 + 2s + 3t$

2. $A(1, 2, -1)$, $B(2, 1, 1)$, $C(3, 1, 4)$

$\overrightarrow{AB} = (1, -1, 2) = \vec{a}$

$\overrightarrow{BC} = (1, 0, 3) = \vec{b}$

$\vec{r} = (1, 2, -1) + s(1, -1, 2) + t(1, 0, 3), s, t \in \mathbf{R}$

$\vec{b} \times \vec{a} = (1, 0, 3) \times (1, -1, 2) = (3, 1, -1)$

$Ax + By + Cz + D = 0$

$(3)x + (1)y + (-1)z + D = 0$

$3(1) + (2) - 1(-1) + D = 0$

$D = -6$

$3x + y - z - 6 = 0$

$\overrightarrow{AC} = (2, -1, 5) = \vec{c}$

$\overrightarrow{BC} = (1, 0, 3) = \vec{b}$

$\vec{r} = (1, 2, -1) + s(2, -1, 5) + t(1, 0, 3), t, s \in \mathbf{R}$

$\vec{b} \times \vec{c} = (1, 0, 3) \times (2, -1, 5) = (3, 1, -1)$

$Ax + By + Cz + D = 0$

$(3)x + (1)y + (-1)z + D = 0$

$3(1) + (2) - 1(-1) + D = 0$

$D = -6$

$3x + y - z - 6 = 0$

Both Cartesian equations are the same regardless of which vectors are used.

3. a. Answers may vary. For example:
$A(-3, 2, 8)$, $B(4, 3, 9)$

$\overrightarrow{AB} = (7, 1, 1) = \vec{a}$

$\vec{r} = (4, 3, 9) + t(7, 1, 1), t \in \mathbf{R}$

$x = x_0 + ta, y = y_0 + tb, z = z_0 + tc$

$x = 4 + 7t, y = 3 + t, z = 9 + t, t \in \mathbf{R}$

$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$

$\frac{x - 4}{7} = \frac{y - 3}{1} = \frac{z - 9}{1}$

b. Answers may vary. For example:
$A(-3, 2, 8), B(4, 3, 9), C(-2, -1, 3)$

$\overrightarrow{AB} = (7, 1, 1) = \vec{a}$

$\overrightarrow{CB} = (6, 4, 6) = (3, 2, 3) = \vec{b}$

$\vec{r} = (4, 3, 9) + t(7, 1, 1) + s(3, 2, 3), t, s \in \mathbf{R}$

$x = x_0 + ta_1 + tb_1, y = y_0 + ta_2 + tb_2,$
$z = z_0 + ta_3 + tb_3$

$x = 4 + 7t + 3s, y = 3 + t + 2s,$
$z = 9 + t + 3s, t, s \in \mathbf{R}$

c. There are no symmetric equations, because there are two parameters.

4. A line passing through $A(7, 1, -2)$ and perpendicular to the plane with the equation $2x - 3y + z - 1 = 0$. Since the line is perpendicular to the plane, the normal of the plane is the lines vector.

$\vec{m} = (2, -3, 1)$

$\vec{r} = \vec{r_0} + t\vec{m}$

$\vec{r} = (7, 1, -2) + t(2, -3, 1), t \in \mathbf{R}$

$x = 7 + 2t, y = 1 - 3t, z = -2 + t$

$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$

$\frac{x - 7}{2} = \frac{y - 1}{-3} = \frac{z + 2}{1}$

5. a. $P(0, 1, -2)$

$\vec{n} = (-1, 3, 3)$

$Ax + By + Cz + D = 0$

$(-1)(x - 0) + (3)(y - 1) + (3)(z + 2) = 0$

$-x + 3y - 3 + 3z + 6 = 0$

$x - 3y - 3z - 3 = 0$

b. $A(3, 0, 1), B(0, 1, -1)$

$\overrightarrow{AB} = (-3, 1, -2)$

$\vec{n} = (1, -1, -1)$

$\vec{n} \times \overrightarrow{AB} = (3, 5, -2)$

$Ax + By + Cz + D = 0$

$(3)x + (5)y + (-2)z + D = 0$

$(3)(3) + (5)(0) + (-2)(1) + D = 0$

$D = -7$

$3x + 5y - 2z - 7 = 0$

c. $A(1, 2, 1), B(2, 1, 4)$

$\overrightarrow{AB} = (1, -1, 3)$

$\vec{n} = (1, 0, 0)$

$\overrightarrow{AB} \times \vec{n} = (0, 3, 1)$

$Ax + By + Cz + D = 0$

$(0)x + (3)y + (1)z + D = 0$

$(0)(1) + (3)(2) + (1)(1) + D = 0$

$D = -7$

$3y + z - 7 = 0$

6. $\vec{r} = (3, 7, 1) + t(2, 2, 3), t \in \mathbf{R}$

$(2, 2, 3) \cdot (a, b, c) = \vec{n}$

$2a + 2b + 3c = 0$

$a = 19, b = -7, c = -8$

$\vec{n} = (19, -7, -8)$

$Ax + By + Cz + D = 0$

$19x - 7y - 8z + D = 0$

$19(0) - 7(0) - 8(0) + D = 0$

$D = 0$

$19x - 7y - 8z = 0$

7. Since the plane is parallel to the yz-plane, its direction vectors are (0, 1, 0) and (0, 0, 1).

$A = (-1, 2, 1)$

$\vec{r} = \vec{r_0} + t\vec{a} + s\vec{b}$

$\vec{a} = (0, 1, 0), \vec{b} = (0, 0, 1)$

$\vec{r} = (-1, 2, 1) + t(0, 1, 0) + s(0, 0, 1)t, s \in \mathbf{R}$

$x = -1, y = 2 + t, z = 1 + s$

8. $A = (4, -3, 2)$

$\vec{r} = (2, 3, 2) + t(1, 1, 4), t \in \mathbf{R}$

$\vec{a} = (1, 1, 4), \vec{b} = [(4 - 2), (-3 - 3), (2 - 2)]$

$\vec{a} = (1, 1, 4), \vec{b} = (2, -6, 0)$

$\vec{a} \times \vec{b} = (24, 8, -8) = (3, 1, -1)$

$Ax + By + Cz + D = 0$

$(3)x + (1)y + (-1)z + D = 0$

$3(4) + 1(-3) - 1(2) + D = 0$

$D = -7$

$3x + y - z - 7 = 0$

9. $L_1: \vec{r} = (4, 4, 5) + s(5, -4, 6), s \in \mathbf{R}$

$L_2: \vec{r} = (4, 4, 5) + s(2, -3, -4), s \in \mathbf{R}$

$\vec{a} = (5, -4, 6), \vec{b} = (2, -3, -4)$

$\vec{a} \times \vec{b} = (34, 32, -7)$

$Ax + By + Cz + D = 0$

$(34)x + (32)y + (-7)z + D = 0$

$34(4) + 32(4) - 7(5) + D = 0$

$D = -229$

$34x + 32y - 7z - 229 = 0$

10. Answers may vary. For example: Since the line is perpendicular to the plane. The normal of the plane is the directional vector of the line.
$A(2, 3, -3)$
$3x - 2y + z = 0$
$\vec{n} = (3, -2, 1)$
$\vec{r} = (2, 3, -3) + s(3, -2, 1), s \in \mathbf{R}$
$x = 2 + 3s, y = 3 - 2s, z = -3 + s$
$\frac{x - 2}{3} = \frac{y - 3}{-2} = \frac{z + 3}{1}$

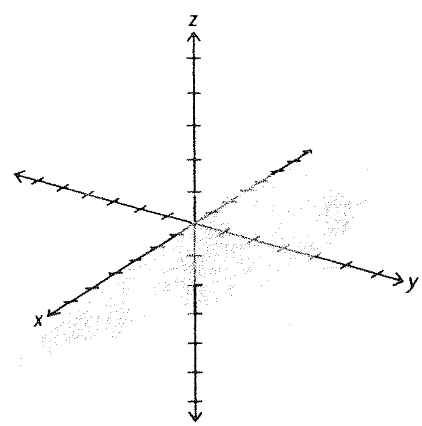
11. Answers may vary. For example: Use the dot product and cross product to find two points that are orthogonal to the normal of the plane. Then use any point from the plane.
$3x + 2y - z + 6 = 0$
$\vec{a} \cdot (3, 2, -1) = 0$
$(a, b, c) \cdot (3, 2, -1) = 0$
$3a + 2b - c = 0$
$\vec{a} = (1, 0, 3)$
$(3, 2, -1) \times (1, 0, 3) = (6, -10, -2)$
$= (3, -5, -1)$
$\vec{r} = (0, 0, 6) + s(1, 0, 3) + t(3, -5, -1), s, t \in \mathbf{R}$
$x = s + 3t, y = -5t, z = 6 + 3s - t$

12. Answers may vary. For example: The x-intercept is $(-3.5, 0, 0)$ and z-intercept is $(0, 0, 7)$. Find the directional vector from these points and use a point one of the intercepts.
$A = (-3.5, 0, 0), B = (0, 0, 7)$
$\vec{v} = [(0 - 3.5), (0 - 0), (7 - 0)]$
$\vec{v} = (3.5, 0, 7) = (1, 0, 2)$
$\vec{r} = r_0 + ta, t \in \mathbf{R}$
$\vec{r} = (0, 0, 7) + t(1, 0, 2), t \in \mathbf{R}$
$x = t, y = 0, z = 7 + 2t$

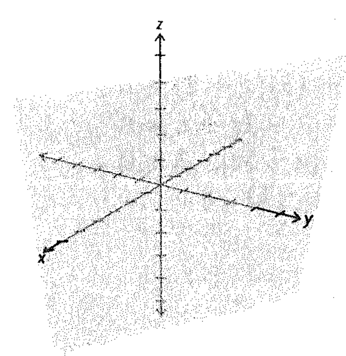
13. The two direction vectors for these lines are
$\vec{a} = (1, -3, -5)$
$\vec{b} = (2, -6, -10) = 2\vec{a}$
So the lines L_1 and L_2 are parallel (they aren't the same line, as $(3, -4, 1)$, a point on L_1, is not a point on L_2). Take one of the direction vectors for the plane to be the vector $\vec{a} = (1, -3, -5)$, and find another by computing the vector with tail at $(3, -4, 1)$ (a point on L_1) and head at $(7, -1, 0)$ (a point on L_2). This is the vector
$\vec{v} = (7, -1, 0) - (3, -4, 1)$
$= (4, 3, -1)$

The point $(3, -4, 1)$ is on the plane, so the vector equation of the plane is
$\vec{r} = (3, -4, 1) + s(1, -3, -5) + t(4, 3, -1)$,
$s, t \in \mathbf{R}$.
The parametric form for the plane is
$x = 3 + s + 4t$,
$y = -4 - 3s + 3t$
$z = 1 - 5s - t, s, t \in \mathbf{R}$
Finally, to find the Cartesian equation of the plane, compute the cross product of the direction vectors.
$\vec{a} \times \vec{v} = (1, -3, -5) \times (4, 3, -1)$
$= (-3(-1) - (3)(-5), 4(-5) - 1(-1), 1(3) - 4(-3))$
$= (18, -19, 15)$
So the Cartesian equation is of the form
$18x - 19y + 15z + D = 0$.
To find the value of D, substitute in the point on the plane $(3, -4, 1)$.
$18(3) - 19(-4) + 15(1) + D = 0$
$D = -145$
So the Cartesian equation is
$18x - 19y + 15z - 145 = 0$

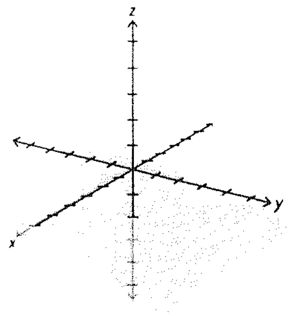
14. a.



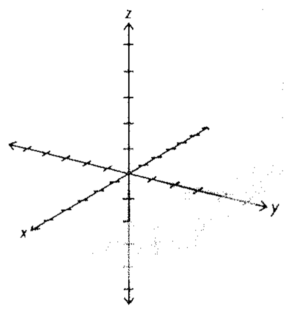
b.



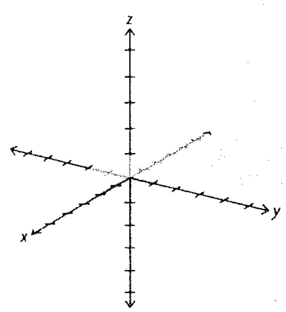
c.



d.



e.



15. a. Answers may vary. For example:

$P(1, -2, 5)$

$Q(3, 1, 2)$

$\overrightarrow{PQ} = \vec{p} = (2, 3, -3)$

$L: \vec{r} = 2ti + (4t + 3)j + (t + 1)k$

$= (0, 3, 1) + t(2, 4, 1)$

$\vec{a} = (2, 4, 1)$

$\vec{r} = (3, 1, 2) + t(2, 4, 1) + s(2, 3, -3), t, s \in \mathbf{R}$

$x = 3 + 2t + 2s, y = 1 + 4t + 3s, z = 2 + t - 3s$

$\vec{a} \times \vec{p} = (2, 3, -3) \times (2, 4, 1) = (15, -8, 2)$

$Ax + By + Cz + D = 0$

$(15)x + (-8)y + (2)z + D = 0$

$15(3) - 8(1) + 2(2) + D = 0$

$D = -41$

$15x - 8y + 2z - 41 = 0$

b. Answers may vary. For example: The normal of the plane is the direction vector of the line, since it is perpendicular to the plane. Then find using the Cartesian form of a plane.

$A(1, 1, 2)\ B = (2, 1, -6), C = (-2, 1, 5)$

$\overrightarrow{BC} = (-4, 0, 11)$

$Ax + Bx + Cx + D = 0$

$(-4)x + (0)y + (11)z + D = 0$

$-4(1) + 11(2) + D = 0$

$D = -18$

$-4x + 11z - 18 = 0$

c. Answers may vary. For example: Since the plane is parallel to the z-axis, one of its direction vectors is (0, 0, 1).

$A(4, 1, -1), B(5, -2, 4)$

$\overrightarrow{AB} = (1, -3, 5)$

$\vec{r} = (4, 1, -1) + t(1, -3, 5) + s(0, 01), t, s \in \mathbf{R}$

$x = 4 + t, y = 1 - 3t, z = -1 + 5t + s$

$(1, -3, 5) \cdot (0, 0, 1) = (-3, -1, 0) = (3, 1, 0)$

$Ax + By + Cz + D = 0$

$(3)x + (1)y + (0)z + D = 0$

$3(4) + 1(1) + D = 0$

$D = -13$

$3x + y - 13 = 0$

d. Answers may vary. For example:

$A(1, 3, -5), B(2, 6, 4), C(3, -3, 3)$

$\overrightarrow{AB} = (1, 3, 9)$

$\overrightarrow{BC} = (1, -9, -1)$

$\vec{r} = r_0 + t\vec{a} + s\vec{b}$

$\vec{r} = (1, 3, -5) + t(1, 3, 9) + s(1, -9, -1), t, s \in \mathbf{R}$

$x = 1 + t + s, y = 3 + 3t - 9s, z = -5 + 9t - s$

$\overrightarrow{AB} \times \overrightarrow{BC} = (78, 10, -12)$

$Ax + By + Cz + D = 0$

$(78)x + (10)y + (-12)z + D = 0$

$78(1) + 10(3) - 12(-5) + D = 0$

$D = -168$

$78x + 10y - 12z - 168 = 0$

16. They are in the same plane because both planes have the same normal vectors and Cartesian equations.

$L_1: \vec{r} = (1, 2, 3) + s(-3, 5, 21) + t(0, 1, 3), s, t \in \mathbf{R}$

$L_2: \vec{r} = (1, -1, -6) + u(1, 1, 1) + v(2, 5, 11),$

$u, v \in \mathbf{R}$

$(-3, 5, 21) \times (0, 1, 3) = (-6, 9, -3) = (2, -3, 1)$

$(1, 1, 1) \times (2, 5, 11) = (6, -9, 3) = (2, -3, 1)$

$Ax + By + Cz + D = 0$

$2x - 3y + z + D = 0$

$2(1) - 3(2) + (3) + D = 0$

$D = 1$

$2(1) - 3(-1) + (-6) + D = 0$

$D = 1$

$2x - 3y + z + 1 = 0$

17. A point B on the line L_2 will have coordinates $B(2 + 2t, 1 + t, 2 - t), t \in \mathbf{R}$ Then

$\overrightarrow{AB} = (2 + 2t, 1 + t, 2 - t) - (5, 4, -3)$
$= (-3 + 2t, -3 + t, 5 - t)$

For this vector to be perpendicular to L_2, it would have zero dot product with the direction vector for L_2, $\vec{v} = (2, 1, -1)$. So

$0 = \vec{v} \cdot \overrightarrow{AB}$
$= (2, 1, -1) \cdot (-3 + 2t, -3 + t, 5 - t)$
$= -6 + 4t - 3 + t - 5 + t$
$= -14 + 6t$

So $t = \frac{14}{6} = \frac{7}{3}$, and the point B is

$$B\left(2 + 2\left(\frac{7}{3}\right), 1 + \frac{7}{3}, 2 - \frac{7}{3}\right) = B\left(\frac{20}{3}, \frac{10}{3}, -\frac{1}{3}\right).$$

18. a. The plane is parallel to the z-axis through the points $(3, 0, 0)$ and $(0, -2, 0)$.
b. The plane is parallel to the y-axis through the points $(6, 0, 0)$ and $(0, 0, -2)$.
c. The plane is parallel to the x-axis through the points $(0, 3, 0)$ and $(0, 0, -6)$.

19. a. To determine which points lie on the line, just see if there is a t-value such that the coordinate works.

$x = 2t, y = 3 + t, z = 1 + t$
$A(2, 4, 2)$
$2 = 2t, 4 = 3 + t, 2 = 1 + t$
$t = 1$
$B(-2, 2, 1)$
$-2 = 2t, 2 = 3 + t, 1 = 1 + t$
There is no value of t that satisfies the equations.
$C(4, 5, 2)$
$4 = 2t, 5 = 3 + t, 2 = 1 + t$
There is no value of t that satisfies the equations.
$D(6, 6, 2)$
$6 = 2t, 6 = 3 + t, 2 = 1 + t$
There is no value of t that satisfies the equations.
Only A lies on the line.

b. $x = 2t, y = 3 + t, z = 1 + t$
$a = 2t, b = 3 + t, -3 = 1 + t$
$t = -4$
$a = 2t = -8$
$b = 3 + t = -1$

20. a. $L_1: \frac{x - 1}{1} = \frac{y - 3}{5}$

$L_2: \frac{x - 2}{2} = \frac{1 - y}{3}$

$$\cos\theta = \frac{|n_1 \cdot n_2|}{|n_1||n_2|}$$

$n_1 = (1, 5),\ n_2 = (2, -3)$

$$\cos\theta = \frac{13}{(\sqrt{26})(\sqrt{13})}$$

$\theta = 45.0°$

b. $y = 4x + 2, y = -x + 3$
$n_1 = (1, 4)$
$n_2 = (1, -1)$

$$\cos\theta = \frac{|n_1 \cdot n_2|}{|n_1||n_2|}$$

$$\cos\theta = \frac{3}{(\sqrt{17})(\sqrt{2})}$$

$\theta = 59.0°$

c. $L_1: x = -1 + 3t, y = 1 + 4t, z = -2t$
$L_2: x = -1 + 2s, y = 3s, z = -7 + s$

$$\cos\theta = \frac{|n_1 \cdot n_2|}{|n_1||n_2|}$$

$n_1 = (3, 4, -2), n_2 = (2, 3, 1)$

$$\cos\theta = \frac{16}{(\sqrt{29})(\sqrt{14})}$$

$\theta = 37.4°$

d. $L_1: (x, y, z) = (4, 7, -1) + t(4, 8, -4)$
$L_2: (x, y, z) = (1, 5, -4) + t(-1, 2, 3)$

$$\cos\theta = \frac{|n_1 \cdot n_2|}{|n_1||n_2|}$$

$n_1 = (4, 8, -4), n_2 = (-1, 2, 3)$

$$\cos\theta = \frac{0}{(\sqrt{96})(\sqrt{14})}$$

$\theta = 90°$

21. a. $L_1: 2x + 3y - z + 9 = 0$
$L_2: x + 2y + 4 = 0$

$$\cos\theta = \frac{|n_1 \cdot n_2|}{|n_1||n_2|}$$

$n_1 = (2, 3, -1),$
$n_2 = (1, 2, 0)$

$$\cos\theta = \frac{6}{(\sqrt{14})(\sqrt{5})}$$

$\theta = 44.2°$

b. $L_1: x - y - z - 1 = 0$
$L_2: 2x + 3y - z + 4 = 0$

$$\cos\theta = \frac{|n_1 \cdot n_2|}{|n_1||n_2|}$$

$n_1 = (1, -1, -1), n_2 = (2, 3, -1)$

$$\cos\theta = \frac{0}{(\sqrt{3})(\sqrt{14})}$$

$\theta = 90°$

22. a. i. The given line is not parallel to the plane because $(3, 0, 2)$ is a point on the line and the plane.

ii. Substitute the expressions for the components of the parametric equation of the line into the equation of the plane.

$4(-3t) + (-5 + 2t) - (-10t) - 10 = 0$

$-12t - 5 + 2t + 10t - 10 = 0$

$-15 = 0$

This last statement is never true. So the line and the plane have no points in common. Therefore, the line is parallel to the plane.

iii. Use the symmetric equation to rewrite x and z in terms of y.

$x = -4y - 23$

$z = -y - 6$

Substitute into the equation of the plane.

$4(-4y - 23) + y - (-y - 6) - 10 = 0$

$-16y - 92 + y + y + 6 - 10 = 0$

$-14y - 96 = 0$

This equation has a solution. Therefore, the line and plane have a point in common and are not parallel.

b. i. Substitute the expressions for the components of the parametric equation of the line into the equation of the plane.

$4(3 + t) + (-2t) - (2 + 2t) - 10 = 0$

$12 + 4t - 2t - 2 - 2t - 10 = 0$

$0 = 0$

This last statement is always true. So every point on the line is also in the plane. Therefore, the line lies in the plane.

ii. The line is parallel to the plane, and so does not lie in it.

iii. $(5, -7, 1)$ is a point that lies on the line that does not lie in the plane. Therefore, the line does not lie in the plane.

23. $(x, y, z) = (4, 1, 6) + p(3, -2, 1) + q(-6, 6, -1)$

$(x, y, z) = (4, 1, 6) + 4(3, -2, 1) + 2(-6, 6, -1)$

$(x, y, z) = (4, 5, 8) \neq (4, 5, 6)$

24. One direction vector for the plane is $(3, -1, 1)$. $(2, 4, 1)$ and $(1, 4, 4)$ are on the plane, so another direction vector is $(2, 4, 1) - (1, 4, 4) = (1, 0, -3)$. So the parametric equations are $x = 1 + s + 3t$, $y = 4 - t$, $z = 4 - 3s + t$, $s, t \in \mathbf{R}$.

25. A plane has two parameters, because a plane goes in two different directions unlike a line that only goes in one direction.

26. This equation will always pass through the origin, because you can always set $s = 0$ and $t = -1$ to obtain $(0, 0, 0)$.

$(x, y, z) = (a, b, c) + s(d, e, f) + t(a, b, c)$

$s = 0, t = -1$

$(x, y, z) = (a, b, c) + 0(d, e, f) - 1(a, b, c)$

$(x, y, z) = (a - a, b - b, c - c) = (0, 0, 0)$

27. a. They do not form a plane, because these three points are collinear.

$\vec{r} = (-1, 2, 1) + t(3, 1, -2)$

b. They do not form a plane, because the point lies on the line.

$\vec{r} = (4, 9, -3) + t(1, -4, 2)$

$\vec{r} = (4, 9, -3) + 4(1, -4, 2)$

$= (8, -7, 5)$

28. If a is the x-intercept, b is the y-intercept, and c is the z-intercept, this means that $(a, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$ are points on the plane. So

$\vec{u} = (a, 0, 0) - (0, 0, c)$

$= (a, 0, -c)$

$\vec{v} = (0, b, 0) - (0, 0, c)$

$= (0, b, -c)$

are direction vectors for the plane. So a normal for this plane is

$\vec{u} \times \vec{v} = (a, 0, -c) \times (0, b, -c)$

$= (0(-c) - b(-c), 0(-c) - a(-c),$
$a(b) - 0(0))$

$= (bc, ac, ab)$

So the Cartesian equation of the plane is of the form $bcx + acy + abz + D = 0$

Substitute the x-intercept, $(a, 0, 0)$, into this equation to determine the value of D.

$bc(a) + ac(0) + ab(0) + D = 0$

$D = -abc$

So the Cartesian equation of this plane is

$bcx + acy + abz - abc = 0$ or

$bcx + acy + abz = abc$

29. If the normal vector is $(6, -5, 12)$, then the Cartesian equation of the plane will be of the form

$6x - 5y + 12z + D = 0$

To determine the value of D, substitute the point $(5, 8, -3)$ (which is on the plane) into this equation.

$6(5) - 5(8) + 12(-3) + D = 0$

$D = 46$

So the Cartesian equation of the plane is

$6x - 5y + 12z + 46 = 0$.

30. a., b. $A(1, -3, 2)$, $B(-2, 4, -2)$, $C(3, 2, 1)$

$\overrightarrow{AB} = (-3, 7, -4)$

$\overrightarrow{BC} = (5, -2, 3)$

$\vec{r} = \vec{r}_0 + t\vec{a} + s\vec{b}, t, s \in \mathbf{R}$

$\vec{r} = (1, -3, 2) + t(-3, 7, -4) + s(5, -2, 3),$
$t, s \in \mathbf{R}$

$x = 1 - 3t + 5s, y = -3 + 7t - 2s,$
$z = 2 - 4t + 3s$

c. To find the Cartesian equation of the plane, a normal vector is needed. This can be found by computing the cross product of the direction vectors found in parts a. and b.

$$\begin{aligned}\overrightarrow{AB} \times \overrightarrow{BC} &= (-3, 7, -4) \times (5, -2, 3)\\ &= (7(3) - (-2)(-4), 5(-4) \\ &\quad - (-3)(3), (-3)(-2) - 5(7))\\ &= (13, -11, -29)\end{aligned}$$

So the Cartesian equation has the form $13x - 11y - 29z + D = 0$.

Since $(1, -3, 2)$ is a point on this plane, we can substitute it in to determine the value of D.

$$\begin{aligned}13(1) - 11(-3) - 29(2) + D &= 0\\ D &= 12\end{aligned}$$

So the Cartesian equation of this plane is $13x - 11y - 29z + 12 = 0$.

d. Substituting $(3, 5, -4)$ into the Cartesian equation found in part **c.**, we get

$13(3) - 11(5) - 29(-4) + 12 = 100 \neq 0$

This means that $(3, 5, -4)$ is not on the plane.

31. a. The normal vector to the given plane is $(4, -2, 5)$, so any plane parallel to this one must have this same normal vector. So if a parallel plane contains the point $(0, 0, 0)$, it will have the form $4x - 2y + 5z + D = 0$.

Substitute in the point $(0, 0, 0)$ to determine the value of D.

$$\begin{aligned}4(0) - 2(0) + 5(0) + D &= 0\\ D &= 0\end{aligned}$$

So the Cartesian equation of this plane is $4x - 2y + 5z = 0$.

b. Reasoning as in part **a.**, if we want the point $(-1, 5, -1)$ to be in our parallel plane we find D in the following way:

$$\begin{aligned}4(-1) - 2(5) + 5(-1) + D &= 0\\ D &= 19\end{aligned}$$

So the Cartesian equation of the plane in this case is $4x - 2y + 5z + 19 = 0$.

c. Reasoning as in parts a. and b., if we want the point $(2, -2, 2)$ to be in our parallel plane we find D in the following way:

$$\begin{aligned}4(2) - 2(-2) + 5(2) + D &= 0\\ D &= -22\end{aligned}$$

So the Cartesian equation of the plane in this case is $4x - 2y + 5z - 22 = 0$.

32. a. The direction vector for L_1 is $(2, 1)$ and for L_2 is $(-2, -1) = -1(2, 1)$. This means that L_1 and L_2 are parallel, and since they have the point $(11, 0)$ in common (take $t = 3$ in L_1 and $s = 6$ in L_2), these lines are coincident. So the angle between them is $\theta = 0°$.

b. The parametric equations of these lines are

$L_1: x = -3 + 3t, y = -1 + 4t, t \in \mathbf{R}$

$L_2: x = 6 + 3s, y = 2 - 2s, s \in \mathbf{R}$

So a point of intersection satisfies

$-3 + 3t = 6 + 3s$

$-1 + 4t = 2 - 2s$

or

$3t - 3s = 9$

$4t - 2s = 3$

or

$$\begin{aligned}t &= s + 3\\ 4t + 2s &= 3\\ 4(s + 3) + 2s &= 3\\ 6s &= -9\\ s &= -\frac{3}{2}\\ t &= s + 3\\ &= -\frac{3}{2} + 3\\ &= \frac{3}{2}\end{aligned}$$

So the point of intersection is

$$\begin{aligned}x &= -3 + 3\left(\frac{3}{2}\right)\\ &= \frac{3}{2}\\ y &= -1 + 4\left(\frac{3}{2}\right)\\ &= 5\end{aligned}$$

The point of intersection is $\left(\frac{3}{2}, 5\right)$ at $s = -\frac{3}{2}$ (for L_2) and $t = \frac{3}{2}$ (for L_1).

The direction vector for L_1 is $(3, 4)$, and for L_2 is $(3, -2)$. So the angle θ between these lines satisfies

$$\begin{aligned}\cos\theta &= \frac{(3, 4) \cdot (3, -2)}{|(3, 4)||(3, -2)|}\\ \theta &= \cos^{-1}\left(\frac{(3, 4) \cdot (3, -2)}{|(3, 4)||(3, -2)|}\right)\\ &= \cos^{-1}\left(\frac{1}{5\sqrt{3}}\right)\\ &\doteq 86.82°\end{aligned}$$

It would also have been correct to report the supplement of this angle, or roughly 93.18°, as the answer in this case.

33. a. $P(1, 3, 5)$

$\vec{r} = \vec{r}_0 + t\vec{a}$

$\vec{r} = (1, 3, 5) + t(-2, -4, -10), t \in \mathbf{R}$

$x = 1 - 2t, y = 3 - 4t, z = 5 - 10t$

$\frac{x-1}{-2} = \frac{y-3}{-4} = \frac{z-5}{-10}$

b. $P(1, 3, 5), Q(-7, 9, 3)$

$\overrightarrow{PQ} = (-8, 6, -2)$

$\vec{r} = \vec{r}_0 + t\vec{a}$

$\vec{r} = (1, 3, 5) + t(-8, 6, -2), t \in \mathbf{R}$

$x = 1 - 8t, y = 3 + 6t, z = 5 - 2t$

$\frac{x-1}{-8} = \frac{x-3}{6} = \frac{x-5}{-2}$

c. $P(1, 3, 5)$

$\overrightarrow{RS} = (-6, -13, 14)$

$\vec{r} = \vec{r}_0 + t\vec{a}$

$\vec{r} = (1, 3, 5) + t(-6, -13, 14), t \in \mathbf{R}$

$x = 1 - 6t, y = 3 - 13t, z = 5 + 14t$

$\frac{x-1}{-6} = \frac{x-3}{-13} = \frac{x-5}{14}$

d. Since its parallel to the x-axis, its direction vector is $(1, 0, 0)$.

$P(1, 3, 5)$,

$\vec{n} = (1, 0, 0)$

$\vec{r} = \vec{r}_0 + t\vec{a}$

$\vec{r} = (1, 3, 5) + t(1, 0, 0), t \in \mathbf{R}$

$x = 1 + t, y = 3, z = 5$

e. Find a perpendicular vector use the dot product.

$(-3, 4, -6) \cdot (a, b, c) = 0$

$-3a + 4b - 6c + 0$

$a = 0, b = 6, c = 4$

$\vec{n} = (0, 6, 4)$

$\vec{r} = (1, 3, 5) + t(0, 6, 4), t \in \mathbf{R}$

f. Since the line is perpendicular to the plane, the line's directional vector is the normal of the plane. Use the cross product to find the vector.

$A(4, 2, 1), B(3, -4, 2), C(-3, 2, 1)$

$\overrightarrow{AB} = (-1, -6, 1)$

$\overrightarrow{BC} = (-6, 6, -1)$

$\overrightarrow{AB} \times \overrightarrow{BC} = (0, -7, -42) = (0, 1, 6) = \vec{n}$

$\vec{r} = (1, 3, 5) + t(0, 1, 6)$

$x = 1, y = 3 + t, z = 5 + 6t$

34. a. This plane will be of the form

$2x - 4y + 5z + D = 0$.

To find D, substitute in $P(-2, 6, 1)$.

$2(-2) - 4(6) + 5(1) + D = 0$

$D = 23$

So the Cartesian equation of the plane is

$2x - 4y + 5z + 23 = 0$.

b. The direction vector for this line is $(3, -5, 2)$ (we can use this as one of the direction vectors for the plane), and a point on this line is $(4, -2, 1)$. So a second direction vector for the plane will be

$\vec{v} = (4, -2, 1) - P(-2, 0, 6)$

$= (6, -2, -5)$

So a normal vector for this plane is

$(3, -5, 2) \times (6, -2, -5) = ((-5)(-5) - (-2)(2), 6(2) - 3(-5), 3(-2) - 6(-5))$

$= (29, 27, 24)$

The Cartesian equation of this plane has the form

$29x + 27y + 24z + D = 0$.

Substitute in $P(-2, 0, 6)$ to determine D.

$29(-2) + 27(0) + 24(6) + D = 0$

$D = -86$

The Cartesian equation of this plane is

$29x + 27y + 24z - 86 = 0$.

c. This plane, being parallel to the xy-plane, is completely determined by a fixed z-coordinate (the x- and y- coordinates are allowed to be anything at all). Since it passes through the point $P(3, 3, 3)$, the equation of this plane is $z = 3$. Written in Cartesian form, this is $z - 3 = 0$.

d. Since this plane is to be parallel to $3x + y - 4z + 8 = 0$, it will have the same normal vector, $(3, 1, -4)$. So this plane will be of the form $3x + y - 4z + D = 0$.

Since $P(-4, 2, 4)$ is on this plane, we can substitute this in to determine the value of D.

$3(-4) + 2 - 4(4) + D = 0$

$D = 26$

So the Cartesian equation of this plane is

$3x + y - 4z + 26 = 0$.

e. Since this plane is perpendicular to the yz-plane, it is completely determined by its intersection with the yz-plane, which will be a line with y-intercept 4 and z-intercept -2. This means that y and z are related by $y = mz + 4$ because of the y-intercept of 4. We can find the value of m by using the z-intercept of -2.

$0 = m(-2) + 4$

$m = 2$

So y and z are related via $y = 2z + 4$, and the Cartesian equation of the plane is $y - 2z - 4 = 0$. (x is allowed to be anything here.)

f. A normal vector, (A, B, C), for this plane will be perpendicular to the normal vector for the plane

$x - 2y + z = 6$, which is $(1, -2, 1)$. Also, (A, B, C) will be perpendicular to the direction vector for the line contained in the plane we seek. This direction vector is $(3, 1, 2)$, and so this means we can take

$$(A, B, C) = (1, -2, 1) \times (3, 1, 2) = ((-2)(2) - (1)(1), 3(1) - 1(2), 1(1) - 3(-2)) = (-5, 1, 7)$$

So the Cartesian equation will have the form $-5x + y + 7z + D = 0$.

Since is on this plane (take in the line this plane is to contain), we can substitute this in to determine the value of D.

$$-5(2) + (-1) + 7(-1) + D = 0$$
$$D = 18$$

So the Cartesian equation of this plane is $-5x + y + 7z + 18 = 0$.

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1. a. i $\overrightarrow{AB}$ and $\overrightarrow{AC}$ can be the direction vectors for this plane and $A(1, 2, 4)$ can be the origin point.

$$\overrightarrow{AB} = (2, 0, 3) - (1, 2, 4) = (1, -2, -1)$$
$$\overrightarrow{AC} = (4, 4, 4) - (1, 2, 4) = (3, 2, 0)$$

This gives a vector equation of $\vec{r} = (1, 2, 4) + s(1, -2, -1) + t(3, 2, 0), s, t \in \mathbf{R}$.

The corresponding parametric equation for this plane is $x = 1 + s + 3t$, $y = 2 - 2s + 2t$, $z = 4 - s$, $s, t \in \mathbf{R}$.

ii. The corresponding Cartesian equation is found by taking the cross product of the two direction vectors.

$$\overrightarrow{AB} \times \overrightarrow{AC} = ((-2)0 - (-1)2, (-1)3 - (1)0, (1)2 - (-2)3) = (2, -3, 8)$$

So $(2, -3, 8)$ is a normal vector for the plane, so the plane has the form $2x - 3y + 8z + D = 0$, for some constant D. To find D, we know that A(1, 2, 4) is a point on the plane, so $2(1) - 3(2) + 8(4) + D = 0$. So $28 + D = 0$, or $D = -28$. So the Cartesian equation for the plane is $2x - 3y + 8z - 28 = 0$.

b. A point (x, y, z) is on the plane if and only if $2x - 3y + 8z - 28 = 0$. Since $2(1) - 3(-1) + 8\left(-\frac{1}{2}\right) - 28 = -27 \neq 0$, the point $\left(1, -1, -\frac{1}{2}\right)$ is not on the plane.

2. a. Since $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ for all (x, y, z) on the plane, it holds true for the given points. So $\frac{2}{a} + \frac{0}{b} + \frac{0}{c} = 1$ or $a = 2$. Similarly $\frac{0}{a} + \frac{3}{b} + \frac{0}{c} = 1$ and $\frac{0}{a} + \frac{0}{b} + \frac{4}{c} = 1$ implies that $b = 3$ and $c = 4$. So the equation of the plane is $\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1$.

b. If both sides are multiplied by the least common multiple of the denominators, then an equivalent equation for the plane is $6x + 4y + 3z = 12$. Hence $(6, 4, 3)$ is a normal vector for this plane.

3. a. Since the origin is a point on the plane and $(2, 1, 3) + 0(1, 2, 5) = (2, 1, 3)$ is a point on the plane, $(2, 1, 3)$ is a direction vector for the plane. $(2, 1, 3) + 1(1, 2, 5) = (3, 3, 8)$ is a point on the plane and $(2, 1, 3)$ is another point on the plane, so $(3, 3, 8) - (2, 1, 3) = (1, 2, 5)$ is a directional vector for the plane as well. $(2, 1, 3)$ and $(1, 2, 5)$ are not collinear, because the ratios between the coordinates are not equal. Since the origin is a point on the plane, a vector equation for the plane is $\vec{r} = s(2, 1, 3) + t(1, 2, 5)$, $s, t \in \mathbf{R}$.

b. To find the Cartesian equation for the plane, the normal vector is determined by the cross product of the two direction vectors from part **a**.

$$(2, 1, 3) \times (1, 2, 5) = ((1)5 - (3)2, (3)1 - (2)5, (2)2 - (1)1) = (-1, -7, 3)$$

So the Cartesian equation for the plane has the form $-x - 7y + 3z + D = 0$, for some constant D. Since the origin is a point on the plane, $-(0) - 7(0) + 3(0) + D = 0$, so $D = 0$. Thus the equation is $-x - 7y + 3z = 0$.

4. a. $(2, 0, -3)$ and $(5, 1, -1)$ are each direction vectors for the planes. The vectors are not collinear since the ratios of the coordinates are not equal. $(4, -3, 5)$ is a point on the plane, so a vector equation for the plane is $\vec{r} = (4, -3, 5) + s(2, 0, -3) + t(5, 1, -1)$, $s, t \in \mathbf{R}$.

b. To find the Cartesian equation for the plane, the normal vector is determined by the cross product of the two direction vectors from part **a**.

$$(2, 0, -3) \times (5, 1, -1) = ((0)(-1) - (-3)1, (-3)5 - (2)(-1), (2)1 - (0)5) = (3, -13, 2)$$

So the Cartesian equation for the plane has the form $3x - 13y + 2z + D = 0$, for some constant D.

Since the $(4, -3, 5)$ is a point on the plane,
$3(4) - 13(-3) + 2(5) + D = 0$, so
$61 + D = 0$. So $D = -61$. Thus the equation is
$3x - 13y + 2z - 61 = 0$.

5. a. The line intersects the yz-plane when $x = 0$.
If $x = 0$, then $\frac{y-4}{-2} = z = \frac{0-2}{4} = -\frac{1}{2}$, so $y = \left(-\frac{1}{2}\right)(-2) + 4 = 5$ and $z = -\frac{1}{2}$. Thus the point is $\left(0, 5, -\frac{1}{2}\right)$.

b. The direction vector $(4, -2, 1)$ is the same, so the equivalent symmetric equation for the line is
$\frac{x}{4} = \frac{y-5}{-2} = z + \frac{1}{2}$.

6. a. The angle between two planes is determined by the dot product of their normal vectors. The normal vector of the first plane is $(1, 1, -1)$ and the normal vector of the second plane is $(1, -1, 1)$.
$(1, 1, -1) \cdot (1, -1, 1) = -1$ and
$|(1, 1, -1)| = \sqrt{3}$. So the angle between the planes is $\cos^{-1}\left(\frac{-1}{\sqrt{3}\sqrt{3}}\right) \doteq 109.5°$ The acute angle is $70.5°$.

b. i. The planes are parallel if and only if the corresponding normal vectors are parallel. The normal vector of the first plane is $(2, -1, k)$ and the normal vector of the second plane is $(k, -2, 8)$. The vectors are parallel if and only if the ratios between the coordinates are equal. Suppose $\frac{k}{2} = \frac{-2}{-1} = 2$, so then $k = 4$. So the vectors can be parallel only when $k = 4$. Since $\frac{8}{4} = 2$ as well, the vectors are parallel at $k = 4$.

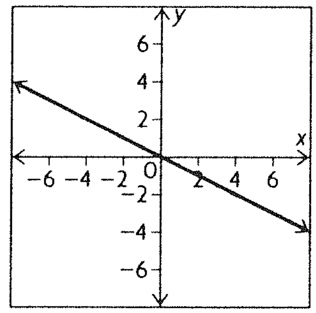
ii. The planes are perpendicular when their normal vectors are perpendicular. The vectors are perpendicular when their dot product is equal to zero.

$$\begin{aligned}(2, -1, k) \cdot (k, -2, 8) &= 2k - 1(-2) + 8k \\ &= 10k + 2\end{aligned}$$

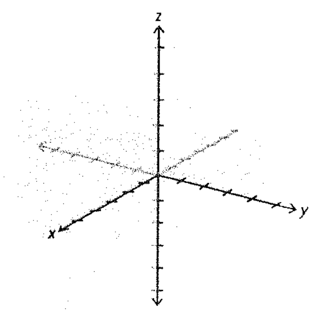
So if $k = -\frac{1}{5}$, then the dot product of the two normal vectors is equal to zero. Hence the planes are perpendicular at $k = -\frac{1}{5}$.

c. The first plane in **b.** intersects the y-axis at the point $(0, d, 0)$, where d satisfies $2(0) - d + k(0) = 5$. So $d = -5$. The second plane in **b.** intersects the y-axis at the point $(0, e, 0)$, where e satisfies $k(0) - 2e + 8(0) = 9$. So $e = -4.5$. Since the planes intersect the y-axis only once and the points are different, the equations can never represent the same plane.

7. a.



b. The equation for the plane can be written as $2x + y + 0z = 0$. So for any real number t, $2(0) + (0) + 0(t) = 0$, so the point $(0, 0, t)$ is on the graph. So the z-axis is on the plane. Also the plane cuts across the xy-plane along the line $2x + y = 0$. So the origin is a point, as well as $(-2, 1, 0)$.



c. The equation for the plane can be written as $Ax + By + 0z = 0$. For any real number t, $A(0) + B(0) + 0(t) = 0$, so $(0, 0, t)$ is on the plane. Since this is true for all real numbers, the z-axis is on the plane.

CHAPTER 9

Relationships Between Points, Lines, and Planes

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1. a. Yes; $(2, -5) = (10, -12) + t(8, -7)$
$(2, -5) = (10, -12) + 1(8, -7)$
b. No; $12(1) + 5(2) - 13 = 9 \neq 0$
c. Yes; $(7, -3, 8) = (1, 0, -4) + t(2, -1, 4)$
$(7, -3, 8) = (1, 0, -4) + 3(2, -1, 4)$
d. No; $(1, 0, 5) = (2, 1, -2) + t(4, -1, 2)$
$(-1, -1, 7) \neq t(4, -1, 2)$
There is no value of t that satisfies the equation.
2. Answers may vary. For example:
a. Vector: $\vec{m} = (7, 3) - (2, 5) = (5, -2)$
$\vec{r} = (2, 5) + t(5, -2), t \in \mathbf{R}$
Parametric: $x = 2 + 5t, y = 5 - 2t, t \in \mathbf{R}$
b. Vector: $\vec{m} = (4, -7) - (-3, 7) = (7, -14)$
$\vec{r} = (-3, 7) + t(7, -14), t \in \mathbf{R}$
Parametric: $x = -3 + 7t, y = 7 - 14t, t \in \mathbf{R}$
c. Vector: $\vec{m} = (-3, -11) - (-1, 0)$
$= (-2, -11)$
$\vec{r} = (-1, 0) + t(-2, -11), t \in \mathbf{R}$
Parametric: $x = -1 + -2t, y = -11t, t \in \mathbf{R}$
d. Vector: $\vec{m} = (6, -7, 0) - (1, 3, 5)$
$= (5, -10, -5)$
$\vec{r} = (1, 3, 5) + t(5, -10, -5), t \in \mathbf{R}$
Parametric: $x = 1 + 5t, y = 3 - 10t, z = 5 - 5t, t \in \mathbf{R}$
e. Vector: $\vec{m} = (-1, 5, 2) - (2, 0, -1)$
$= (-3, 5, 3)$
$\vec{r} = (2, 0, -1) + t(-3, 5, 3), t \in \mathbf{R}$
Parametric: $x = 2 - 3t, y = -5t, z = -1 + 3t, t \in \mathbf{R}$
f. Vector: $\vec{m} = (12, -5, -7) - (2, 5, -1)$
$= (10, -10, -6)$
$\vec{r} = (2, 5, -1) + t(10, -10, -6), t \in \mathbf{R}$
Parametric: $x = 2 + 10t, y = 5 - 10t, z = -1 - 6t, t \in \mathbf{R}$
3. a. Since $\vec{n} = (2, 6, -1)$, the Cartesian equation of the plane is of the form $2x + 6y - z + D = 0$, where D is to be determined. Since $P_0(4, 1, -3)$ is on the plane, it must satisfy the equation. So $2(4) + 6(1) - 1(-3) + D = 8 + 6 + 3 + D = 17 + D = 0$. $D = -17$, and the equation of the plane is $2x + 6y - z - 17 = 0$.
b. Since $\vec{n} = (0, 7, 0)$, the Cartesian equation of the plane is of the form $7y + D = 0$, where D is to be determined. Since $P_0(-2, 0, 5)$ is on the plane, it must satisfy the equation. So $7(0) + D = 0 + D = 0$ thus $D = 0$. The equation of the plane is $7y = 0$, or $y = 0$.
c. Since $\vec{n} = (4, -3, 0)$, the Cartesian equation of the plane is of the form $4x - 3y + D = 0$, where D is to be determined. Since $P_0(3, -1, -2)$ is on the plane, it must satisfy the equation. So $4(3) - 3(-1) + D = 12 + 3 + D = 15 + D = 0$. $D = -15$, and the equation of the plane is $4x + 3y - 15 = 0$.
d. Since $\vec{n} = (6, 5, -3)$, the Cartesian equation of the plane is of the form $6x - 5y + 3z + D = 0$, where D is to be determined. Since $P_0(0, 0, 0)$ is on the plane, it must satisfy the equation. So $6(0) - 5(0) + 3(0) + D = 0$, or $D = 0$. The equation of the plane is $6x - 5y + 3z = 0$.
e. Since $\vec{n} = (11, -6, 0)$, the Cartesian equation of the plane is of the form $11x - 6y + D = 0$, where D is to be determined. Since $P_0(4, 1, 8)$ is on the plane, it must satisfy the equation. So $11(4) - 6(1) + D = 44 - 6 + D = 38 + D = 0$. $D = -38$, and the equation of the plane is $11x - 6y - 38 = 0$.
f. Since $\vec{n} = (1, 1, -1)$, the Cartesian equation of the plane is of the form $x + y + -z + D = 0$, where D is to be determined. Since $P_0(2, 5, 1)$ is on the plane, it must satisfy the equation. So $2 + 5 - 1 + D = 6 + D = 0$. $D = -6$, and the equation of the plane is $x + y - z - 6 = 0$.
4. Start by writing the given line in parametric form: $(x, y, z) = (2 + s + 2t, 1 - s, 3s - 5t)$, so $x = 2 + s + 2t$, $y = 1 - s$, and $z = 3s - 5t$. Solving for s in each component, we get $s = 1 - y$ and substituting this into $z = 3s - 5t$ gives $z = 3(1 - y) - 5t = 3 - 3y - 5t$.

So now $-3 + 3y + z = -5t$ and $t = \frac{3 - 3y - z}{5}$.
Finally, substituting both equations for s and t into $x = 2 + s + 2t$, we get

$$x = 2 + (1 - y) + 2\left(\frac{3 - 3y - z}{5}\right).$$

Rearranging, we get

$$5x = 10 + 5 - 5y + 6 - 6y - 2z$$
$$5x + 11y + 2z - 21 = 0.$$

5. L_1 is not parallel to the plane because $(3, 0, 2)$ is a point on the line and the plane. Substitute the expressions for the components of the parametric equation of the line into the equation of the plane.

$$4(3 + t) + (-2t) - (2 + 2t) - 10 = 0$$
$$12 + 4t - 2t - 2 - 2t - 10 = 0$$
$$0 = 0$$

This last statement is always true. So every point on the line is also in the plane. Therefore, the line lies on the plane.
For L_2 substitute the expressions for the components of the parametric equation of the line into the equation of the plane.

$$4(-3t) + (-5 + 2t) - (-10t) - 10 = 0$$
$$-12t - 5 + 2t + 10t - 10 = 0$$
$$-15 = 0$$

This last statement is never true. So the line and the plane have no points in common. Therefore, L_2 is parallel to the plane. The line cannot lie on the plane.
For L_3 use the symmetric equation to rewrite x and z in terms of y.

$$x = -4y - 23$$
$$z = -y - 6$$

Substitute into the equation of the plane.

$$4(-4y - 23) + y - (-y - 6) - 10 = 0$$
$$-16y - 92 + y + y + 6 - 10 = 0$$
$$-14y - 96 = 0$$

This equation has a solution. Therefore, L_3 and the plane have a point in common and are not parallel. However, $(5, -7, 1)$ is a point that lies on the line that does not lie on the plane. Therefore, L_3 does not lie in the plane.

6. a. A normal vector to this plane is determined by calculating the cross product of the position vectors, $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$$\overrightarrow{AB} = (2, 0, 0) - (1, 0, -1) = (1, 0, 1)$$
$$\overrightarrow{AC} = (6, -1, 5) - (1, 0, -1) = (5, -1, 6)$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = ((0 \cdot 6) - (1 \cdot -1), (1 \cdot 5) - (1 \cdot 6), (1 \cdot -1) - (0 \cdot 5))$$
$$= (0 + 1, 5 - 6, -1 - 0)$$
$$= (1, -1, -1) = \vec{n}.$$

If $P(x, y, z)$ is any point on the plane, then $\overrightarrow{AP} = (x - 1, y, z + 1)$, and if the normal to the plane is $(1, -1, -1)$, then
$(x - 1, y, z + 1) \cdot (1, -1, -1) = 0$, so
$x - 1 - y - z - 1 = 0$ and thus,
$x - y - z - 2 = 0$

b. $\overrightarrow{PQ} = (6, 4, 0) - (4, 1, -2) = (2, 3, 2)$
$\overrightarrow{PR} = (0, 0, -3) - (4, 1, -2) = (-4, -1, -1)$
$\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$

$$= (3(-1) - 2(-1)), 2(-4) - 2(-1), 2(-1) - 3(4))$$
$$= (-3 + 2, -8 + 2, -2 + 12) = (-1, -6, 10)$$

Since $(-1, -6, 10) = -1(1, 6, -10)$, we will use $(1, 6, -10)$ as the normal vector so that the coefficient of x is positive. If $P(x, y, z)$ is any point on the plane, then $\overrightarrow{AP} = (x - 4, y - 1, z + 2)$, and if the normal to the plane is $(1, 6, -10)$, then
$(x - 4, y - 1, z + 2) \cdot (1, 6, -10) = 0$,
so $x - 4 + 6y - 6 - 10z - 20 = 0$.
and thus $x + 6y - 10z - 30 = 0$.

7. Answers may vary. For example: One direction vector is $\vec{m} = (2, -1, 6) - (1, -4, 3) = (1, 3, 3)$. Now we need to find a normal to the plane such that $\vec{n} \cdot \vec{m} = 0$. So $(1, 3, 3) \cdot (a, 0, c) = 0$. Now we have $a + 3c = 0$. A possible solution to this is $a = 3, c = -1$. So $\vec{n} = (3, 0, -1)$ and the Cartesian equation of the plane is $3x - z = 0$. Since the plane is parallel to the y-axis, $(0, 1, 0)$ is another direction vector for the plane. Therefore, a vector equation for the plane is
$\vec{r} = (1, -4, 3) + t(1, 3, 3) + s(0, 1, 0), s, t \in \mathbf{R}$.

8. We are given the point $A(-1, 3, 4)$. We need to find a normal vector $\vec{n} = (a, b, c)$ such that
$a(x + 1) + b(y - 3) + c(z - 4) + d = 0$.
The normal vector also must be perpendicular to the two planes and their normals, $(2, -1, 3)$ and $(5, 1, -3)$. One possible solution for the normal is $\vec{n} = (0, 3, 1)$. So we have

$$3(y - 3) + z - 4 = 0$$
$$3y + z - 9 - 4 = 0$$

And the equation of the plane is $3y + z = 13$.

9.1 The Intersection of a Line with a Plane and the Intersection of Two Lines, pp. 496–498

1. a. First, show the parametric equations as $x = 1 + 5s$, $y = 2 + s$, $z = -3 + s$. Then the plane can be written as $\pi: x - 2y - 3z = 6$, and the vector equation of the line is $\vec{r} = (1, 2, -3) + s(5, 1, 1), s \in \mathbf{R}$.

b. When we substitute the parametric equations into the Cartesian equation for the plane, we get

$(1 + 5s) - 2(2 + s) - 3(-3 + s) = 6$

$1 - 4 + 9 + 5s - 2s - 3s = 6 - 0s = 6$

Note that by finishing the solution, we get $0s = 0$. Since any real number will satisfy this equation, we have an infinite number of solutions, and this line lies on the plane.

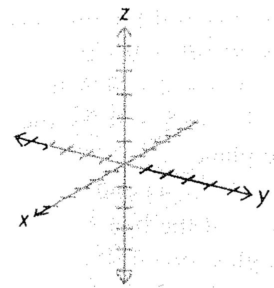
2. a. A line and a plane can intersect in three ways:
Case 1: The line and the plane have zero points of intersection. This occurs when the lines are not incidental, meaning they do not intersect.
Case 2: The line and the plane have only one point of intersection. This occurs when the line crosses the plane at a single point.
Case 3: The line and the plane have an infinite number of intersections. This occurs when the line is coincident with the plane, meaning the line lies on the plane.

b. Assume that the line and the plane have more than one intersection, but not an infinite number. For simplicity, assume two intersections. At the first intersection, the line crosses the plane. In order for the line to continue on, it must have the same direction vector. If the line has already crossed the plane, then it continues to move away from the plane, and can not intersect again. So the line and the plane can only intersect zero, one, or infinitely many times.

3. a. The line $\vec{r} = s(1, 0, 0)$ is the x-axis.

b. The plane $y = 1$ has the form $\vec{r} = (x, 1, z)$, where x, and z are any values in $\mathbf{R}$. So the plane is parallel to the xz-plane, but just one unit away to the right.

c.



d. There are no intersections between the line and the plane.

4. a. For $x + 4y + z - 4 = 0$, if we substitute the parametric equations, we have

$(-2 + t) + 4(1 - t) + (2 + 3t) + 4$

$= -2 + 4 + 2 + t - 4t + 3t - 4$

$= 0t + 0$

$= 0$. All values of t give a solution to the equation, so all points on the line are also on the plane.

b. For the plane $2x - 3y + 4z - 11 = 0$, we can substitute the parametric equations derived from $\vec{r} = (1, 5, 6) + t(1, -2, -2)$:

$x = 1 + t$, $y = 5 - 2t$, $z = 6 - 2t$.

So we have $2(1 + t) - 3(5 - 2t) + 4(6 - 2t) - 11$

$= 2 - 15 + 24 - 11 + 2t + 6t - 8t$

$= 0t + 0$

$= 0$

Similar to part a., all values of t give a solution to this equation, so all points on the line are also on the plane.

5. a. First, we should determine the parametric equations from the vector form: $x = -1 - s$, $y = 1 + 2s$, $z = 2s$. Substituting these into the equation of the plane, we get

$2(-1 - s) - 2(1 + 2s) + 3(2s) - 1$

$= -2 - 2 - 1 - 2s - 4s + 6s$

$= -5 + 0s$

Since there are no values of s such that $-5 = 0$, this line and plane do not intersect.

b. Substituting the parametric equations into the equation of the plane, we get

$2(1 + 2t) - 4(-2 + 5t) + 4(1 + 4t) - 13$

$2 + 8 + 4 - 13 + 4t - 20t + 16t$

$= 1 + 0t$

Since there are no values of t such that $1 = 0$, there are no solutions, and the plane and the line do not intersect.

6. a. The direction vector is $\vec{m} = (-1, 2, 2)$ and the normal is $\vec{n} = (2, -2, 3)$, so if the line and the plane meet at right angles, $\vec{m} \cdot \vec{n} = 0$. So $(-1 \cdot 2) + (2 \cdot -2) + (2 \cdot 3) = -2 - 4 + 6 = 0$, but $2(-1) - 2(1) + 3(0) - 1 = -5 \neq 0$. So the point on the line is not on the plane.
b. The direction vector is $\vec{m} = (2, 5, 4)$ and the normal is $\vec{n} = (2, -4, 4)$, so if the line and the plane meet at right angles, $\vec{m} \cdot \vec{n} = 0$. So $(2 \cdot 2) + (5 \cdot -4) + (4 \cdot 4) = 4 - 20 + 16 = 0$, but $2(1) - 4(-2) + 4(1) - 13 = 1 \neq 0$. So the point on the line is not on the plane.
7. a. If the line and the plane intersect, then they are equal at a particular point p. So we must substitute the parametric equations into the equation of the plane, and then solve for p.
$(-1 + 6p) + 2(3 + p) - (4 - 2p) + 29$
$= -1 + 6 - 4 + 6p + 2p + 2p + 29$
$= 30 + 10p$
$= 0$. So now $-10p = 30$ and $p = -3$.
Now we must find the point at which the line and plane intersect. To do this, just substitute $p = -3$ into the vector form of the line:
$(-1, 3, 4) + -3(6, 1, -2) = (-19, 0, 10)$.
b. If the line and the plane intersect, then they are equal at a particular point p. So we must substitute the parametric equations into the equation of the plane, and then solve for p.
$x = 1 + 4s, y = -2 - s, z = 3 + s$
$2(1 + 4s) + 7(-2 - s) + (3 + s) + 15$
$= 2 - 14 + 3 + 15 + 8s - 7s + s$
$= 6 + 2s$
$= 0$. So now $-2s = 6$ and $s = -3$. Now we must find the point at which the line and plane intersect. To do this, just substitute $s = -3$ into the vector form of the line:
$(1, -2, 3) + -3(4, -1, 1) = (-11, 1, 0)$
8. a. Comparing the x and y components in L_1 and L_2, we have
$3 + 4s = 4 + 13t$
$1 - s = 1 - 5t$
We can easily solve for one of the variables by using the second equation: $s = 5t$. Substituting this back into the first equation: $3 + 20t = 4 + 13t$ so $1 = 7t$ and thus $t = \frac{1}{7}$. So now we must solve for s: $3 + 4s = 4 + \frac{13}{7}$ and $s = \frac{20}{28} = \frac{5}{7}$. Placing these back into the equations for L_1 and L_2:

$$L_1: (3, 1, 5) + \frac{5}{7}(4, -1, 2) = \left(\frac{41}{7}, \frac{2}{7}, \frac{45}{7}\right)$$
$$L_2: \left(4 + \frac{13}{7}, 1 - \frac{5}{7}, \frac{5}{7}\right) = \left(\frac{41}{7}, \frac{2}{7}, \frac{5}{7}\right)$$

The points must be equal for intersection to occur, so there is no intersection and the lines are skew.
b. If we compare the z components of the two lines, we see $2 = 8 - 6s$ or $s = 1$. Substituting this back into the x component (the y component would work just as well), we have $3 + m = -3 + 7(1) = 4$, or $m = 1$. So now we can substitute m and s back into the equations for the line, and we get
$L_3: (3, 7, 2) + (1, -6, 0) = (4, 1, 2)$
$L_4: (-3, 2, 8) + (7, -1, -6) = (4, 1, 2)$
So $(4, 1, 2)$ is the only point of intersection between these two lines.
9. a. Comparing the y and z components of each vector equation, we get the system of equations:
$3 - 2p = 3 - 2q$
$4 + 3p = -4 + 11q$
Note that from the first equation, $p = q$. So the second equation becomes $4 + 3q = -4 + 11q$. Solving for q, we get $q = 1$. So from the earlier relation, $p = 1$. Placing these two values back into the vector equations, we get
$(-2, 3, 4) + (6, -2, 3) = (4, 1, 7)$
$(-2, 3, -4) + (6, -2, 11) = (4, 1, 7)$
This shows that these two lines intersect at $(4, 1, 7)$.
b. Comparing the x and z components of each vector equation, we get the system of equations:
$4 + r = 2 + s$
$6 + 4r = -8 + 5s$
Note that from the first equation, $s = 2 + r$. So the second equation becomes $6 + 4r = 2 + 5r$. Solving for r, we get $r = 4$. So from the earlier relation, $s = 6$. Placing these two values back into the vector equations, we get
$(4, 1, 6) + 4(1, 0, 4) = (8, 1, 22)$
$(2, 1, -8) + 6(1, 0, 5) = (8, 1, 22)$
This shows that these two lines intersect at $(8, 1, 22)$.
c. Comparing the x and z components of each vector equation, we get the system of equations:
$2 + m = -2 + 3p$
$1 + m = 1 - p$
Note that from the second equation, $m = -p$. So the first equation becomes $2 - p = -2 + 3p$. Solving for p, we get $p = 1$. So from the earlier relation, $m = -1$. Placing these two values back into the vector equations, we get

$(2, 2, 1) - (1, 1, 1) = (1, 1, 0)$
$(-2, 2, 1) + (3, -1, -1) = (1, 1, 0)$
This shows that these two lines intersect at $(1, 1, 0)$.
d. Comparing the x and y components of each vector equation, we get the system of equations:
$1 + 0m = 2 + s$
$2 + 4m = 3 - 2s$
Note that from the first equation, $s = -1$. So the second equation becomes $2 + 4m = 5$.
Solving for m, we get $m = \frac{3}{4}$. Placing these two values back into the vector equations, we get

$$(9, 1, 2) - \frac{3}{4}(5, 0, 4) = \left(\frac{21}{4}, 1, -1\right)$$

$(8, 2, 3) - (4, 1, -2) = (4, 1, 5)$
The two lines do not intersect, so they are skew.
10. At the point where the line intersects the z-axis, the point $Q(0, 0, q)$ equals the vector equation. So for the x component, $-3 + 3s = 0$ or $s = 1$. Substituting this into the vector equation, we get $(-3, 2, 1) + (3, -2, 7) = (0, 0, 8)$. So $q = 8$.
11. a. Comparing the x components, we get $-2 + 7s = -30 + 7t$, which can be reduced to $28 + 7s = 7t$ or $s - t = 4$. Comparing the other components, the same equation results.
b. From L_1, we see that at $(-2, 3, 4)$, $s = 0$. When this occurs, $t = 4$. Substituting this into L_2, we get $(-30, 11, -4) + 4(7, -2, 2) = (-2, 3, 4)$. Since both of these lines have the same direction vector and a common point, the lines are coincidental.
12. a. First, we must determine the values of s and t. So comparing the x and z components, we get
$-3 + s = 1 - 3t$
$1 + s = 2 + 8t$
From the second equation, $s = 1 + 8t$. Substituting this back into the first equation,
$-3 + 1 + 8t = 1 - 3t$ or $t = \frac{3}{11}$.
Substituting back into the second equation,
$-3 + s = 1 - \frac{9}{11} = \frac{2}{11}$, and solving for s,
$s = \frac{2}{11} + 3 = \frac{35}{11}$. Now we can solve for k. Compare the y components after substituting s and t.

$$8 - \frac{35}{11} = 4 + \frac{3}{11}k$$

$53 = 44 + 3k$
or $k = 3$.
b. The lines intersect when $s = \frac{35}{11}$. The point of intersection is $\left(-3 + \frac{35}{11}, 8 - \frac{35}{11}, 1 + \frac{35}{11}\right)$ or $\left(\frac{2}{11}, \frac{53}{11}, \frac{46}{11}\right)$.

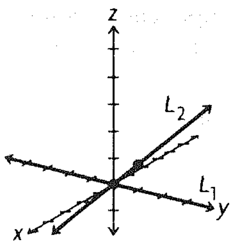
13. On the xz-plane, the point A has the coordinates $(x, 0, z)$, for any x, z. Similarly, on the yz-plane, the point B has the coordinates $(0, y, z)$ for any y, z. Now the task is to find the required values of s for these points. Starting with the x component of point B, we have $0 = -8 + 2s$ or $s = 4$. So point B is $(-8, -6, -1) + 4(2, 2, 1) = (0, 2, 3)$. For point A, we need the y coordinate to equal 0. So $0 = -6 + 2s$ or $s = 3$. So point A is
$(-8, -6, -1) + 3(2, 2, 1) = (-2, 0, 2)$.
Now we need to find the distance.

$$\begin{aligned} d &= \sqrt{(0 - (-2))^2 + (2 - 0)^2 + (3 - 2)^2} \\ &= \sqrt{4 + 4 + 1} \\ &= \sqrt{9} \\ &= 3 \end{aligned}$$

14. a. Comparing the y and z components of each vector equation, we get the system of equations:
$1 + 0p = -1 - 2q$
$1 - p = 1 - 2q$
Note that from the first equation, $2 = -2q$ or $q = -1$. So the second equation becomes $1 - p = 1 + 2$ or $p = -2$.
Placing these two values back into the vector equations to find the intersection point A, we get
$(2, 1, 1) - 2(4, 0, -1) = (-6, 1, 3)$
$(3, -1, 1) - (9, -2, -2) = (-6, 1, 3)$
Thus, the intersection point is $(-6, 1, 3)$.
b. A point on the xy plane has the form $(x, y, 0)$. If such a point is $(-6, 1, 0)$ then the distance from this point is $d = \sqrt{0 + 0 + 3^2} = 3$.
15. a. Comparing the x and y components of each vector equation, we get the system of equations:
$-1 + 5s = 4 + 0t$
$3 - 2s = -1 + 2t$
Note that from the first equation, $5 = 5s$ or $s = 1$. So the second equation becomes $3 - 2 = -1 + 2t$ or $t = 1$. Placing these two values back into the vector equations to find the intersection point A, we get
$(-1, 3, 2) + (5, -2, 10) = (4, 1, 12)$
$(4, -1, 1) + (0, 2, 11) = (4, 1, 12)$
Thus, the intersection point is $(4, 1, 12)$.
b. We need to find a vector (a, b, c) such that
$5a - 2b + 10c = 0$
$2b + 11c = 0$
A possible solution to the second equation is $(a, 11, -2)$. If we substitute this into the first equation, we get $5a - 22 - 20 = 0 \rightarrow 5a = 42$.

We can use this to get a solution of $\left(\frac{42}{5}, 11, -2\right)$. To eliminate the fraction, we get $(42, 55, -10)$. So the vector equation is $\vec{r} = (4, 1, 12) + t(42, 55, -10)$, $t \in \mathbf{R}$.

16. a.



b. The only point of intersection is at the origin $(0, 0, 0)$.

c. If $p = 0$ and $q = 0$, the intersection occurs at $(0, 0, 0)$.

17. a. Represent the lines parametrically, and then substitute into the equation for the plane.
For the first equation, $x = t$, $y = 7 - 8t$, $z = 1 + 2t$. Substituting into the plane equation, $2t + 7 - 8t + 3 + 6t - 10 = 0$. Simplifying, $0t = 0$. So the line lies on the plane.
For the second line, $x = 4 + 3s$, $y = -1$, $z = 1 - 2s$ Substituting into the plane equation, $8 + 6s - 1 + 3 - 6s - 10 = 0$. Simplifying, $0s = 0$. This line also lies on the plane.

b. Compare the x and y components:
$4 + 3s = t$
$7 - 8t = -1$
From the second equation, $t = 1$. Substituting back into the first equation, $4 + 3s = 1$, or $s = -1$.
Determine the point of intersection:
$(1, 7 - 8, 1 + 2) = (1, -1, 3)$
$(4 - 3, -1, 1 + 2) = (1, -1, 3)$
The point of intersection is $(1, -1, 3)$.

18. Answers may vary. For example:
$\vec{r} = (2, 0, 0) + p(2, 0, 1), p \in \mathbf{R}$

9.2 Systems of Equations, pp. 507–509

1. a. linear
b. not linear
c. linear
d. not linear

2. Answers may vary. For example:
$x + y + 2z = -15$
a. $x + 2y + z = -3$
$2x + y + z = -10$

b. Subtract the first equation from the second, and subtract twice the first equation from the third.
$x + y + 2z = -15$
$0x - y + z = -12$
$0x - y - 3z = 20$
Add the second and third equation.
$x + y + 2z = -15$
$0x - y + z = -12$
$0x + 0y - 4z = 32$
From the third equation, $z = -8$.
Substitute z back into the second equation,
$-y - 8 = -12$
$-y = -12 + 8 = -4$
So $y = 4$. Now substitute y and z back into the first equation.
$x + 4 + 2(-8) = x - 12 = -15$
And so $x = -3$. Thus the solution is $(-3, 4, -8)$ as expected.

3. a. $-7 - 3(5) + 4\left(\frac{3}{4}\right) = -7 - 15 + 3 = -19$
$-7 - 8\left(\frac{3}{4}\right) = -7 - 6 = -13$
$-7 + 2(5) = 3$
Yes, $\left(-7, 5, \frac{3}{4}\right)$ is a solution.

b.
$3(-7) - 2(5) + 16\left(\frac{3}{4}\right) = -21 - 10 + 12 = -19$
$3(-7) - 2(5) = -21 - 10 = -31$
$\neq -23$
$8(-7) - 5 + 4\left(\frac{3}{4}\right) = -56 - 5 + 3 = -58$
Because the second equation fails to produce an equality, $\left(-7, 5, \frac{3}{4}\right)$ is not a solution.

4. a. Solve for y. $y = -3$
The solution is $(-2, -3)$.

b. Multiply the second equation by 6
$3x + 5y = -21$
$x - 3y = 7$
Add 3 times the first equation to 5 times the second equation.
$3x + 5y = -21$
$14x = -28$
From the second equation, $x = -2$.
Substituting x back into the first equation,
$3(-2) + 5y = -21$
$5y = -15$
So $y = -3$.
The two systems are equivalent because they have the same solution.

5. a. Add the second equation to 5 times the first equation.

$2x - y = 11$

$11x = 66$

Solve for x in the second equation, $x = 6$. Substitute x back into the first equation

$2(6) - y = 11$

$-y = 11 - 12 = -1$

So $y = 1$

Therefore, the solution is $(6, 1)$.

b. Subtract three times the first equation from twice the second equation.

$2x + 5y = 19$

$0x - 7y = -35$

From the second equation, $y = 5$.

Substitute y back into the first equation.

$2x + 5(5) = 19$

$2x = 19 - 25 = -6$

So $x = -3$

Therefore, the solution is $(-3, 5)$.

c. Add the second equation to 3 times the first equation to the second equation

$-x + 2y = 10$

$0x + 11y = 33$

From the second equation, $y = 3$.

Substitute y back into the first equation.

$-x + 2(3) = 10$

$-x = 4$

So $x = -4$.

Therefore the solution is $(-4, 3)$.

6. a. These two lines are parallel, and therefore cannot have an intersection.

b. The second equation is five times the first, therefore the lines are coincident.

7. a. Let $x = t$. So $2t - y = 3$ then $y = 2t - 3$.

b. Let $x = t$, $y = s$. Then $t - 2s + z = 0$ and $z = 2s + t$.

8. a. If $x = t$, $y = -2t - 11$, then $y = -2x - 11$ and so $2x + y = -11$ is the required linear equation.

b. $2x + y = -11$

$2(3t + 3) + (-6t - 17) = 6t - 6t + 6 - 17$

$= -11$

9. a. The two equations will have no solutions when $k \neq 12$, since they will be parallel should this occur.

b. It is impossible to have only one solution for these two equations. They have exactly the same direction vector. They will never intersect at exactly one place.

c. The two equations will have infinitely many solutions when $k = 12$. When this occurs, the two equations are coincident.

10. a. There are infinitely many solutions to this equation. This is reason why it is represented graphically as a line.

b. Let $x = t$. So $2t + 4y = 11$, then $4y = 11 - 2t$ and $y = \frac{11}{4} - \frac{1}{2}t$, $t \in \mathbf{R}$

c. This equation will not have any integer solutions because the left hand side is an even function and the right side is an odd function.

11. a. Add the second equation to -2 times the first.

$x + 3y = a$

$0x - 3y = b - 2a$

Divide the second equation by -3 to get $y = -\frac{1}{3}b + \frac{2}{3}a$. Now substitute this back into the first equation.

$$x + 3\left(-\frac{1}{3}b + \frac{2}{3}a\right) = a$$

$$x - b + 2a = a$$

$$x = -a + b$$

b. Since they have different direction vectors, these two equations are not parallel or coincident and will intersect somewhere.

12. a. Add the third equation to the first to eliminate z.

$x + y + z = 0$

$x - y + 0z = 1$

$x + 2y + 0z = -5$

Add twice the second equation to the third equation to eliminate

Add twice the second equation to the third equation to eliminate y.

$x + y + z = 0$

$x - y + 0z = 1$

$3x + 0y + 0z = -3$

Divide the third equation by -3 to get $x = -1$.

Now substitute into the second equation.

$-1 - y = 1$

$y = -2$

Finally, substitute x and y to get

$-1 + -2 + z = 0$

So $z = 3$. Therefore, the solution is $(-1, -2, 3)$.

b. Add the first equation to -2 times the second, and add the first equation to -2 times the third.

$2x - 3y + z = 6$

$0x - 5y - 3z = -56$

$0x - y + 3z = 40$

Now add the second equation to -1 times the third.

$2x - 3y + z = 6$

$0x - 5y - 3z = -56$

$0x - 4y + 0z = -16$

From the third equation, $y = 4$.
Now substitute this into the second equation.

$$-5(4) - 3z = -56$$
$$-3z = -36$$
$$z = 12$$

Now substitute these two values back into the first equation.

$$2x - 3(4) + 12 = 6$$
$$2x = 6, x = 3$$

So the solution is $(3, 4, 12)$.

c. Add the second equation to -1 times the third.

$$x + y + 0z = 10$$
$$0x + y + z = -2$$
$$-x + y + 0z = 2$$

Add the third equation to the first equation.

$$x + y + 0z = 10$$
$$0x + y + z = -2$$
$$0x + 2y + 0z = 12$$

So $y = 6$. Now substitute into the other two equations.

$$x + 6 = 10 \rightarrow x = 4$$
$$6 + z = -2 \rightarrow z = -8$$

So the solution is $(4, 6, -8)$.

d. To eliminate fractions, multiply each of the equations by 60.

$$20x + 15y + 12z = 840$$
$$15x + 12y + 20z = -1260$$
$$12x + 20y + 15z = 420$$

Add 3 times the first equation to -4 times the second, and add 3 times the first equation to -5 times the third.

$$20x + 15y + 12z = 840$$
$$0x - 3y - 44z = 7560$$
$$0x - 55y - 39z = 420$$

Now add 55 times the second equation to -3 times the third equation.

$$20x + 15y + 12z = 840$$
$$0x - 3y - 44z = 7560$$
$$0x + 0y - 2303z = 414\,540$$

Divide the third equation through by -2303 to get $z = -180$. Substituting z back into the second equation.

$$-3y - 44(-180) = 7560 \rightarrow -3y = -360$$

So $y = 120$. Now substitute these two values back into the first equation.

$$20x + 15(120) + 12(-180) = 840$$
$$20x = 840 - 1800 + 2160 = 1200$$

So $x = 60$. Therefore the solution is $(60, 120, -180)$.

e. Note that if $2x - y = 0 \rightarrow y = 2x$, and $2z - x = 0 \rightarrow z = \frac{1}{2}x$. So we substitute these two relations into the second equation.

$$2(2x) - \frac{1}{2}x = \frac{7}{2}x = 7 \rightarrow x = 2$$

So now $z = 1$, $y = 4$, and the solution is $(2, 4, 1)$.

f. Add the first equation to -2 times the second equation.

$$x + y + 2z = 13$$
$$-2x + 0y - 7z = -38$$
$$2x + 0y + 6z = 32$$

Add the second and third equations.

$$x + y + 2z = 13$$
$$-2x + 0y - 7z = -38$$
$$0x + 0y - z = -6$$

So from the third equation, $z = 6$.
Substituting into the second equation,

$$-2x - 42 = -38$$
$$-2x = 4 \rightarrow x = -2$$

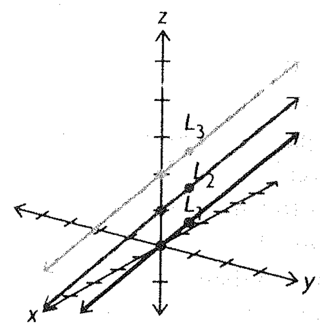
Finally, substituting both values into the first equation.

$$-2 + y + 12 = 13 \rightarrow y = 3.$$

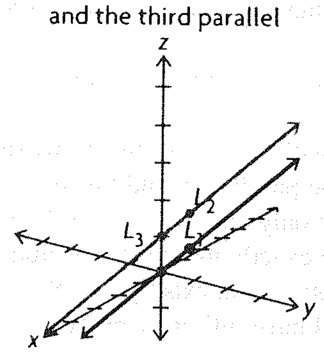
So the final solution is $(-2, 3, 6)$.

13. Answers may vary. For example:

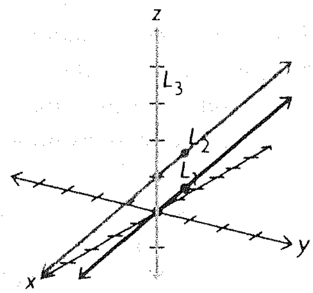
a. Three lines parallel



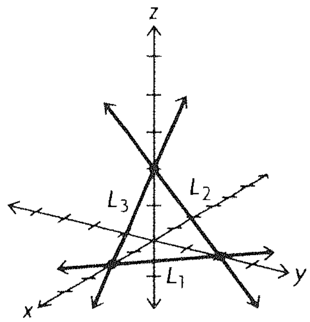
Two lines coincident and the third parallel



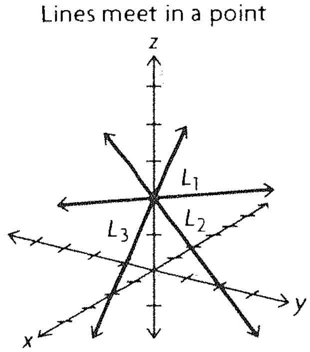
Two parallel lines cut by the third line



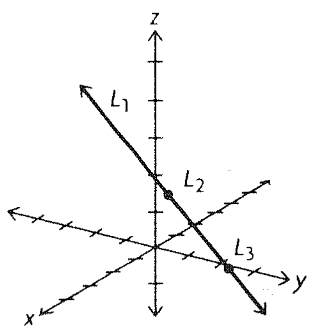
The lines form a triangle



b. Lines meet in a point



c. Three coincident lines



14. a. Add -1 times the first equation and the second equation. Add -1 times the first equation and the third equation.

$$x + y + z = a$$
$$0x + 0y - z = b - a$$
$$-x + 0y + 0z = c - a$$

So $z = a - b$, $x = a - c$. Then substitute into the first equation.

$$a - c + y + a - b = a$$
$$y = -a + b + c$$

So the final solution is $(a - c, -a + b + c, a - b)$.

15. a. For two equations to have no solutions, they must be parallel—meaning it must have a the same direction vector. So if $k = 2$, then the lines are parallel.

b. If two equations have an infinite number of solutions, then the lines must be coincident. One way to do this is if the second equation is a multiple of the first equation. To achieve this, $k = -2$.

c. For two equations to have a unique solution, they must have only one intersection. From a., we saw that if $k = 2$, the lines are parallel, and if $k = -2$, then they are coincident. Since the only other option is for the lines to have a unique solution, $k \neq \pm 2$.

9.3 The Intersection of Two Planes, pp. 516–517

1. a. This means that the two equations represent planes that are parallel and not coincident.

b. Answers may vary. For example: $x - y + z = 1$, $x - y + z = -2$

2. a. The solution to the system of equations is: $x = \frac{1}{2} + \frac{1}{2}s - t$, $y = s$, $z = t$, $s, t \in \mathbf{R}$. The two planes are coincident.

b. Answers may vary. For example:
$x - y + z \Rightarrow -1$; $2x - 2y + 2z = -2$

3. a. $2z = -4 \Rightarrow z = -2$.

$$x - y + (-2) \Rightarrow -1$$
$$x - y \Rightarrow 1.$$

$x = 1 + s$, $y = s$, $z = -2$, $s \in \mathbf{R}$

The two planes intersect in a line.

b. Answers may vary. For example:
$x - y + z = -1$; $x - y - z = 3$.

4. a. ① $2x + y + 6z = p$; ② $x + my + 3z = q$

For the planes to be coincident equation ② must be a multiple of equation ①. Since the coefficients of x and z in equation ① are twice that of the x and z coefficients in equation ② all of the coefficients and constants in equation ② must be half of the corresponding coefficients in equation ①. So:

$m = \frac{1}{2}$, $p = 2q$, $q = 1$, and $p = 2$.

The value for m is unique, but p just has to be twice q and arbitrary values can be chosen.

b. For parallel planes all of the coefficients of the variables must be multiples of each other, but the constant terms must differ by a different constant. So a possible solution is:

$m = \frac{1}{2}$, $q = 1$, and $p = 3$.

The value for m is again unique but p and q can be arbitrarily chosen as long as $p \neq 2q$.

c. For the two planes to intersect at right angles the two normal vectors, $\vec{n_1} = (2, 1, 6)$ and $\vec{n_2} = (1, m, 3)$, must satisfy:

$\vec{n_1} \cdot \vec{n_2} = 0.$

$\vec{n_1} \cdot \vec{n_2} = 2 + m + 18 = 0$

$m = -20$. This value is unique, since only one value was found to satisfy the given conditions.

d. From **c.** we know that in order to intersect in right angles $m = -20$. Choose $p = 1$, $q = 1$.

The value for m is unique from the solution to **c.**, but the values for p and q can be arbitrary since the only value which can change the angle between the planes is m.

5. a. Letting $z = s$:

$$y = -3s.$$
$$x + 2(-3s) - 3s = 0.$$
$$x = 9s$$

The solution is:

$x = 9s$, $y = -3s$, $z = s$, $s \in \mathbf{R}$

b. Letting $y = t$.

$$t + 3z = 0$$
$$3z = -t$$
$$z = -\frac{1}{3}t.$$
$$x + 2t - 3\left(-\frac{1}{3}\right)t = 0$$
$$x + 3t = 0$$
$$x = -3t.$$

The solution is:

$x = -3t$, $y = t$, $z = -\frac{1}{3}t$; $t \in \mathbf{R}$.

c. Since t is an arbitrary real number we can express t as:

$t = -3s$; $s \in \mathbf{R}$.

Substituting this into the solution for **b.** shows that the two solutions are equivalent.

6. a. Equation ② is twice that of equation ①, so they represent intersecting coincident planes.

b. The coefficients of each variable are the same, but the constant terms are different, so the equations represent non-intersecting parallel planes.

c. The coefficients of the x and z variables are the same but the y coefficients are different. So the equations represent planes that intersect in a line.

d. The coefficients of each variable from equation ① to ② are not the same multiple. Therefore the equations represent planes that intersect in a line.

e. The intersection is a line by the same reasoning as **d.**

f. The intersection is a line by the same reasoning as **d.**

7. a. $x = 1 - s - t$, $y = s$, $z = t$, $s, t \in \mathbf{R}$

b. There is no solution since the planes are parallel.

c. ① − ②:

$$-2y = 4$$
$$y = -2.$$
$$x - 2 + 2z = -2$$
$$x + 2z = 0$$
$$x = -2z.$$

$x = -2s$, $y = -2$, $z = s$, $s \in \mathbf{R}$.

d. Let $z = s$; $s \in \mathbf{R}$.

From ②:

$$x = y + 6.$$
$$(y + 6) + y + 2s = 4$$
$$2y + 2s = -2$$
$$y = -s - 1.$$

$x = -s + 5$, $y = -s - 1$, $z = s$, $s \in \mathbf{R}$.

e. $-2 \cdot$ ②: $2x - 4y - 2z = -2$

Adding ①:

$4x - 5y = 0.$

$$x = \frac{5}{4}y.$$

Let $y = s$, $s \in \mathbf{R}$.

$$2\left(\frac{5}{4}s\right) - s + 2z = 2$$
$$\frac{3}{2}s + 2z = 2$$
$$z = 1 - \frac{3}{4}s.$$

$x = \frac{5}{4}s$, $y = s$, $z = 1 - \frac{3}{4}s$, $s \in \mathbf{R}$

f. $x - y + 2(4) = 0$

$$x = y - 8.$$

$x = s - 8$, $y = s$, $z = 4$, $s \in \mathbf{R}$.

8. a. The system will have an infinite number of solutions for any value of k. When $k = 2$ equation ② will be twice that of ① so the solution is a plane:

$x = 1 - s - 2t$, $y = s$, $z = t$, $s, t \in \mathbf{R}$.

For any other value of k the solution will be a line. For example $k = 0$:

$$2y = -4z$$
$$y = -2z.$$

$x + (-2z) + 2z = 1$

$x = 1.$

$x = 1, y = -2s, z = s, s \in \mathbf{R}.$

b. No there is no value of k for which the system will not have a solution. The only time when there is no solution is when the corresponding coefficients for each variable differ by a common multiple between equations, and the constant terms differ by a different multiple. The only way the first condition is satisfied is when $k = 2$, but when this happens the constant terms differ by the same factor as the variables, namely 2.

9. The line of intersection of the two planes:
π_1: $2x - y + z = 0$, π_2: $y + 4z = 0$ is:

$$y = -4z$$
$$2x - (-4z) + z = 0$$
$$2x = -5z$$
$$x = -\frac{5}{2}z.$$

$x = -\frac{5}{2}s, y = -4s, z = s, s \in \mathbf{R}.$

The direction vector is $\left(-\frac{5}{2}, -4, 1\right)$ or $(-5, -8, 2)$.

$\vec{r_1} = s(-5, -8, 2), s \in \mathbf{R}$. Since the line we are looking for is parallel to this line, we know that the direction vector must be the same. The line passes through $(-2, 3, 6)$ and has direction vector $(-5, -8, 2)$. The equation of the line is
$\vec{r_2} = (-2, 3, 6) + s(-5, -8, 2), s \in \mathbf{R}.$

10. The line of intersection of the two planes, $2x - y + 2z = 0$ and $2x + y + 6z = 4$ is:

$$4x + 8z = 4$$
$$x = 1 - 2z.$$
$$2(1 - 2z) - y + 2z = 0$$
$$2 - y - 2z = 0$$
$$y = 2 - 2z.$$

$x = 1 - 2s, y = 2 - 2s, z = s, s \in \mathbf{R}.$

In order for the a line to be contained in the plane we need to check that the values for x, y, and z always satisfy the plane equation:

$$5x + 3y + 16z - 11 = 0.$$
$$5(1 - 2s) + 3(2 - 2s) + 16(s) - 11 = 0$$
$$5 + 6 - 11 - 10s - 6s + 16s = 0$$

$0 = 0$. Since this is true the line is contained in the plane.

11. a. π_1: $2x + y - 3z = 3$, π_2: $x - 2y + z = -1$.

$\pi_1 - 2\pi_2$: $5y - 5z = 5$

$$y = 1 + z.$$
$$2x + (1 + z) - 3z = 3$$
$$2x - 2z = 2$$
$$x = 1 + z.$$

$x = 1 + s, y = 1 + s, z = s, s \in \mathbf{R}.$

b. L meets the xy-plane when $z = 0$.

$x = 1, y = 1$. $A = (1, 1, 0)$.

L meets the z-axis when both x and y are zero:

$s = -1.$

$z = -1.$

$B = (0, 0, -1)$

The length of AB is therefore:

$\sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$ or about 1.73.

12. The line with equation $x = -2y = 3z$ has parametric equations: $x = s, y = -\frac{1}{2}s, z = \frac{1}{3}s, s \in \mathbf{R}$.

This has the equivalent vector form:

$$\vec{r} = s\left(1, -\frac{1}{2}, \frac{1}{3}\right), s \in \mathbf{R}.$$

The line of intersection of the two planes $x - y + z = 1$ and $2y - z = 0$ is:

$$y = \frac{1}{2}z$$
$$x - \frac{1}{2}z + z = 1$$
$$x = 1 - \frac{1}{2}z.$$

$x = 1 - \frac{1}{2}t, y = \frac{1}{2}t, z = t, t \in \mathbf{R}$. Which has a vector equation of:

$\vec{r} = (1, 0, 0) + t\left(-\frac{1}{2}, \frac{1}{2}, 1\right), t \in \mathbf{R}$. The vector equation of the plane with the given properties is thus:

$$\vec{r} = (1, 0, 0) + t\left(-\frac{1}{2}, \frac{1}{2}, 1\right) + s\left(1, -\frac{1}{2}, \frac{1}{3}\right), s, t \in \mathbf{R}.$$

The normal vector for the plane is then:

$$\left(-\frac{1}{2}, \frac{1}{2}, 1\right) \times \left(1, -\frac{1}{2}, \frac{1}{3}\right) = \left(\frac{1}{2} \cdot \frac{1}{3}\right) - \left(1 \cdot -\frac{1}{2}\right), 1 \cdot 1 - \left(-\frac{1}{2} \cdot \frac{1}{3}\right), -\frac{1}{2}\left(-\frac{1}{2}\right) - \frac{1}{2} \cdot 1 = \left(\frac{2}{3}, \frac{7}{6}, -\frac{1}{4}\right).$$

Or equivalently $(8, 14, -3)$.

The Cartesian equation is then:

$8x + 14y - 3z + D = 0$, and must contain the point $(1, 0, 0)$.

$$8(1) + D = 0.$$
$$D = -8.$$

$8x + 14y - 3z - 8 = 0.$

Mid-Chapter Review, pp. 518–519

1. a. $\vec{r} = (4, -3, 15) + t(2, -3, 5), t \in \mathbf{R}$

$x = 4 + 2t, y = -3 - 3t, z = 15 + 5t$

$0 = 15 + 5t$

$t = -3$

$x = 4 + 2(-3), y = -3 - 3(-3),$
$z = 15 + 5(-3)$
$x = -2, y = 6, z = 0$
$(-2, 6, 0)$
b. $\vec{r} = (4, -3, 15) + t(2, -3, 5), t \in \mathbf{R}$
$x = 4 + 2t, y = -3 - 3t, z = 15 + 5t$
$0 = -3 - 3t$
$t = -1$
$x = 4 + 2(-1), y = -3 - 3(-1),$
$z = 15 + 5(-1)$
$x = 2, y = 0, z = 10$
$(2, 0, 10)$
c. $\vec{r} = (4, -3, 15) + t(2, -3, 5), t \in \mathbf{R}$
$x = 4 + 2t, y = -3 - 3t, z = 15 + 5t$
$0 = 4 + 2t$
$t = -2$
$x = 4 + 2(-2), y = -3 - 3(-2),$
$z = 15 + 5(-2)$
$x = 0, y = 3, z = 5$
$(0, 3, 5)$
2. a.–e. Answers may vary. For example:
$A(2, 1, 3), B(3, -2, 5), C(-8, -5, 7)$
$a = (-2.5, -3.5, 6)$
$b = (-3, -2, 5)$
$c = (2.5, -0.5, 4)$
$m_1 = (Aa) = (-4.5, -4.5, 3) = (3, 3, -2)$
$m_2 = (Bb) = (-6, 0, 0) = (1, 0, 0)$
$m_3 = (Cc) = (10.5, 4.5, -3) = (7, 3, -2)$
Then substitute in the point and the direction vector to find the equation of the line.
$A(2, 1, 3), B(3, -2, 5), C(-8, -5, 7)$
$m_1 = (Aa) = (-4.5, -4.5, 3) = (3, 3, -2)$
$m_2 = (Bb) = (-6, 0, 0) = (1, 0, 0)$
$m_3 = (Cc) = (10.5, 4.5, -3) = (7, 3, -2)$
$A: \vec{r} = (2, 1, 3) + t(3, 3, -2), t \in \mathbf{R}$
$x = 2 + 3t, y = 1 + 3t, z = 3 - 2t, t \in \mathbf{R}$
$B: \vec{r} = (3, -2, 5) + t(1, 0, 0), t \in \mathbf{R}$
$x = 3 + t, y = -2, z = 5, t \in \mathbf{R}$
$C: \vec{r} = (-8, -5, 7) + t(7, 3, -2), t \in \mathbf{R}$
$x = -8 + 7t, y = -5 + 3t, z = 7 - 2t, t \in \mathbf{R}$
$A: x = 2 + 3t, y = 1 + 3t, z = 3 - 2t, t \in \mathbf{R}$
$B: x = 3 + t, y = -2, z = 5, t \in \mathbf{R}$
$C: x = -8 + 7t, y = -5 + 3t, z = 7 - 2t, t \in \mathbf{R}$
$y = -2 = 1 + 3t$
$t = -1$
$x = 2 + 3(-1), y = 1 + 3(-1),$
$z = 3 - 2(-1)$
$x = -1, y = -2, z = 5$
$(-1, -2, 5)$
$A: x = 2 + 3t, y = 1 + 3t, z = 3 - 2t, t \in \mathbf{R}$
$B: x = 3 + t, y = -2, z = 5, t \in \mathbf{R}$
$C: x = -8 + 7t, y = -5 + 3t, z = 7 - 2t, t \in \mathbf{R}$
$y = -2 = -5 + 3t$
$t = 1$
$x = -8 + 7(1), y = -5 + 3(1), z = 7 - 2(1)$
$x = -1, y = -2, z = 5$
$(-1, -2, 5)$
The three medians meet at $(-1, -2, 5)$.
3. a. $L_1: 5x + y + 2z + 15 = 0$
$L_2: 4x + y + 2z + 8 = 0$
$L_1 - L_2: x + 7 = 0$
So $x = -7$.
$L_1: y + 2z - 20 = 0$
$L_2: y + 2z - 20 = 0$
$z = t,$
$y + 2(t) - 20 = 0$
$y = 20 - 2t$
$\vec{r} = (-7, 20, 0) + t(0, -2, 1), t \in \mathbf{R}$
b. $L_1: 4x + 3y + 3z - 2 = 0$
$L_2: 5x + 2y + 3z + 5 = 0$
$2L_1 - 3L_2: -7x - 3z - 19 = 0$
$z = 7t,$
$-7x - 3(7t) - 19 = 0,$
$$x = -3t - \frac{19}{7}$$
$$4\left(-3t - \frac{19}{7}\right) + 3y + 3(7t) - 2 = 0$$
$$y = -3t + \frac{30}{7}$$
$$\vec{r} = \left(-\frac{19}{7}, \frac{30}{7}, 0\right) + t(3, 3, -7), t \in \mathbf{R}$$
c. $L_1: \vec{r} = (-7, 20, 0) + t(0, -2, 1), t \in \mathbf{R}$
$$L_2: \vec{r} = \left(-\frac{19}{7}, \frac{30}{7}, 0\right) + t(3, 3, -7), t \in \mathbf{R}$$
$L_1: x = -7, y = 20 - 2t, z = t$
$$L_2: x = -\frac{19}{7} + 3t, y = \frac{30}{7} + 3t, z = -7t$$
$$-\frac{19}{7} + 3t = -7, t = -\frac{30}{21}$$
$$x = -\frac{19}{7} + 3\left(-\frac{30}{21}\right), y = \frac{30}{7} + 3\left(-\frac{30}{21}\right),$$
$$z = -7\left(-\frac{30}{21}\right)$$
$x = -7, y = 0, z = 10$
$(-7, 0, 10)$
4. a. $\pi_1: 3x + y + 7z + 3 = 0$
$\pi_2: x - 13y - 3z - 38 = 0$

$13\pi_1 + \pi_2: 40x + 88z + 1 = 0$

$z = t,$

$40x + 88(t) + 1 = 0$

$x = -\frac{11t}{5} - \frac{1}{40}$

$3\left(-\frac{11t}{5} - \frac{1}{40}\right) + y + 7(t) + 3 = 0$

$y = -\frac{2t}{5} - \frac{117}{40}$

$x = -\frac{11t}{5} - \frac{1}{40}, y = -\frac{2t}{5} - \frac{117}{40}, z = t, t \in \mathbf{R}$

b. $\pi_1: x - 3y + z + 11 = 0$

$\pi_2: 6x - 13y + 8z - 28 = 0$

$-6\pi_1 + \pi_2: 5y + 2z - 94 = 0$

$z = s.$

$5y + 2(s) - 94 = 0$

$y = -\frac{2}{5}s + \frac{94}{5}$

$x - 3\left(-\frac{2}{5}s + \frac{94}{5}\right) + (s) + 11 = 0$

$x = -\frac{11}{5}s + \frac{227}{5}$

$x = -\frac{11}{5}s + \frac{227}{5}, y = -\frac{2}{5}s + \frac{94}{5}, z = s, s \in \mathbf{R}$

c. The lines found in 4. a. and 4. b. do not intersect, because they are in parallel planes.

5. a. For there to be no solution the lines must be inconsistent with each other.

$L_1: x + ay = 9$

$L_2: ax + 9y = -27$

$\frac{1}{a} = \frac{a}{9}$

$a = \pm 3$

For $a = 3$:

$L_1: x + 3y = 9$

$L_2: 3x + 9y = -27$

For $a = -3$, the equations are equivalent.

So there is no solution when $a = 3$.

b. To have an infinite number of solutions, the lines must be proportional.

$L_1: x + ay = 9$

$L_2: ax + 9y = -27$

$-3(x + ay = 9) = -3x - 3ay = -27$

$L_1: -3x - 3ay = -27$

$L_2: ax + 9y = -27$

$a = -3$

c. The system has one solution when $a \neq 3$ or $a \neq -3$, because other values lead to an infinite number of solutions or no solution.

6. $L_1: \frac{x - 11}{2} = \frac{y - 4}{-4} = \frac{z - 27}{5} = s$

$L_2: x = 0, y = 1 - 3t, z = 3 + 2t, t \in \mathbf{R}$

$L_1: x = 2s + 11, y = -4s + 4, z = 27 + 5s$

$x = 0 = 2s + 11,$

$s = -5.5$

$y = -4(-5.5) + 4, z = 27 + 5(-5.5)$

$x = 0, y = 26, z = -0.5$

$y = 26 = 1 - 3t, t = -\frac{25}{3}$

$z = -0.5 = 3 + 2t, t = -\frac{7}{4}$

Since there is no t-value that satisfies the equations, there is no intersection, and these lines are skew.

7. a. $L_1: \frac{x - 5}{2} = y - 2 = \frac{z + 4}{-3} = s$

$L_2: (x - 3, y - 20, z - 7) = t(2, -4, 5), t \in \mathbf{R}$

$L_1: x = 2s + 5, y = s + 2, z = -3s - 4$

$L_2: x = 2t + 3, y = -4t + 20, z = 5t + 7$

$x = 2t + 3 = 2s + 5$

$y = s + 2 = -4t + 20$

$z = -3s - 4 = 5t + 7$

$L_3: 2t - 2s - 2 = 0$

$L_4: 4t + s - 18 = 0$

$L_5: 5t + 3s + 11 = 0$

$L_3 + 2L_4: 10t - 38 = 0, t = 3.8$

$3L_3 + 2L_5: 16t + 16 = 0, t = -1$

b. Since there is no t-value that satisfies the equations, there is no intersection, and these lines are skew.

8. $L_1: x = 1 + 2s, y = 4 - s, z = -3s, s \in \mathbf{R}$

$L_2: x = -3, y = t + 3, z = 2t, t \in \mathbf{R}$

$x = -3 = 1 + 2s$

$s = -2$

$x = -3, y = 6, z = 6$

$(-3, 6, 6)$

9. a. $L_1: \vec{r} = (5, 1, 7) + s(2, 0, 5), s \in \mathbf{R}$

$L_2: \vec{r} = (-1, -1, 3) + t(4, 2, -1), t \in \mathbf{R}$

$L_1: x = 5 + 2s, y = 1, z = 7 + 5s$

$L_2: x = -1 + 4t, y = -1 + 2t, z = 3 - t$

$y = 1 = -1 + 2t,$

$t = 1$

$x = -1 + 4(1), y = -1 + 2(1),$

$z = 3 - (1)$

$x = 3, y = 1, z = 2$

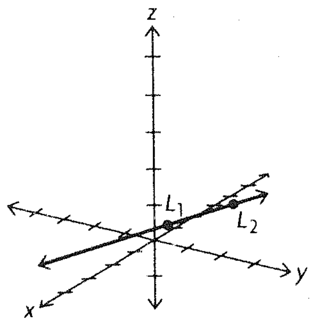
$(3, 1, 2)$

b. $L_1: \vec{r} = (2, -1, 3) + s(5, -1, 6), s \in \mathbf{R}$

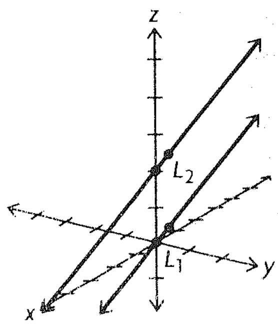
$L_2: \vec{r} = (-8, 1, -9) + t(5, -1, 6), t \in \mathbf{R}$

These lines are the same, so either one of these lines can be used as their intersection.

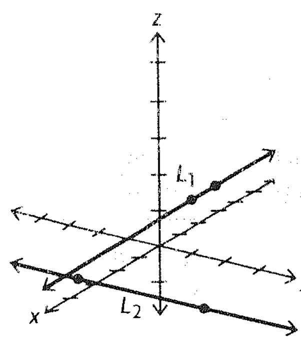
10. a. Answers may vary. For example:
i. coincident



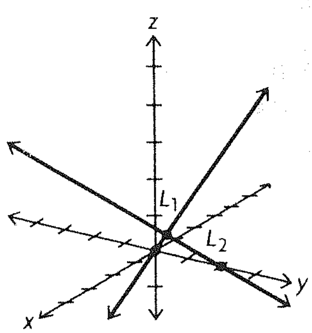
ii. parallel and distinct



iii. skew



iv. intersect in a point



b. i. When lines are the same, they are a multiple of each other.
ii. When lines are parallel, one equation is a multiple of the other equation, except for the constant term.
iii. When lines are skew, there are no common solutions to make each equation consistent.
iv. When the solution meets in a point, there is only one unique solution for the system.

11. a. A line and plane have an infinite number of points of intersection when the line lies in the plane.
b. Answers may vary. For example:
$\vec{r} = t(3, -5, -3), t \in \mathbf{R}$
$\vec{r} = t(3, -5, -3) + s(1, 1, 1), t, s \in \mathbf{R}$

12. a. ① $2x + 3y = 30$
② $x - 2y = -13$
Equation ① − (2 × equation ②): $7y = 56$
$y = 8$
$2x + 24 = 30$
$x = 3$
$(3, 8)$

b. ① $x + 4y - 3z + 6 = 0$
② $2x + 8y - 6z + 11 = 0$
There is no solution to this system, because the planes are parallel, but one plane lies above the other.

c. ① $x - 3y - 2z = -9$
② $2x - 5y + z = 3$
③ $-3x + 6y + 2z = 8$
Equation ① + (2 × equation ②): $5x - 13y = -3$
Equation ② + (equation ③): $-2x + 3y = -1$

$$\begin{array}{r} 2(5x - 13y = -3) \\ +\ 5(-2x + 3y = -1) \\ \hline -11y = -11 \end{array}$$

$y = 1$
$5x - 13(1) = -3$
$x = 2$
$(2) - 3(1) - 2z = -9$
$z = 4$
$(2, 1, 4)$

13. a. The two lines intersect at a point.
b. The two planes are parallel and do not meet.
c. The three planes intersect at a point.

14. a. $L{:}(x - y = 1) + (y + z = -3)$
$= x + z = -2$

$L_1{:}\ y - z = 0, x = -\frac{1}{2}$

$x + z = -2$

$\left(-\frac{1}{2}\right) + z = -2$

$z = -\frac{3}{2}$

$y - z = 0$

$$y - \left(-\frac{3}{2}\right) = 0$$
$$y = -\frac{3}{2}$$
$$\left(-\frac{1}{2}, -\frac{3}{2}, -\frac{3}{2}\right)$$

b. $\cos\theta = \frac{|n \cdot n_1|}{|n||n_1|}$
$n = (1, 1, -1)$
$n_1 = (0, 1, 1)$
$\cos\theta = \frac{0}{|\sqrt{3}||\sqrt{2}|}$
$\theta = 90°$

c. $(0, 1, 1) \times (1, 1, -1) = (-2, 1, -1)$
$= (2, -1, 1)$
$Ax + By + Cz + D = 0$
$2x - y + z + D = 0$
$$2\left(\frac{-1}{2}\right) - \left(\frac{-3}{2}\right) + \left(\frac{-3}{2}\right) + D = 0$$
$D = 1$
$2x - y + z + 1 = 0$

9.4 The Intersection of Three Planes, pp. 531–533

1. a. ① $x - 3y + z = 2$
② $0x + y - z = -1$
③ $0x + 0y + 3z = -12$
The system can be solved by first solving equation ③ for z. Thus,
$3z = -12$
$z = -4$
If we use the method of back substitution, we can substitute $z = -4$ into equation ② and solve for y.
$y - (-4) = -1$
$y = -5$
If we substitute $y = -5$ and $z = -4$ into equation ① we obtain the value of x.
$x - 3(-5) - 4 = 2$ or $x = -9$
The three planes intersect at the point with coordinates $(-9, -5, -4)$
Check:
Substituting into equation ①:
$x - 3y + z = -9 + 15 - 4 = 2$
Substituting into equation ②:
$0x + y - z = -5 + 4 = -1$
Substituting into equation ③: $0x + 0y + 3z = -12$

b. This solution is the point at which all three planes meet.

2. a. ① $x - y + z = 4$
② $0x + 0y + 0z = 0$
③ $0x + 0y + 0z = 0$
The answer may vary depending upon the constant you multiply the equations by. For example,
$2 \times (x - y + z = 4) = 2x - 2y + 2z = 8$
$3 \times (x - y + z = 4) = 3x - 3y + 3z = 12$
$3x - 3y + 3z = 12$ and $2x - 2y + 2z = 8$ are equations that could work.

b. These three planes are intersecting in one single plane, because all three equations can be changed into one equivalent equation. They are coincident planes.

c. Setting $x = t$ and $y = s$ leads to
$t - s + z = 4$ or $z = s - t + 4$, $s, t \in \mathbf{R}$

d. Setting $y = t$ and $z = s$ leads to
$x - t + s = 4$ or $x = t - s + 4$, $s, t \in \mathbf{R}$

3. a. ① $2x - y + 3z = -2$
② $x - y + 4z = 3$
③ $0x + 0y + 0z = 1$
The answer may vary depending upon the constants and equations you use to determine your answer. For example,
Equation ① + equation ② + equation ③ =
$(2x - y + 3z = -2)$
$+ (x - y + 4z = 3)$
$+ (0x + 0y + 0z = 1)$
$3x - 2y + 7z = 2$
or
2 × equation ② − equation ③ =
$(2x - 2y + 8z = 6)$
$- (0x + 0y + 0z = 1)$
$2x - 2y + 8z = 5$
$2x - y + 3z = -2$, $x - y + 4z = 3$, and $3x - 2y + 7z = 2$ is one system of equations that could produce the original system composed of equations ①, ②, and ③.
$2x - y + 3z = -2$, $x - y + 4z = 3$, and $2x - 2y + 8z = 5$ is another system of equations that could produce the original system composed of equations ①, ②, and ③.

b. The systems have no solutions.

4. a. ① $x + 2y - z = 4$
② $x + 0y - 2z = 0$
③ $2x + 0y + 0z = -6$
The system can be solved by first solving equation ③ for x. So,

$2x = -6$
$x = -3$
If we use the method of back substitution, we can substitute $x = -3$ into equation ② and solve for z.
$-3 - 2z = 0$

$$z = -\frac{3}{2}$$

If we substitute $x = -3$ and $z = -\frac{3}{2}$ into equation ① we obtain the value of y.

$$-3 + 2y + \frac{3}{2} = 4 \text{ or } y = \frac{11}{4}$$

The equations intersect at the point with coordinates $(-3, \frac{11}{4}, -\frac{3}{2})$
Check:
Substituting into equation ①:
$x + 2y - z = -3 + \frac{22}{4} + \frac{3}{2} = 4$
Substituting into equation ②:
$x + 0y - 2z = -3 + 3 = 0$
Substituting into equation ③: $2x + 0y + 0z = -6$
b. This solution is the point at which all three planes meet.
5. a. ① $2x - y + z = 1$
② $x + y - z = -1$
③ $-3x - 3y + 3z = 3$
Since equation ③ = −equation ②, equation ② and equation ③ are consistent or lie in the same plane. Equation ① meets this plane in a line.
b. Adding equation ② and equation ① creates an equivalent equation, $3x = 0$ or $x = 0$. Substituting $x = 0$ into equation ① and equation ② gives equation ④ $z - y = 1$ and equation ⑤ $y - z = -1$. Equations ④ and ⑤ indicate the problem has infinite solutions. Substituting $y = t$ into equation ④ or ⑤ leads to
$x = 0, y = t$, and $z = 1 + t, t \in \mathbf{R}$
Check:

$$2(0) - s + (s + 1) = 1$$
$$0 + s - (s + 1) = -1$$
$$-3(0) - 3(s) + 3(s + 1) = 3$$

6. ① $2x + 3y - 4z = -5$
② $x - y + 3z = -201$
③ $5x - 5y + 15z = -1004$
There is no solution to this system of equations, because if you multiply equation ② by 5 you obtain a new equation, $5x - 5y + 15z = -1005$, which is inconsistent with equation ③.
7. a. Yes when this equation is alone, this is true, because any constants can be substituted into the variables in the equation $0x + 0y + 0z = 0$ and the equation will always be consistent.
b. Answers may vary. For example: To obtain a no solution and an equation with $0x + 0y + 0z = 0$, you must have two equal planes and one parallel distinct plane. For example one solution is:
$x + y + z = 2$
$2x + 2y + 2z = 4$
$3x + 3y + 3z = 12$
8. a. ① $2x + y - z = -3$
② $x - y + 2z = 0$
③ $3x + 2y - z = -5$
2 × equation ② + equation ③ = $5x + 0y + 0z = -5$ which gives $x = -1$.
Equation ① + equation ② = $3x + 0y + 1z = -3$. Substituting $x = 1$ into this equation leads to: $3(-1) + z = -3$ or $z = 0$.
Substituting $z = 0$ and $x = -1$ into equation ① gives: $2(-1)y - 0 = -3$ or $y = -1$. $(-1, -1, 0)$ is the point at which the three planes meet.
Check:
Substituting into equation ①:
$2x + y - z = -2 - 1 + 0 = -3$
Substituting into equation ②:
$x - y + 2z = -1 + 1 + 0 = 0$
Substituting into equation ③:
$3x + 2y - z = -3 - 2 + 0 = -5$
b. ① $\frac{x}{3} - \frac{y}{4} + z = \frac{7}{8}$
② $2x + 2y - 3z = -20$
③ $x - 2y + 3z = 2$
Equation ② + equation ③ = $3x + 0y + 0z = -18$ which gives $x = -6$.
Equation ③ − 3 × Equation ① = $-\frac{5}{4}y = -\frac{5}{8}$ or $y = \frac{1}{2}$. Substituting $x = -6$ and $y = \frac{1}{2}$ into equation ③ leads to:

$$-6 - 2\left(\frac{1}{2}\right) + 3z = 2 \text{ or } z = 3.$$

$(-6, \frac{1}{2}, 3)$ is the point at which the three planes meet.
Check:
Substituting into equation ①:
$\frac{x}{3} - \frac{y}{4} + z = -2 - \frac{1}{8} + 3 = \frac{7}{8}$
Substituting into equation ②:
$2x + 2y - 3z = -12 + 1 - 9 = -20$
Substituting into equation ③:
$x - 2y + 3z = -6 - 1 + 9 = 2$
c. ① $x - y = -199$
② $x + z = -200$
③ $y - z = 201$
Equation ② + equation ③ = equation ④ $= x + y = 1$

Equation ④ + equation ① = $2x = -198$ or $x = -99$. Substituting $x = -99$ into equation ① leads to:
$-99 - y = -199$ or $y = 100$. Substituting $x = -99$ into equation ②, you obtain:
$-99 + z = -200$ or $z = -101$
$(-99, 100, -101)$ is the point at which the three planes meet.
Check:
Substituting into equation ①:
$x - y = -99 - (100) = -199$
Substituting into equation ②:
$x + z = -99 - 101 = -200$
Substituting into equation ③:
$y - z = 100 - (-101) = 201$

d. ① $x - y - z = -1$
② $y - 2 = 0$
③ $x + 1 = 5$
Rearranging equation ② gives $y = 2$. Solving for x in equation ③ gives $x = 4$.
Substituting $x = 4$ and $y = 2$ into equation ① leads to:
$4 - 2 - z = -1$ or $z = 3$.
$(4, 2, 3)$ is the point at which all three planes meet.

9. a. ① $x - 2y + z = 3$
② $2x + 3y - z = -9$
③ $5x - 3y + 2z = 0$
Equation ③ + equation ② = equation ④
$= 7x + 1z = -9$.
Setting $z = t, x = -\frac{1}{7}t - \frac{9}{7}$
Equation ② − 2 × equation ① = equation ⑤
$= 7y + -3z = -15$.
Setting $z = t, y = -\frac{15}{7} + \frac{3}{7}t$
$x = -\frac{1}{7}t - \frac{9}{7}, y = -\frac{15}{7} + \frac{3}{7}t$, and $z = t, t \in \mathbf{R}$ The planes intersect in a line.

b. ① $x - 2y + z = 3$
② $x + y + z = 2$
③ $x - 3y + z = -6$
Equation ③ − equation ② = $-4y = -8$ or $y = 2$
Equation ③ − equation ① = $-1y = -9$ or $y = 9$
Since the solutions for y are different from these two equations, there is no solution to this system of equations.

c. ① $x - y + z = -2$
② $x + y + z = 2$
③ $x - 3y + z = -6$
Equation ① + equation ② = equation ④
$= 2x + 2y = 0$.
Setting $z = t, x = -t$
Using $z = t$ and $x = -t$. Solve equation ①
$-t - y + t = -2$ or $y = 2$
$x = -t, y = 2$, and $z = t, t \in \mathbf{R}$
The planes intersect in a line.

10. a. ① $x - y + z = 2$
② $2x - 2y + 2z = 4$
③ $x + y - z = -2$
Equation ① + equation ③ = equation ④
$= 2x = 0$ or $x = 0$.
Setting $z = t$.
Equation ①: $0 - y + t = 2$ or $y = t - 2$
$x = 0, y = t - 2$, and $z = t, t \in \mathbf{R}$

b. ① $2x - y + 3z = 0$
② $4x - 2y + 6z = 0$
③ $-2x + y - 3z = 0$
Equation ① + equation ③ = equation ④
$= 2x = 0$ or $x = 0$.
Setting $y = t$ and $z = s$, equation ①:
$2x - t + 3s = 0$ or $x = \dfrac{t - 3s}{2}$
$x = \dfrac{t - 3s}{2}, y = t$, and $z = s, s, t \in \mathbf{R}$

11. a. ① $x + y + z = 1$
② $x - 2y + z = 0$
③ $x - y + z = 0$
Equation ① − equation ③ = equation ④
$= 2y = 1$ or $y = \frac{1}{2}$
Equation ② − equation ③ = equation ⑤
$= -y = 0$ or $y = 0$
Since the y-variable is different in equation ④ and equation ⑤, the system is inconsistent and has no solution.

b. Answers may vary. For example: If you use the normals from equations ①, ②, and ③, you can determine the direction vectors from the equations' coefficients.
$\overrightarrow{n_1} = (1, 1, 1)$
$\overrightarrow{n_2} = (1, -2, 1)$
$\overrightarrow{n_3} = (1, -1, 1)$
$m_1 = \overrightarrow{n_1} \times \overrightarrow{n_2} = (3, 0, -3)$
$m_2 = \overrightarrow{n_1} \times \overrightarrow{n_3} = (2, 0, -2)$
$m_3 = \overrightarrow{n_2} \times \overrightarrow{n_3} = (-1, 0, 1)$

c. The three lines of intersection are parallel and are pairwise coplanar, so they form a triangular prism.

d. $\overrightarrow{n_1} \times \overrightarrow{n_2}$ is perpendicular to $\overrightarrow{n_3}$. So since, $(\overrightarrow{n_1} \times \overrightarrow{n_2}) \cdot \overrightarrow{n_3} = 0$, a triangular prism forms.

12. a. ① $x - y + 3z = 3$
② $x - y + 3z = 6$
③ $3x - 5z = 0$

Equation ① and equation ② have the same set of coefficients and variables, however, equations ① equals 3 while equation ② equals 6, which means there is no possible solution.

b. ① $5x - 2y + 3z = 1$
② $5x - 2y + 3z = -1$
③ $5x - 2y + 3z = 13$

All three equations equal different numbers so there is no possible solution.

c. ① $x - y + z = 9$
② $2x - 2y + 2z = 18$
③ $2x - 2y + 2z = 17$

Equation ② equals 18 while equation ③ equals 17, which means there is no possible solution.

d. The coefficients of equation ① are half the coefficients of equation ②, but the constant term is not half the other constant term.

13. a. ① $2x - y - z = 10$
② $x + y + 0z = 7$
③ $0x + y - z = 8$

Equation ① − 2 × equation ② − equation ③: $-4y = -12$ or $y = 3$. Substituting $y = 3$ into equation ② and equation ③ gives:
$x + 3 + 0z = 7$ or $x = 4$
$0x + 3 - z = 8$ or $z = -5$
$(4, 3, -5)$

b. ① $2x - y + z = -3$
② $x + y - 2z = 1$
③ $5x + 2y - 5z = 0$

Equation ① + equation ②: $3x - z = -2$.

Setting $z = t$, $x = \frac{t - 2}{3}$

Equation ① − 2 × equation ②: $-3y + 5z = -5$.

Setting $z = t$, $y = \frac{5t + 5}{3}$

$x = \frac{t - 2}{3}, y = \frac{5t + 5}{3}, z = t, t \in \mathbf{R}$

c. ① $x + y - z = 0$
② $2x - y + z = 0$
③ $4x - 5y + 5z = 0$

Equation ① + equation ②: $3x = 0$ or $x = 0$
Setting $x = 0$ and $z = t$ in equation ② gives,
$2(0) - y + t = 0$ or $y = t$
$x = 0, y = t, z = t, t \in \mathbf{R}$

d. ① $x - 10y + 13z = -4$
② $2x - 20y + 26z = -8$
③ $x - 10y + 13z = -8$

If you multiply equation ② by two, you obtain $2x - 20y + 26z = -16$. Since equation ② and equation ③ equal different numbers, there is no solution to this system.

e. ① $x - y + z = -2$
② $x + y + z = 2$
③ $3x + y + 3z = 2$

Equation ① + equation ②: $-2y = -4$ or $y = 2$
Setting $y = 2$ and $z = t$ in equation ①,
$x - 2 + t = -2$ or $x = -t$
$x = -t, y = 2, z = t, t \in \mathbf{R}$

f. ① $x + y + z = 0$
② $x - 2y + 3z = 0$
③ $2x - y + 3z = 0$

Equation ① − equation ② = equation ④
$= 3y - 2z = 0$
Equation ③ − 2 × equation ② − equation ⑤
$= 3y - 3z = 0$
Equation ④ − equation ⑤: $z = 0$
Setting $z = 0$ in equation ① and equation ②,
Equation ⑥ $= x + y = 0$
Equation ⑦ $= x - 2y = 0$
Equation ⑥ − equation ⑦: $3y = 0$ or $y = 0$
Setting $y = 0$ and $z = 0$ in equation ① leads to $x = 0$
$(0, 0, 0)$

14. a. First, reorder these equations so that equation ② is first, equation ③ is second, and equation ① last.

① $x - y + z = p$
② $4x + qy + z = 2$
③ $2x + y + z = 4$

To eliminate x from the last two equations, subtract 4 times equation ① from equation ②, and subtract 2 times equation ① from equation ③.

① $x - y + z = p$
② $(q + 4)y - 3z = 2 - 4p$
③ $3y - z = 4 - 2p$

There will be an infinite number of solutions if $q + 4 = 9$ and $3(4 - 2p) = 2 - 4p$ because then equation ② will be 3 times equation ③. This means that $p = q = 5$.

b. Based on what was found in part **a.**, substituting in $p = q = 5$ we will arrive at the equivalent system

① $x - y + z = 5$
② $9y - 3z = -18$
③ $3y - z = -6$

which is really the same as

① $x - y + z = 5$
② $3y - z = -6$

Letting $z = t$, we see that equation ② delivers

$y = \frac{1}{3}(t - 6)$

$= \frac{1}{3}t - 2$

and so equation ① gives

$x = \frac{1}{3}(t - 6) - t + 5$

$= -\frac{2}{3}t + 3$

So the parametric equation of the line of intersection is

$x = -\frac{2}{3}t + 3, y = \frac{1}{3}t - 2, z = t, t \in \mathbf{R}.$

15. a. First, eliminate x from two of these equations. To make things easier, switch equation ① with equation ②, and multiply equation ③ by 2.

① $2x + y + z = -4$

② $4x + 3y + 3z = -8$

③ $6x - 4y + (2m^2 - 12)z = 2m - 8$

Now eliminate x from the last two equations by using proper multiples of the first equation.

① $2x + y + z = -4$

② $y + z = 0$

③ $-7y + (2m^2 - 15)z = 2m + 4$

Now eliminate y from the third equation by using a proper multiple of the second equation.

① $2x + y + z = -4$

② $y + z = 0$

③ $(2m^2 - 8)z = 2m + 4$

If $2m^2 - 8 = 0$ (the coefficient of z in the third equation), then $m = \pm 2$. However, if $m = 2$, the third equation would become $0z = 8$, which has no solutions. So there is no solution if $m = 2$.

b. Working with what was found in part **a.**, if $m \neq \pm 2$, then the third equation in the equivalent system found there will have a unique solution for z, namely

$z = \frac{2m + 4}{2m^2 - 8}$,

and back-substituting into the other two equations will give unique solutions for x and y also. So there is a unique solution if $m \neq \pm 2$.

c. Again using the equivalent system found in part **a.**, setting $m = -2$ will deliver the third equation $0z = 0$, which allows for z to be anything at all. So $m = -2$ will give an infinite number of solutions.

16. a. ① $\frac{1}{a} + \frac{1}{b} - \frac{1}{c} = 0$

② $\frac{2}{a} + \frac{3}{b} + \frac{2}{c} = \frac{13}{6}$

③ $\frac{4}{a} - \frac{2}{b} + \frac{3}{c} = \frac{5}{2}$

Equation ② − 2 × equation ①:

$\frac{1}{b} + \frac{4}{c} = \frac{13}{6}$ = equation ④

Equation ③ − 4 × equation ①: $-\frac{6}{b} + \frac{7}{c}$

$m_3 = \vec{n} \times \vec{n_1} = (-1, 0, 1) = \frac{5}{2}$ = equation ⑤

Equation ⑤ + 6 × equation ④:

$\frac{31}{c} = 15.5$ or $c = 2$

Substituting $c = 2$ into equation ④:

$\frac{1}{b} + 2 = \frac{13}{6}$ or $b = 6$

Substituting $c = 2$ and $b = 6$ into equation ①:

$\frac{1}{a} + \frac{1}{6} - \frac{1}{2} = 0$ or $a = 3$

(3, 6, 2)

9.5 The Distance from a Point to a Line in R^2 and R^3, pp. 540–541

1. a. $3x + 4y - 5 = 0$

$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$

$d = \frac{|3(-4) + 4(5) - 5|}{\sqrt{3^2 + 4^2}}$

$= \frac{3}{5}$

b. $5x - 12y + 24 = 0$

$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$

$d = \frac{|5(-4) - 12(5) + 24|}{\sqrt{5^2 + (-12)^2}}$

$= \frac{56}{13}$ or 4.31

c. $9x - 40y = 0$

$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$

$d = \frac{|9(-4) - 40(5)|}{\sqrt{9^2 + (40)^2}}$

$= \frac{236}{\sqrt{1681}}$ or 5.76

2. a. $2x - y + 1 = 0$ and $2x - y + 6 = 0$

In order to find the distance between these two parallel lines, you must first find a point on one of

the lines. It is easiest to find a point where the line crosses the x or y-axis.
$2(0) - y + 1 = 0$ or $y = 1$ which corresponds to the point $(0, 1)$.

$$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$

$$d = \frac{|2(0) - 1(1) + 6|}{\sqrt{2^2 + (-1)^2}}$$

$$= \frac{5}{\sqrt{5}} \text{ or } 2.24$$

b. $7x - 24y + 168 = 0$ and $7x - 24y - 336 = 0$
In order to find the distance between these two parallel lines, you must first find a point on one of the lines. It is easiest to find a point where the line crosses the x or y-axis.
$7(0) - 24y + 168 = 0$ or $y = 7$ which corresponds to the point $(0, 7)$

$$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$

$$d = \frac{|7(0) - 24(7) - 336|}{\sqrt{7^2 + (-24)^2}}$$

$$= \frac{504}{25} \text{ or } 20.16$$

3. a. $\vec{r} = (-1, 2) + s(3, 4), s \in \mathbf{R}$
We start by writing the given equation of the line in parametric form. Doing so gives $x = -1 + 3s$, $y = 2 + 4s$. We construct a vector from $R(-2, 3)$ to a general point on the line.

$$\vec{a} = [-2 - (-1 + 3s), 3 - (2 + 4s)]$$
$$= (-1 - 3s, 1 - 4s).$$
$$(3, 4) \cdot (-1 - 3s, 1 - 4s) = 0$$
$$(-3 - 9s) + (4 - 16s) = 0$$
$$s = \frac{1}{25}$$

This means that the minimal distance between $R(-2, 3)$ and the line occurs when $s = \frac{1}{25}$.
This point corresponds to $\left(-\frac{22}{25}, \frac{54}{25}\right)$. The distance between this point and $(-2, 3)$ is 1.4.
b. $\vec{r} = (1, 0) + t(5, 12), t \in \mathbf{R}$
We start by writing the given equation of the line in parametric form. Doing so gives $x = 1 + 5t$, $y = 12t$. We construct a vector from $R(-2, 3)$ to a general point on the line.

$$\vec{a} = [-2 - (1 + 5t), 3 - (12t)]$$
$$= (-3 - 5t, 3 - 12t).$$
$$(5, 12) \cdot (-3 - 5t, 3 - 12t) = 0$$
$$(-15 - 25t) + (36 - 144t) = 0$$
$$t = \frac{21}{169}$$

This means that the minimal distance between $R(-2, 3)$ and the line occurs when $t = \frac{21}{169}$.
This point corresponds to $\left(\frac{274}{169}, \frac{252}{169}\right)$. The distance between this point and $(-2, 3)$ is about 3.92.
c. $\vec{r} = (1, 3) + p(7, -24), p \in \mathbf{R}$
We start by writing the given equation of the line in parametric form. Doing so gives $x = 1 + 7p$, $y = 3 - 24p$. We construct a vector from $R(-2, 3)$ to a general point on the line.

$$\vec{a} = [-2 - (1 + 7p), 3 - (3 - 24p)]$$
$$= (-3 - 7p, 24p).$$
$$(7, -24) \cdot (-3 - 7p, 24p) = 0$$
$$(-21 - 49p) + (-576p) = 0$$
$$p = -\frac{21}{625}$$

This means that the minimal distance between $R(-2, 3)$ and the line occurs when $p = -\frac{21}{625}$.
This point corresponds to $\left(\frac{478}{625}, \frac{2379}{625}\right)$.
The distance between this point and $(-2, 3)$ is about 2.88.

4. a. $d = \dfrac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$
If you substitute in the coordinates $(0, 0)$, the formula changes to $d = \dfrac{|A(0) + B(0) + C|}{\sqrt{A^2 + B^2}}$, which reduces to $d = \dfrac{|C|}{\sqrt{A^2 + B^2}}$.
b. $3x - 4y - 12 = 0$ and $3x - 4y + 12 = 0$

$$d(L_1) = \frac{|C|}{\sqrt{A^2 + B^2}} = \frac{|-12|}{\sqrt{3^2 + (-4)^2}}$$
$$= \frac{12}{5}$$

$$d(L_2) = \frac{|C|}{\sqrt{A^2 + B^2}} = \frac{|12|}{\sqrt{3^2 + (-4)^2}}$$
$$= \frac{12}{5}$$

The distance between these parallel lines is $\frac{12}{5} + \frac{12}{5} = \frac{24}{5}$, because one of the lines is below the origin and the other is above the origin.
c. $3x - 4y - 12 = 0$ and $3x - 4y + 12 = 0$
In order to find the distance between these two parallel lines, you must first find a point on one of the lines. It is easiest to find a point where the line crosses the x or y-axis.

$3(0) - 4y - 12 = 0$ or $y = -3$ which corresponds to the point $(0, 3)$.

$$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$

$$d = \frac{|3(0) - 4(-3) + 12|}{\sqrt{3^2 + (-4)^2}}$$

$$= \frac{24}{5}$$

Both the answers to 4.b. and 4.c. are the same.

5. a. $\vec{r} = (-2, 1) + s(3, 4), s \in \mathbf{R}$

$\vec{r} = (1, 0) + t(3, 4), t \in \mathbf{R}$

First find a random point on one of the lines. We will use $(-2, 1)$ from the first equation. We start by writing the second equation in parametric form. Doing so gives $x = 1 + 3t$, $y = 4t$. We construct a vector from $P(-2, 1)$ to a general point on the line.

$\vec{a} = [-2 - (1 + 3t), 1 - (4t)]$

$= (-3 - 3t, 1 - 4t)$.

$(3, 4) \cdot (-3 - 3t, 1 - 4t) = 0$

$(-9 - 9t) + (4 - 16t) = 0$

$$t = -\frac{1}{5}$$

This means that the minimal distance between $P(-2, 1)$ and line occurs when $t = -\frac{1}{5}$. This point corresponds to $(\frac{2}{5}, -\frac{4}{5})$. The distance between this point and $(-2, 1)$ is 3

b. $\frac{x - 1}{4} = \frac{y}{-3}$ and $\frac{x}{4} = \frac{y + 1}{-3}$

First change one equation into a Cartesian equation, which leads to $3x + 4y - 3 = 0$ and take a point from the other equation such as $(4, -4)$.

$$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$

$$d = \frac{|3(4) + 4(-4) - 3|}{\sqrt{3^2 + 4^2}}$$

$$= \frac{7}{5} \text{ or } 1.4$$

c. $2x - 3y + 1 = 0$ and $2x - 3y - 3 = 0$

In order to find the distance between these two parallel lines, you must first find a point on one of the lines. It is easiest to find a point where the line crosses the x or y-axis.

$2(0) - 3y - 3 = 0$ or $y = -1$ which corresponds to the point $(0, -1)$.

$$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$

$$d = \frac{|2(0) - 3(-1) + 1|}{\sqrt{2^2 + (-3)^2}}$$

$$= \frac{4}{\sqrt{13}} \text{ or } 1.11$$

d. $5x + 12y = 120$ and $5x + 12y + 120 = 0$

In order to find the distance between these two parallel lines, you must first find a point on one of the lines. It is easiest to find a point where the line crosses the x or y-axis.

$5(0) + 12y = 120$ or $y = 10$ which corresponds to the point $(0, 10)$.

$$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$

$$d = \frac{|5(0) + 12(10) + 120|}{\sqrt{5^2 + 12^2}}$$

$$= \frac{240}{13} \text{ or } 18.46$$

6. a. $P(1, 2, -1)$ $\vec{r} = (1, 0, 0) + s(2, -1, 2), s \in \mathbf{R}$

We start by writing the given equation of the line in parametric form. Doing so gives $x = 1 + 2s$, $y = -s$, and $z = 2s$. We construct a vector from $P(1, 2, -1)$ to a general point on the line.

$\vec{a} = [1 - (1 + 2s), 2 - (-s), -1 - (2s)]$

$= (-2s, 2 + s, -1 - 2s)$.

$(2, -1, 2) \cdot (-2s, 2 + s, -1 - 2s) = 0$

$(-4s) + (-2 - s) + (-2 - 4s) = 0$

$$s = -\frac{4}{9}$$

This means that the minimal distance between $P(1, 2, -1)$ and the line occurs when $s = -\frac{4}{9}$. This point corresponds to $(\frac{1}{9}, \frac{4}{9}, -\frac{8}{9})$. The distance between this point and $P(1, 2, -1)$ is 1.80.

b. $P(0, -1, 0)$ $\vec{r} = (2, 1, 0) + t(-4, 5, 20), t \in \mathbf{R}$

We start by writing the given equation of the line in parametric form. Doing so gives $x = 2 - 4t$, $y = 1 + 5t$, and $z = 20t$. We construct a vector from $P(0, -1, 0)$ to a general point on the line.

$\vec{a} = [0 - (2 - 4t), -1 - (1 + 5t), 0 - (20t)]$

$= (-2 + 4t, -2 - 5t, 20t)$.

$(-4, 5, 20) \cdot (-2 + 4t, -2 - 5t, -20t) = 0$

$(8 - 16t) + (-10 - 25t) + (-400t) = 0$

$$t = -\frac{2}{441}$$

This means that the minimal distance between $P(0, -1, 0)$ and the line occurs when $t = -\frac{2}{441}$.

This point corresponds to $\left(\frac{890}{441}, \frac{431}{441}, -\frac{40}{441}\right)$. The distance between this point and $P(0, -1, 0)$ is 2.83.

c. $P(2, 3, 1)$ $\vec{r} = p(12, -3, 4), p \in \mathbf{R}$

We start by writing the given equation of the line in parametric form. Doing so gives $x = 12p$, $y = -3p$, and $z = 4p$. We construct a vector from $P(2, 3, 1)$ to a general point on the line.

$$\vec{a} = [2 - (12p), 3 - (-3p), 1 - (4p)]$$
$$= (2 - 12p, 3 + 3p, 1 - 4p).$$
$$(12, -3, 4) \cdot (2 - 12p, 3 + 3p, 1 - 4p) = 0$$
$$(24 - 144p) + (-9 - 9p) + (4 - 16p) = 0$$
$$p = \frac{19}{169}$$

This means that the minimal distance between $P(2, 3, 1)$ and the line occurs when $p = \frac{19}{169}$. This point corresponds to $\left(\frac{228}{169}, -\frac{57}{169}, \frac{76}{169}\right)$. The distance between this point and $P(2, 3, 1)$ is 3.44.

7. a. $\vec{r} = (1, 1, 0) + s(2, 1, 2), s \in \mathbf{R}$

$\vec{r} = (-1, 1, 2) + t(2, 1, 2), t \in \mathbf{R}$

First find a random point on one of the lines. We will use $P(-1, 1, 2)$ from the second equation. We then write the first equation in parametric form. Doing so gives $x = 1 + 2s$, $y = 1 + s$, and $z = 0 + 2s$. We construct a vector from $P(-1, 1, 2)$ to a general point on the line.

$$\vec{a} = [-1 - (1 + 2s), 1 - (1 + s), 2 - 2s]$$
$$= (-2 - 2s, 2 - 2s).$$
$$(2, 1, 2) \cdot (-2 - 2s, -s, 2 - 2s) = 0$$
$$(-4 - 4s) + (-s) + (4 - 4s) = 0$$
$$s = 0$$

This means that the minimal distance between $P(-1, 1, 2)$ and line occurs when $s = 0$. This point corresponds to $(1, 1, 0)$. The distance between this point and $(-1, 1, 2)$ is 2.83

b. $\vec{r} = (3, 1, -2) + m(1, 1, 3), m \in \mathbf{R}$

$\vec{r} = (1, 0, 1) + n(1, 1, 3), n \in \mathbf{R}$

First find a random point on one of the lines. We will use $P(1, 0, 1)$ from the second equation. We then write the first equation in parametric form. Doing so gives $x = 3 + m$, $y = 1 + m$, and $z = -2 + 3m$. We construct a vector from $P(1, 0, 1)$ to a general point on the line.

$$\vec{a} = [1 - (3 + m), 0 - (1 + m), 1 - (-2 + 3m)]$$
$$= (-2 - 3m, -1 - m, 3 - 3m).$$
$$(1, 1, 3) \cdot (-2 - 3m, -1 - m, 3 - 3m) = 0$$
$$(-2 - 3m) + (-1 - m) + (9 - 9m) = 0$$
$$m = \frac{6}{13}$$

This means that the minimal distance between $P(1, 0, 1)$ and line occurs when $m = \frac{6}{13}$. This point corresponds to $\left(\frac{45}{13}, \frac{19}{13}, -\frac{6}{13}\right)$. The distance between this point and $(1, 0, 1)$ is 3.28

8. a. $\vec{r} = (1, -1, 2) + s(1, 3, -1), s \in \mathbf{R}$

First we write the equation in parametric form. Doing so gives $x = 1 + s$, $y = -1 + 3s$, and $z = 2 - s$. We construct a vector from $P(2, 1, 3)$ to a general point on the line.

$$\vec{a} = [2 - (1 + s), 1 - (-1 + 3s), 3 - (2 - s)]$$
$$= (1 - s, 2 - 3s, 1 + s).$$
$$(1, 3, -1) \cdot (1 - s, 2 - 3s, 1 + s) = 0$$
$$(1 - s) + (6 - 9s) + (1 + s) = 0$$
$$s = \frac{6}{11}$$

This means that the minimal distance between $P(2, 1, 3)$ and line occurs when $s = \frac{6}{11}$. This point corresponds to $\left(\frac{17}{11}, \frac{7}{11}, \frac{16}{11}\right)$.

b. The distance between $\left(\frac{17}{11}, \frac{7}{11}, \frac{16}{11}\right)$ and $(2, 1, 3)$ is 1.65.

9. First, find the line L of intersection between the planes

① $x - y + 2z = 2$

② $x + y - z = -2$

Subtract the first equation from the second to eliminate x and get the equivalent system

① $x - y + 2z = 2$

② $2y - 3z = -4$

Let $z = t$. Then the second equation gives

$$2y = 3t - 4$$
$$y = \frac{3}{2}t - 2$$

So substituting these into the first equation gives

$$x = y - 2z + 2$$
$$= \left(\frac{3}{2}t - 2\right) - 2t + 2$$
$$= -\frac{1}{2}t$$

So the equation of the line of intersection for these two planes in parametric form is

$x = -\frac{1}{2}t$, $y = \frac{3}{2}t - 2$, $z = t$, $t \in \mathbf{R}$.

The direction vector for this line is $\left(-\frac{1}{2}, \frac{3}{2}, 1\right)$, which is parallel to $(-1, 3, 2)$. So, to make things easier, the parametric form of this line of intersection could also be expressed as

$x = -t$, $y = 3t - 2$, $z = 2t$, $t \in \mathbf{R}$

In vector form, this is the same as

$\vec{r} = (0, -2, 0) + t(-1, 3, 2), t \in \mathbf{R}$.
Since $Q(0, -2, 0)$ is on this line,
$\overrightarrow{QP} = (-1, 2, -1) - (0, -2, 0)$
$= (-1, 4, -1)$
So the distance from $P(-1, 2, -1)$ to the line of intersection is

$$d = \frac{|(-1, 3, 2) \times (-1, 4, -1)|}{|(-1, 3, 2)|}$$
$$= \frac{|(-11, -3, -1)|}{|(-1, 3, 2)|}$$
$$= \sqrt{\frac{131}{14}}$$
$$\doteq 3.06$$

To find the point on the line that gives this minimal distance, let (x, y, z) be a point on the line. Then, using the parametric equations,
$(x, y, z) = (-t, 3t - 2, 2t)$
So the distance from P to this point is
$\sqrt{(x + 1)^2 + (y - 2)^2 + (z + 1)^2}$
$= \sqrt{(1 - t)^2 + (3t - 4)^2 + (2t + 1)^2}$
$= \sqrt{14t^2 - 22t + 18}$
To get the minimal distance, set this quantity equal to $\sqrt{\frac{131}{14}}$.

$$\sqrt{14t^2 - 22t + 18} = \sqrt{\frac{131}{14}}$$
$$14t^2 - 22t + 18 = \frac{131}{14}$$
$$196t^2 - 308t + 252 = 131$$
$$196t^2 - 308t + 121 = 0$$
$$t = \frac{308 \pm \sqrt{0}}{392}$$
$$= \frac{11}{14}$$

So the point on the line at minimal distance from P is
$$(x, y, z) = (-t, 3t, -2, 2t)$$
$$= \left(-\frac{11}{14}, 3\left(\frac{11}{14}\right) - 2, 2\left(\frac{11}{14}\right)\right)$$
$$= \left(-\frac{11}{14}, \frac{5}{14}, \frac{22}{14}\right)$$

10. A point on the line
$\vec{r} = (0, 0, 1) + s(4, 2, 1), s \in \mathbf{R}$,
has parametric equations
$x = 4s, y = 2s, z = 1 + s, s \in \mathbf{R}$.
Let this point be called
$Q(4s, 2s, 1 + s)$. Then
$\overrightarrow{QA} = (2, 4, -5) - (4s, 2s, 1 + s)$
$= (2 - 4s, 4 - 2s, -6 - s)$
If Q is at minimal distance from A, then this vector will be perpendicular to the direction vector for the line, $(4, 2, 1)$. This means that
$0 = (2 - 4s, 4 - 2s, -6 - s) \cdot (4, 2, 1)$
$= 10 - 21s$
$$s = \frac{10}{21}$$
So the point Q on the line at minimal distance from A is
$$Q(4s, 2s, 1 + s) = Q\left(4\left(\frac{10}{21}\right), 2\left(\frac{10}{21}\right), 1 + \frac{10}{21}\right)$$
$$= Q\left(\frac{40}{21}, \frac{20}{21}, \frac{31}{21}\right)$$
Also
$$\overrightarrow{QA} = \left(2 - \frac{40}{21}, 4 - \frac{20}{21}, -5 - \frac{31}{21}\right)$$
$$= \left(\frac{2}{21}, \frac{64}{21}, -\frac{136}{21}\right)$$
So the point A' will satisfy
$$\overrightarrow{QA'} = -\overrightarrow{QA}$$
$$= \left(-\frac{2}{21}, -\frac{64}{21}, \frac{136}{21}\right)$$
$$= A'(a, b, c) - Q$$
$$= \left(a - \frac{40}{21}, b - \frac{20}{21}, c - \frac{31}{21}\right)$$
So $a = \frac{38}{21}$, $b = -\frac{44}{21}$, and $c = \frac{167}{21}$. That is,
$A'\left(\frac{38}{21}, -\frac{44}{21}, \frac{167}{21}\right)$.

11. a. Think of H as being the origin, E as being on the x-axis, D as being on the z-axis, and G as being on the y-axis. That is,
$H(0, 0, 0)$
$E(3, 0, 0)$
$G(0, 2, 0)$
$D(0, 0, 2)$
and so on for the other points as well. Then line segment HB has direction vector
$B(3, 2, 2) - H(0, 0, 0) = (3, 2, 2)$.
Also, $\overrightarrow{HA} = (3, 0, 2)$. So the distance formula says that the distance between A and line segment HB is
$$d = \frac{|(3, 2, 2) \times (3, 0, 2)|}{|(3, 2, 2)|}$$
$$= \frac{|4, 0, -6|}{|(3, 2, 2)|}$$
$$= \sqrt{\frac{52}{17}}$$
$$\doteq 1.75$$

b. Vertices D and G will give the same distance to HB because they are equidistant to the segment HB. (This is easy to check with the distance formula used similarly to part **a.** The vertices C, E, and F give different distances than those found in part **a.**)

c. The height of triangle AHB was found in part **a.**, and was $\sqrt{\frac{52}{17}}$. The base length of this triangle is the magnitude of $\overrightarrow{HB} = (3, 2, 2)$, which is $\sqrt{52}$. So the area of this triangle is

$$\frac{1}{2}\left(\sqrt{\frac{52}{17}}\right)(\sqrt{17}) = \frac{1}{2}(\sqrt{52})$$
$$\doteq 3.6 \text{ units}^2$$

9.6 The Distance from a Point to a Plane, pp. 549–550

1. a. Yes the calculations are correct, Point A lies in the plane.

b. The answer 0 means that the point lies in the plane.

2. Use the distance formula.

$$d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

a. The distance from $A(3, 1, 0)$ to the plane $20x - 4y + 5z + 7 = 0$ is

$$d = \frac{|20(3) + -4(1) + 5(0) + 7|}{\sqrt{20^2 + (-4)^2 + 5^2}}$$
$$= 3$$

b. The distance from $B(0, -1, 0)$ to the plane $2x + y + 2z - 8 = 0$ is

$$d = \frac{|2(0) + 1(-1) + 2(0) - 8|}{\sqrt{2^2 + 1^2 + 2^2}}$$
$$= 3$$

c. The distance from $C(5, 1, 4)$ to the plane $3x - 4y - 1 = 0$ is

$$d = \frac{|3(5) + -4(1) + 0(4) - 1|}{\sqrt{3^2 + (-4)^2 + 0^2}}$$
$$= 2$$

d. The distance from $D(1, 0, 0)$ to the plane $5x - 12y = 0$ is

$$d = \frac{|5(1) - 12(0) + 0(0) + 0|}{\sqrt{5^2 + (-12)^2 + 0^2}}$$
$$= \frac{5}{13} \text{ or } 0.38$$

e. The distance from $E(-1, 0, 1)$ to the plane $18x - 9y + 18z - 11 = 0$ is

$$d = \frac{|18(-1) - 9(0) + 18(1) - 11|}{\sqrt{18^2 + (-9)^2 + 18^2}}$$
$$= \frac{11}{27} \text{ or } 0.41$$

3. a. $3x + 4y - 12z - 26 = 0$ and
$3x + 4y - 12z + 39 = 0$

First find a point in the second plane such as $(-3, 0, 0)$. Then use $d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$ to solve.

$$d = \frac{|3(-13) + 4(0) - 12(0) - 26|}{\sqrt{3^2 + 4^2 + (-12)^2}}$$
$$= 5$$

b.
$$\begin{aligned} &3x + 4y - 12z - 26 = 0\\ +\ &3x + 4y - 12z + 39 = 0\\ &6x + 8y - 24z + 13 = 0 \end{aligned}$$

c. Answers may vary. Any point on the plane $6x + 8y - 24z + 13 = 0$ will work, for example $\left(-\frac{1}{6}, 0, \frac{1}{2}\right)$.

4. a. The distance from $P(1, 1, -3)$ to the plane $y + 3 = 0$ is

$$d = \frac{|0(1) + 1(1) + 0(-3) + 3|}{\sqrt{0^2 + (1)^2 + 0^2}}$$
$$= 4$$

b. The distance from $Q(-1, 1, 4)$ to the plane $x - 3 = 0$ is

$$d = \frac{|1(-1) + 0(1) + 0(4) - 3|}{\sqrt{1^2 + 0^2 + 0^2}}$$
$$= 4$$

c. The distance from $R(1, 0, 1)$ to the plane $z + 1 = 0$ is

$$d = \frac{|0(1) + 0(0) + 1(1) + 1|}{\sqrt{0^2 + 0^2 + 1^2}}$$
$$= 2$$

5. First you have to find an equation of a plane to the three points. The equation to this plane is $14x - 28y + 28z - 42 = 0$. Then use $d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$ to solve for the distance.

$$d = \frac{|14(1) - 28(-1) + 28(1) - 42|}{\sqrt{14^2 + (-28)^2 + 28^2}}$$
$$= \frac{2}{3} \text{ or } 0.67$$

6. $3 = \frac{|A(3) - 2(-3) + 6(1) + 0|}{\sqrt{A^2 + (-2)^2 + 6^2}}$

$3\sqrt{(A^2 + 40)} = |3A + 12|$

$\sqrt{(A^2 + 40)} = |A + 4|$

$A^2 + 40 = A^2 + 8A + 16$

$24 = 8A$

$3 = A$

$A = 3$ is the only solution to this equation.

7. These lines are skew lines, and the plane containing the second line, $\vec{r} = (0, 0, 1) + t(1, 1, 0), t \in \mathbf{R}$, that is parallel to the first line will have direction vectors $(1, 1, 0)$ and $(3, 0, 1)$. So a normal to this plane is $(1, 1, 0) \times (3, 0, 1) = (1, -1, -3)$.
So the equation of this plane will be of the form $x - y - 3z + D = 0$. We want the point $(0, 0, 1)$ to be on this plane, and substituting this into the equation above gives $D = 3$. So the equation of the plane containing $\vec{r} = (0, 0, 1) + t(1, 1, 0), t \in \mathbf{R}$ and parallel to the first line is

$x - y - 3z + 3 = 0$.

Since $(0, 1, -1)$ is on the first line, the distance between these skew lines is the same as the distance between this point and the plane just determined.
By the distance formula, this distance is

$d = \frac{|(0) - (1) - 3(-1) + 3|}{\sqrt{1^2 + (-1)^2 + (-3)^2}}$

$= \frac{5}{\sqrt{11}}$

$\doteq 1.51$.

8. a.–b. We will do both of these parts at once.
The two given lines are

$\vec{r} = (1, -2, 5) + s(0, 1, -1), s \in \mathbf{R}$,

$\vec{r} = (1, -1, -2) + t(1, 0, -1), t \in \mathbf{R}$.

By converting to parametric form, a general point on the first line is

$U(1, s - 2, 5 - s)$,

and on the second line is

$V(1 + t, -1, -2 - t)$.

So the vector

$\overrightarrow{UV} = (t, 1 - s, s - t - 7)$.

If the points U and V are those that produce the minimal distance between these two lines, then $\overrightarrow{UV}$ will be perpendicular to both direction vectors, $(0, 1, -1)$ and $(1, 0, -1)$. In the first case, we get

$0 = (t, 1 - s, s - t - 7) \cdot (0, 1, -1)$

$= 8 - 2s + t$

$t = 2s - 8$

In the second case, we get

$0 = (t, 1 - s, s - t - 7) \cdot (1, 0, -1)$

$= 2t - s + 7$

Substituting $t = 2s - 8$ into this second equation, we get

$2(2s - 8) - s + 7 = 0$

$s = 3$

$t = 2s - 8$

$t = -2$

Substituting these values for s and t into U and V, we get

$U(1, 1, 2)$

$V(-1, -1, 0)$

So $U(1, 1, 2)$ is the point on the first line that produces the minimal distance to the second line at point $V(-1, -1, 0)$. This minimal distance is given by

$|\overrightarrow{UV}| = |(-2, -2, -2)|$

$= \sqrt{12}$

$\doteq 3.46$

Review Exercise, pp. 552–555

1. $2x - y = 31, x + 8y = -34, 3x + ky = 38$

$(2x - y = 31) - 2(x + 8y = -34)$

$= 0x - 17y = 99$

$y = -\frac{99}{17}, x = \frac{214}{17}$

$3\left(\frac{214}{17}\right) + k\left(\frac{-99}{17}\right) = 38$

$k = -\frac{4}{99}$

2. ① $x - y = 13$
② $3x + 2y = -6$
③ $x + 2y = -19$

(2 × Equation ①) + equation ② = $5x + 0y = 20$ or $x = 4$. Substituting $x = 4$ into equation ① gives $(4) - y = 13$ or $y = -9$. However, when you substitute this coordinates into the third equation, the third equation is not consistent, so there is no solution to this problem.

3. a. ① $x - y + 2z = 3$
② $2x - 2y + 3z = 1$
③ $2x - 2y + z = 11$

Equation ② − equation ③ = $5z = -10$ or $z = -2$. Substituting $z = -2$ into all of the equations gives

④ $x - y - 4 = 3$
⑤ $2x - 2y - 6 = 1$
⑥ $2x - 2y - 2 = 11$

There are no x and y variables that satisfy these equations, so the answer is no solution.

b. ① $x + y + z = 300$
② $x + y - z = 98$
③ $x - y + z = 100$

Equation ② + equation ③ = $2x = 198$ $x = 99$.
Substituting $x = 99$ into all three equations gives:
④ $y + z = 201$
⑤ $y - z = -1$
⑥ $-y + z = 1$

Equation ④ + equation ⑤ = $2y = 200$ or $y = 100$. You then get $z = 101$ after substituting both x and y into equation ①.
(99, 100, 101)
Check:
① $99 + 100 + 101 = 300$
② $99 + 100 - 101 = 98$
③ $99 - 100 + 101 = 100$

4. a. These four points will lie in the same plane if and only if the line determined by the first two points intersects the line determined by the last two points. The direction vector determined by the first two is
$\vec{a} = (7, -5, 1) - (1, 2, 6)$
$= (6, -7, -5)$
So these first two points determine the line with vector equation
$\vec{r} = (1, 2, 6) + s(6, -7, -5), s \in \mathbf{R}$.
The direction vector determined by the last two points is
$\vec{b} = (-3, 5, 6) - (1, 1, 4)$
$= (-4, 4, 2)$
So these first two points determine the line with vector equation
$\vec{r} = (1, 1, 4) + t(-4, 4, 2), t \in \mathbf{R}$.
Converting these two lines to parametric form, we obtain the equations
① $1 + 6s = 1 - 4t$
② $2 - 7s = 1 + 4t$
③ $6 - 5s = 4 + 2t$
Adding the first and second equations gives $3 - s = 2$, so $s = 1$. Substituting this into the third equation, we get
$1 = 4 + 2t$
$-3 = 2t$
So $t = -\frac{3}{2}$. We need to check this s and t for consistency. Substituting $s = 1$ into the vector equation for the first line gives
$\vec{r} = (1, 2, 6) + (1)(6, -7, -5)$
$= (7, -5, 1)$
as a point on this line. Substituting $t = -\frac{3}{2}$ into the vector equation for the second line gives
$\vec{r} = (1, 1, 4) + \left(-\frac{3}{2}\right)(-4, 4, 2)$
$= (1, 1, 4) + (6, -6, -3)$
$= (7, -5, 1)$
as a point on this line. This means the two lines intersect, and so the four points given lie in the same plane.
b. Direction vectors for the plane containing the four points in part **a.** are $(6, -7, -5)$ and $(-4, 4, 2)$. So a normal to this plane is
$(6, -7, -5) \times (-4, 4, 2) = (6, 8, -4)$.
We will use the parallel normal $(3, 4, -2)$. So the equation of this plane is of the form
$3x + 4y - 2z + D = 0$.
Substitute in the point $(1, 2, 6)$ to find D.
$3(1) + 4(2) - 2(6) + D = 0$
$D = 1$
The equation of the plane is
$3x + 4y - 2z + 1 = 0$.
So, using the distance formula, this plane is distance
$$d = \frac{|3(0) + 4(0) - 2(0) + 1|}{|(3, 4, -2)|}$$
$$= \frac{1}{\sqrt{29}}$$
$\doteq 0.19$
from the origin.

5. Use the distance formula
$$d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$
a. The distance from $A(-1, 1, 2)$ to
$3x - 4y - 12z - 8 = 0$
$$d = \frac{|3(-1) - 4(1) - 12(2) - 8|}{\sqrt{3^2 + (-4)^2 + (-12)^2}}$$
$= 3$
b. The distance from $B(3, 1, -2)$ to
$8x - 8y + 4z - 7 = 0$
$$d = \frac{|8(3) - 8(1) + 4(-2) - 7|}{\sqrt{8^2 + (-8)^2 + (4)^2}}$$
$= \frac{1}{12}$ or 0.08

6. $\vec{r} = (3, 1, 1) + t(2, -1, 2), t \in \mathbf{R}$
$3x - 4y - 5z = 0$
Find the parametric equations from the first equation, then substitute those equations into the second equation. Solve for t. Substitute that t-value into the first equation.
$\vec{r} = (3, 1, 1) + t(2, -1, 2), t \in \mathbf{R}$
$x = 3 + 2t$

$y = 1 - t$
$z = 1 + 2t$
$3(3 + 2t) - 4(1 - t) - 5(1 + 2t) = 0$
t can be any value to satisfy this value, so the two equations intersect along
$\vec{r} = (3, 1, 1) + t(2, -1, 2), t \in \mathbf{R}$.

7. a. ① $3x - 4y + 5z = 9$
② $6x - 9y + 10z = 9$
③ $9x - 12y + 15z = 9$
$3 \times (3x - 4y + 5z = 9) = 9x - 12y + 15z = 27$
There is no solution because the first and third equations are inconsistent.

b. ① $2x + 3y + 4z = 3$
② $4x + 6y + 8z = 4$
③ $5x + y - z = 1$
$2 \times (2x + 3y + 4z = 3) = 4x + 6y + 8z = 6$
There is no solution because the first and second equations are inconsistent.

c. ① $4x - 3y + 2z = 2$
② $8x - 6y + 4z = 4$
③ $12x - 9y + 6z = 1$
$3 \times (4x - 3y + 2z = 2) = 12x - 9y + 6z = 6$
There is no solution because the first and third equations are inconsistent.

8. a. ① $3x + 4y + z = 4$
② $5x + 2y + 3z = 2$
③ $6x + 8y + 2z = 8$
(Equation ①) $-$ (2 $\times$ equation ②)
$= -7x - 5z = 0$
Letting $z = t$, then $x = -\frac{5}{7}t$ and $y = 1 + \frac{2}{7}t$.
$$x = -\frac{5}{7}t, y = 1 + \frac{2}{7}t, z = t, t \in \mathbf{R}$$

b. ① $4x - 8y + 12z = 4$
② $2x + 4y + 6z = 4$
③ $x - 2y - 3z = 4$
(Equation ①) + (4 $\times$ equation ③)
$= 24z = -12$ or $z = -\frac{1}{2}$. Letting $z = -\frac{1}{2}$ creates:

④ $4x - 8y = 10$
⑤ $2x + 4y = 7$
(Equation ①) + (2 $\times$ equation ②) $= 8x = 24$ or $x = 3$. Substituting in $x = 3$ and $z = -\frac{1}{2}$ gives $y = \frac{1}{4}$
$$x = 3, y = \frac{1}{4}, z = -\frac{1}{2}$$

c. ① $x - 3y + 3z = 7$
② $2x - 6y + 6z = 14$
③ $-x + 3y - 3z = -7$
Letting $z = s$, then $y = t$ gives $x - 3t + 3s = 7$ or $x = -3s + 3t + 7$
$x = 3t - 3s + 7, y = t, z = s, s, t \in \mathbf{R}$

9. a. ① $3x - 5y + 2z = 4$
① $6x + 2y - z = 2$
① $6x - 3y + 8z = 6$
(Equation ②) $-$ (2 $\times$ equation ①) $= 12y - 5z = -6$
Setting $z = t$,
$$12y - 5t = -6 \text{ or } y = -\frac{1}{2} + \frac{5}{12}t$$
Substituting these two values into the first equation gives $x = \frac{1}{2} + \frac{1}{36}t$
$$x = \frac{1}{2} + \frac{1}{36}t, y = \frac{-1}{2} + \frac{5}{12}t, z = t, t \in \mathbf{R}$$

b. ① $2x - 5y + 3z = 1$
② $4x + 2y + 5z = 5$
③ $2x + 7y + 2z = 4$
(Equation ②) $-$ (2 $\times$ equation ①)
$= 12y - z = 3$
Setting $z = t$,
$$12y - t = 3 \text{ or } y = \frac{1}{4} + \frac{1}{12}t$$
Substituting these two values into the first equation gives $x = \frac{9}{8} - \frac{31}{24}t$
$$x = \frac{9}{8} - \frac{31}{24}t, y = \frac{1}{4} + \frac{1}{12}t, z = t, t \in \mathbf{R}$$

10. a. $2x + y + z = 6$
$x - y - z = -9$
$3x + y = 2$
The first equation + the second equation gives $3x = -3$ or $x = -1$. Substituting $x = -1$ into the third equation, $3(-1) + y = 2$ or $y = 5$.
Substituting these two values into the first equation, $2(-1) + 5 + z = 6$ or $z = 3$
These three planes meet at the point $(-1, 5, 3)$.

b. ① $2x - y + 2z = 2$
② $3x + y - z = 1$
③ $x - 3y + 5z = 4$
Equation ① + equation ② $= 5x + z = 3$
Equation ③ $-$ (3 $\times$ equation ①) $= -5x - z = -2$.
These two equations are inconsistent, so the planes do not intersect at any point. Geometrically the planes form a triangular prism.

c. ① $2x + y - z = 0$
② $x - 2y + 3z = 0$
③ $9x + 2y - z = 0$

$2 \times$ equation ① + equation ② $= 5x + z = 0$, so $z = -5x$.
Equation ③ − equation ① $= 7x + y = 0$, so $y = -7x$.
Let $x = t$. The intersection of the planes is a line through the origin with equation $x = t, y = -7t, z = -5t, t \in \mathbf{R}$.

11. $\vec{r} = (2, -1, -2) + s(1, 1, -2), s \in \mathbf{R}$
By substituting in different s-values, you can find when the plane intersects the xz-plane when $y = 0$ and the xy-plane when $z = 0$.
The plane intersects the xz-plane at $(3, 0, -4)$ and the xy-plane at $(1, -2, 0)$. Then find the distance between these two points using the distance formula. The distance between these two points is 4.90.

12. a. $x - 2y + z + 4 = 0$
$\vec{r} = (3, 1, -5) + s(2, 1, 0), s \in \mathbf{R}$
$\vec{m} \cdot \vec{n} = (2, 1, 0) \cdot (1, -2, 1) = 0$ Since the line's direction vector is perpendicular to the normal of the plane and the point $(3, 1, -5)$ lies on both the line and the plane, the line is in the plane.

b. $\vec{r} = (7, 5, -1) + t(4, 3, 2), t \in \mathbf{R}$
$\vec{r} = (3, 1, -5) + s(2, 1, 0), s \in \mathbf{R}$
Solve for the parametric equations of both equations and then set them equal to each other.
$L_1: x = 7 + 4t, y = 5 + 3t, z = -1 + 2t$
$L_2: x = 3 + 2s, y = 1 + s, z = -5$
$z = -5 = -1 + 2t, t = -2$
$t = -2, x = -1, y = -1, z = -5$
$t = -2$ corresponds to the point $(-1, -1, -5)$

c. $x - 2y + z + 4 = 0$
$-1 - 2(-1) + (-5) + 4 = 0$
The point $(-1, -1, -5)$ is on the plane since it satisfies the equation of the plane.

d. $\vec{r} = (7, 5, -1) + t(4, 3, 2), t \in \mathbf{R}$
$(A, B, C) \cdot (4, 3, 2) = 0$
$A = 7, B = -2, C = -11$
$7x - 2y - 11z + D = 0$
$D = -50$
$7x - 2y - 11z - 50 = 0$

13. a. $\vec{r} = (3, 0, -1) + t(1, 1, 2), t \in \mathbf{R}$
$A(-2, 1, 1)$
$x = 3 + t, y = t, z = -1 + 2t$
$0 = 3 + t - x, 0 = t - y, 0 = -1 + 2t - z$
$\sqrt{(3 + t - x)^2 + (t - y)^2 + (-1 + 2t - z)^2}$
$\sqrt{(3 + t + 2)^2 + (t - 1)^2 + (-1 + 2t - 1)^2}$
$\sqrt{6t^2 + 30}$
$t = 0$ gives the lowest distance of 5.48

b. $t = 0$ corresponds to the point $(3, 0, -1)$

14. a. $\vec{r} = (1, -1, 1) + t(3, 2, 1), t \in \mathbf{R}$
$\vec{r} = (-2, -3, 0) + s(1, 2, 3), s \in \mathbf{R}$
Set the equations parametric equations equal to each other, and determine either the s or t-value. Find the point that corresponds to this value.
$L_1: x = 1 + 3t, y = -1 + 2t, z = 1 + t$
$L_2: x = -2 + s, y = -3 + 2s, z = 3s$
$x = 1 + 3t = -2 + s$
$y = -1 + 2t = -3 + 2s$
$z = 1 + t = 3s$
$s = 0, t = -1$
$s = 0$ corresponds to the point $(-2, -3, 0)$.

b. $\vec{r} = (1, -1, 1) + t(3, 2, 1), t \in \mathbf{R}$
$\vec{r} = (-2, -3, 0) + s(1, 2, 3), s \in \mathbf{R}$
$P(-2, -3, 0)$
$\vec{n_1} \times \vec{n_2} = (3, 2, 1) \times (1, 2, 3)$
$= (4, -8, 4) = (1, -2, 1)$
$\vec{r} = (-2, -3, 0) + t(1, -2, 1), t \in \mathbf{R}$

15. a. Since the plane we want contains L, we can use the direction vector for L, $(1, 2, -1)$, as one of the plane's direction vectors. Since the plane contains the point $(1, 2, -3)$ (which is on L) and the point $K(3, -2, 4)$, it will contain the direction vector $(3, -2, 4) - (1, 2, -3) = (2, -4, 7)$
To find a normal vector for the plane we want, take the cross product of these two direction vectors.
$(2, -4, 7) \times (1, 2, -1) = (-10, 9, 8)$
So the plane we seek will be of the form
$-10x + 9y + 8z + D = 0$.
To determine the value of D, substitute in the point $(1, 2, -3)$ that is to be on this plane.
$-10(1) + 9(2) + 8(-3) + D = 0$
$D = 16$
The equation of the plane we seek is
$-10x + 9y + 8z + 16 = 0$.

b. Using the distance formula, the distance from $S(1, 1, -1)$ to the plane $-10x + 9y + 8z + 16 = 0$ is

$$d = \frac{|-10(1) + 9(1) + 8(-1) + 16|}{|(-10, 9, 8)|}$$
$$= \frac{7}{\sqrt{245}}$$
$$\doteq 0.45$$

16. a. ① $x + y - z = 1$
② $2x - 5y + z = -1$
③ $7x - 7y - z = k$
Equation ① + equation ② = equation ④
$= 3x - 4y = 0$

Equation ② + equation ③ = equation ⑤
$= 9x - 12y = -1 + k$
For the solution to this system to be a line, equation ④ and equation ⑤ must be the proportional. $k = 1$ makes these two line proportional and the solution to this system a line.
b. In part a., we found that $k = 1$ by arriving at the equivalent system
① $3x - 4y = 0$
② $2x - 5y + z = -1$
③ $9x - 12y = 0$
As the first and third equations are proportional, this is really the same system as
① $3x - 4y = 0$
② $2x - 5y + z = -1$
Letting $x = t$ in the first equation, we see that $y = \frac{3}{4}t$. Substituting these values for x and y into the second equation, we find that
$$z = 5\left(\frac{3}{4}t\right) - 2t - 1$$
$$= \frac{7}{4}t - 1.$$
So the direction vector for the line that solves this system is $(1, \frac{3}{4}, \frac{7}{4})$, which is parallel to $(4, 3, 7)$.
So equivalent parametric equations of this line are
$x = 4t$
$y = 3t$
$z = -1 + 7t, t \in \mathbf{R}$.
So one possible vector equation of this line is
$\vec{r} = (0, 0, -1) + t(4, 3, 7), t \in \mathbf{R}$.
b. In part **a.**, we found that $k = 1$ by arriving at the equivalent system
① $3x - 4y = 0$
② $2x - 5y + z = -1$
③ $9x - 12y = 0$
As the first and third equations are proportional, this is really the same system as
① $3x - 4y = 0$
② $2x - 5y + z = -1$
Letting $x = t$ in the first equation, we see that $y = \frac{3}{4}t$. Substituting these values for x and y into the second equation, we find that
$$z = 5\left(\frac{3}{4}t\right) - 2t - 1$$
$$= \frac{7}{4}t - 1.$$
So the direction vector for the line that solves this system is $\left(1, \frac{3}{4}, \frac{7}{4}\right)$, which is parallel to $(4, 3, 7)$.
So equivalent parametric equations of this line are
$x = 4t$
$y = 3t$
$z = -1 + 7t, t \in \mathbf{R}$.
So one possible vector equation of this line is
$\vec{r} = (0, 0, -1) + t(4, 3, 7), t \in \mathbf{R}$.

17. a. ① $x + 2y + z = 1$
② $2x - 3y - z = 6$
③ $3x + 5y + 4z = 5$
④ $4x + y + z = 8$
Equation ① + equation ② = equation ⑤
$= 3x - y = 7$
(4 × equation ②) + equation ③ = equation ⑥
$= 11x - 7y = 29$
(7 × equation ⑤) + equation ⑥
= equation ⑦ $= -10x = -20y$ or $x = 2$
Substituting into equation ⑤: $6 - y = 7y = -1$.
Substituting into equation ①: $2 + -2 + z = 1$ or $z = 1$.
$(2, -1, 1)$
b. ① $x - 2y + z = 1$
② $2x - 5y + z = -1$
③ $3x - 7y + 2z = 0$
④ $6x - 14y + 4z = 0$
Equation ② − (2 × equation ①)
= equation ⑤ $= -y - z = -3$.
Setting $z = t$.
$-y - t = -3$ or $y = 3 - t$
Substituting $y = 3 - t$ and $z = t$ into equation ①:
$x - 2(3 - t) + t = 1$ or $x = 7 - 3t$
$x = 7 - 3t, y = 3 - t, z = t, t \in \mathbf{R}$

18. ① $\frac{9a}{b} - 8b + \frac{3c}{b} = 4$
② $-\frac{3a}{b} + 4b + \frac{4c}{b} = 3$
③ $\frac{3a}{b} + 4b - \frac{4c}{b} = 3$
$x = \frac{a}{b}, y = b, z = \frac{c}{b}$
① $9x - 8y + 3z = 4$
② $-3x + 4y + 4z = 3$
③ $3x + 4y - 4z = 3$
③ + ② $= 8y = 6$
$y = \frac{3}{4}$
① $9x + 3z = 10$
② $-3x + 4z = 0$
③ $3x - 4z = 0$
① + 3② $= 15z = 10$

$z = \frac{2}{3}, x = \frac{8}{9}$

$y = \frac{3}{4} = b,$

$x = \frac{8}{9} = \frac{a}{b} = \frac{a}{\frac{3}{4}}, a = \frac{2}{3}$

$z = \frac{2}{3} = \frac{c}{b} = \frac{c}{\frac{3}{4}}, c = \frac{1}{2}$

$\left(\frac{2}{3}, \frac{3}{4}, \frac{1}{2}\right)$

19. First put the equation into parametric form. Then substitute the x, y, and z-values into $x + 2y - z + 10 = 0$ to determine t. Then substitute t back into the parametric equations to determine the coordinates.

$\frac{x + 1}{-4} = \frac{y - 2}{3} = \frac{z - 1}{-2} = t$

$x = -4t - 1, y = 3t + 2, z = -2t + 1$

$x + 2y - 3z + 10 = 0$

$(-4t - 1) + 2(3t + 2) - 3(-2t + 1) + 10 = 0$

$t = -\frac{5}{4}$

$x = -4\left(-\frac{5}{4}\right) - 1, y = 3\left(-\frac{5}{4}\right) + 2,$

$z = -2\left(-\frac{5}{4}\right) + 1$

$\left(4, -\frac{7}{4}, \frac{7}{2}\right)$

20. Let $A'(a, b, c)$ denote the image point under this reflection. We want to find a, b, and c. The equation of the plane is $x - y + z - 1 = 0$, so letting $y = s$ and $z = t$, we get $x = 1 - t + s, s, t \in \mathbf{R}$. These are the parametric equations of this plane, so a general point on this plane has coordinates $P(1 - t + s, s, t)$.

So $\overrightarrow{PA} = (1, 0, 4) - (1 - t + s, s, t)$
$= (t - s, -s, 4 - t)$

The normal vector to this plane is $(1, -1, 1)$, and in order for $\overrightarrow{PA}$ to be perpendicular to the plane, it must be parallel to this normal. This means that $\overrightarrow{PA}$ and $(1, -1, 1)$ will have a cross product equal to the zero vector.

$(t - s, -s, 4 - t) \times (1, -1, 1)$
$= (4 - s - t, 4 + s - 2t, 2s - t)$
$= (0, 0, 0)$

So we get the system of equations

① $4 - s - t = 0$
② $4 + s - 2t = 0$
③ $2s - t = 0$

Adding the first two equations gives

$8 - 3t = 0$

$t = \frac{8}{3}$

Substituting this value for t into the third equation gives

$0 = 2s - t$
$= 2s - \frac{8}{3}$

$s = \frac{4}{3}$

Substituting these values for s and t into the equation for $\overrightarrow{PA}$, we get

$\overrightarrow{PA} = (t - s, -s, 4 - t)$
$= \left(\frac{8}{3} - \frac{4}{3}, -\frac{4}{3}, 4 - \frac{8}{3}\right)$
$= \left(\frac{4}{3}, -\frac{4}{3}, \frac{4}{3}\right)$

This is the vector that is normal to the plane, with its head at point $A(1, 0, 4)$ and tail at the point in the plane

$P(1 - t + s, s, t) = P\left(1 - \frac{8}{3} + \frac{4}{3}, \frac{4}{3}, \frac{8}{3}\right)$
$= \left(-\frac{1}{3}, \frac{4}{3}, \frac{8}{3}\right)$

So the vector

$\overrightarrow{PA'} = -\overrightarrow{PA}$
$= \left(-\frac{4}{3}, \frac{4}{3}, -\frac{4}{3}\right)$
$= (a, b, c) - \left(-\frac{1}{3}, \frac{4}{3}, \frac{8}{3}\right)$
$= \left(a + \frac{1}{3}, b - \frac{4}{3}, c - \frac{8}{3}\right)$

This means that $a = -\frac{5}{3}$, $b = -\frac{8}{3}$, and $c = -\frac{4}{3}$. That is, the reflected point is $A'\left(-\frac{5}{3}, \frac{8}{3}, \frac{4}{3}\right)$.

21. a. The first plane has normal $(3, 1, 7)$ and the second has normal $(4, -12, 4)$. Their line of intersection will be perpendicular to both of these normals. So we can take as direction vector the cross product of these two normals.

$(3, 1, 7) \times (4, -12, 4) = (88, 16, -40)$
$= 8(11, 2, -5)$

So let's use $(11, 2, -5)$ as the direction vector for this line of intersection. To find a point on both of these planes, solve for z in the second plane, and substitute this into the equation for the first plane.

$4x - 12y + 4z - 24 = 0$

$$4z = 24 - 4x + 12y$$
$$z = 6 - x + 3y$$
$$0 = 3x + y + 7z + 3$$
$$= 3x + y + 7(6 - x + 3y) + 3$$
$$= -4x + 22y + 45$$

If $y = 0$ in this last equation, then $x = \frac{45}{4}$ and

$$z = 6 - x + 3y$$
$$= 6 - \frac{45}{4} + 3(0)$$
$$= -\frac{21}{4}$$

The point $\left(\frac{45}{4}, 0, \frac{21}{4}\right)$. lies on both planes. So the vector equation of the line of intersection for the first two planes is

$$\vec{r} = \left(\frac{45}{4}, 0, -\frac{21}{4}\right) + t(11, 2, -5), t \in \mathbf{R}.$$

The corresponding parametric form is

$$x = \frac{45}{4} + 11t$$
$$y = 2t$$
$$z = -\frac{21}{4} - 5t, t \in \mathbf{R}.$$

We will use a similar procedure for the other two lines of intersection. For the third plane, the normal vector is $(1, 2, 3)$. So a direction vector for the line of intersection between the first and third planes is

$$(3, 1, 7) \times (1, 2, 3) = (-11, -2, 5)$$
$$= -(11, 2, -5)$$

We may use $(11, 2, -5)$ as the direction vector for this line of intersection. We find a point on both of these planes in the same way as before.

$$x + 2y + 3z - 4 = 0$$
$$x = 4 - 2y - 3z$$
$$0 = 3x + y + 7z + 3$$
$$= 3(4 - 2y - 3z) + y + 7z + 3$$
$$= -6y - 2z + 15$$

Taking $y = 0$ in this last equation, we get $z = \frac{15}{2}$ and

$$x = 4 - 2y - 3z$$
$$= 4 - 2(0) - 3\left(\frac{15}{2}\right)$$
$$= -\frac{37}{2}$$

A point on both the first and third planes is $\left(-\frac{37}{2}, 0, \frac{15}{2}\right)$. So the vector equation for this line of intersection is

$$\vec{r} = \left(-\frac{37}{2}, 0, \frac{15}{2}\right) + t(11, 2, -5), t \in \mathbf{R},$$

and the corresponding parametric equations are

$$x = -\frac{37}{2} + 11t$$
$$y = 2t$$
$$z = \frac{15}{2} - 5t, t \in \mathbf{R}.$$

Finally, we consider the line of intersection between the second and third planes. In this case, a direction vector is

$$(4, -12, 4) \times (1, 2, 3) = (-44, -8, 20)$$
$$= -4(11, 2, -5)$$

We may use $(11, 2, -5)$ as the direction vector for this line of intersection. We find a point on both of these planes in the same way as before.

$$x + 2y + 3z - 4 = 0$$
$$x = 4 - 2y - 3z$$
$$0 = 4x - 12y + 4z - 24$$
$$= 4(4 - 2y - 3z) - 12y + 4z - 24$$
$$= -20y - 8z - 8$$

Taking $y = 0$ in this last equation, we get $z = -1$ and

$$x = 4 - 2y - 3z$$
$$= 4 - 2(0) - 3(-1)$$
$$= 7$$

A point on both the second and third planes is $(7, 0, -1)$. So the vector equation for this line of intersection is

$$\vec{r} = (7, 0, -1) + t(11, 2, -5), t \in \mathbf{R},$$

and the corresponding parametric equations are

$$x = 7 + 11t$$
$$y = 2t$$
$$z = -1 - 5t, \in \mathbf{R}.$$

b. All three lines of intersection found in part a. have direction vector $(11, 2, -5)$, and so they are all parallel. Since no pair of normal vectors for these three planes is parallel, no pair of these planes is coincident.

22. ① $\frac{2}{a^2} + \frac{5}{b^2} + \frac{3}{c^2} = 40$

② $\frac{3}{a^2} - \frac{6}{b^2} - \frac{1}{c^2} = -3$

③ $\frac{9}{a^2} - \frac{5}{b^2} + \frac{4}{c^2} = 67$

$①+3②=④=\frac{11}{a^2}+\frac{-13}{b^2}=31$

$③+4②=⑤=\frac{21}{a^2}+\frac{-29}{b^2}=55$

$21④-11⑤=\frac{46}{b^2}=46, b=+1, b=-1$

$\frac{21}{a^2}+\frac{-29}{1}=55, a=\frac{1}{2}, a=-\frac{1}{2}$

$\frac{2}{0.25}+\frac{5}{1}+\frac{3}{c^2}=40, c=\frac{1}{3}, c=-\frac{1}{3}$

$a=\frac{1}{2}, a=-\frac{1}{2}, b=1, b=-1, c=\frac{1}{3}, c=-\frac{1}{3}$

Because each equation has each of a^2, b^2, and c^2, the possible solutions are all combinations of the positive and negative values for a, b, and c: $\left(\frac{1}{2},1,\frac{1}{3}\right)$, $\left(\frac{1}{2},1,-\frac{1}{3}\right)$, $\left(\frac{1}{2},-1,\frac{1}{3}\right)$, $\left(\frac{1}{2},-1,-\frac{1}{3}\right)$, $\left(-\frac{1}{2},1,\frac{1}{3}\right)$, $\left(-\frac{1}{2},1,-\frac{1}{3}\right)$, $\left(-\frac{1}{2},-1,\frac{1}{3}\right)$, and $\left(\frac{1}{2},-1,-\frac{1}{3}\right)$.

23. The general form of such a parabola is $y=ax^2+bx+c$. We need to determine a, b, and c. Since $(-1,2)$, $(1,-1)$, and $(2,1)$ all lie on the parabola, we get the system of equations

① $a-b+c=2$

② $a+b+c=-1$

③ $4a+2b+c=1$

Adding the first and second equations gives

$a+c=\frac{1}{2}$

Subtracting the first from the second equation gives

$2b=-3$

$b=-\frac{3}{2}$

Using the fact that $a+c=\frac{1}{2}$ and $b=-\frac{3}{2}$ in the third equation gives

$$\begin{aligned}1&=4a+2b+c\\&=3a+2b+(a+c)\\&=3a+2\left(-\frac{3}{2}\right)+\frac{1}{2}\\&=3a-\frac{5}{2}\end{aligned}$$

$\frac{7}{2}=3a$

$a=\frac{7}{6}$

So using once more that $a+c=\frac{1}{2}$, we substitute this value in for a and get

$\frac{1}{2}=a+c$

$=\frac{7}{6}+c$

$c=-\frac{2}{3}$

So the equation of the parabola we seek is

$y=\frac{7}{6}x^2-\frac{3}{2}x-\frac{2}{3}$.

24. The equation of the plane is $4x-5y+z-9=0$, which has normal $(4,-5,1)$. Converting this plane to parametric form gives

$x=s$

$y=t$

$z=9-4s+5t, s,t\in\mathbf{R}$.

So for any point $Y(s,t,9-4s+5t)$ on this plane, we can form the vector

$$\begin{aligned}\overrightarrow{XY}&=(s,t,9-4s+5t)-(3,2,-5)\\&=(s-3,t-2,14-4s+5t)\end{aligned}$$

This vector is perpendicular to the plane when it is parallel to the normal vector $(4,-5,1)$. Two vectors are parallel precisely when their cross product is the zero vector.

$(s-3,t-2,14-4s+5t)\times(4,-5,1)$
$=(68+26t-20s, 59+20t-17s, 23-4t-5s)$
$=(0,0,0)$

So we get the system of equations

① $68+26t-20s=0$

② $59+20t-17s=0$

③ $23-4t-5s=0$

Subtracting four times the third equation from the first equation gives

$42t-24=0$

$t=\frac{4}{7}$

Substituting this value for t into the second equation gives

$$\begin{aligned}0&=59+20t-17s\\&=59+20\left(\frac{4}{7}\right)-17s\end{aligned}$$

$17s=\frac{493}{7}$

$s=\frac{29}{7}$

Substituting these values for s and t into the equation for Y gives

$$Y(s,t,9-4s+5t)=Y\left(\frac{29}{7},\frac{4}{7},9-4\left(\frac{29}{7}\right)+5\left(\frac{4}{7}\right)\right)=\left(\frac{29}{7},\frac{4}{7},-\frac{33}{7}\right)$$

So the point M we wanted is $M\left(\frac{29}{7},\frac{4}{7},-\frac{33}{7}\right)$.

25. $\frac{11x^2 - 14x + 9}{(3x-1)(x^2+1)} = \frac{A}{3x-1} + \frac{Bx + C}{x^2+1}$

$\frac{11x^2 - 14x + 9}{(3x-1)(x^2+1)}$

$= \frac{A(x^2+1) + (Bx + C)3x - 1}{(3x-1)(x^2+1)}$

$11x^2 - 14x + 9 = (A + 3B)x^2 + (3C - B)x + (A - C)$

$A - C = 9, 3C - B = -14, A + 3B = 11$

$B = 3C + 14, A = C + 9$

$A + 3(3C + 14) = 11, A + 9C = -31$

$(C + 9) + 9C = -31$

$10C = -40, C = -4$

$B = 3(-4) + 14 = 2, A = (-4) + 9 = 5$

$A = 5, B = 2, C = -4$

26. a. The vector

$\overrightarrow{EF} = (-1, -4, -6) - (4, 0, 3)$
$= (-5, -4, -3)$

This is a direction vector for the line containing the segment EF. The point $E(-1, -4, -6)$ is on this line, so the vector equation of this line is

$\vec{r} = (-1, -4, -6) + t(-5, -4, -3), t \in \mathbf{R}$.

b. Based on the equation of the line found in part a., a general point on this line is of the form

$J(-1 - 5t, -4 - 4t, -6 - 3t), t \in \mathbf{R}$.

For this general point, the vector

$\overrightarrow{JD} = (3, 0, 7) - (-1 - 5t, -4 - 4t, -6 - 3t)$
$= (4 + 5t, 4 + 4t, 13 + 3t)$

This vector will be perpendicular to the direction vector for the line found in part a. at the point J we seek. This means that

$0 = (4 + 5t, 4 + 4t, 13 + 3t) \cdot (-5, -4, -3)$
$= -5(4 + 5t) - 4(4 + 4t) - 3(13 + 3t)$
$= -75 - 50t$

$t = -\frac{3}{2}$

Substituting this value of t into the equation for the general point on the line in part a.,

$J(-1 - 5t, -4 - 4t, -6 - 3t)$

$= J\left(-1 - 5\left(-\frac{3}{2}\right), -4 - 4\left(-\frac{3}{2}\right), -6 - 3\left(-\frac{3}{2}\right)\right)$

$= \left(\frac{13}{2}, 2, -\frac{3}{2}\right)$

These are the coordinates for the point J we wanted.

c. Using the coordinates for J found in part b.,

$\overrightarrow{JD} = (3, 0, 7) - \left(\frac{13}{2}, 2, -\frac{3}{2}\right)$

$= \left(-\frac{7}{2}, -2, \frac{17}{2}\right)$

This vector forms the height of $\triangle DEF$, and the length of this vector is

$|\overrightarrow{JD}| = \left|\left(-\frac{7}{2}, -2, \frac{17}{2}\right)\right|$

$= \sqrt{\left(-\frac{7}{2}\right)^2 + (-2)^2 + \left(\frac{17}{2}\right)^2}$

$= \sqrt{\frac{177}{72}}$

$\doteq 9.41$

The length of the base of $\triangle DEF$ is

$|\overrightarrow{EF}| = |(-5, -4, -3)|$
$= \sqrt{(-5)^2 + (-4)^2 + (-3)^2}$
$= \sqrt{50}$
$\doteq 7.07$

So the area of $\triangle DEF$ equals

$\frac{1}{2}(\sqrt{50})\left(\sqrt{\frac{177}{2}}\right) = \frac{5}{2}\sqrt{177}$

$\doteq 33.26 \text{ units}^2$

27. $3x - 2z + 1 = 0$

$4x + 3y + 7 = 0$

$(5, -5, 5)$

$\vec{n_1} \times \vec{n_2} = (3, 0, -2) \times (4, 3, 0) = (6, -8, 9)$

$6x - 8y + 9z + D = 0$

$D = -115$

$6x - 8y + 9z - 115 = 0$

Chapter 9 Test, p. 556

1. a. $\vec{r}_1 = (4, 2, 6) + s(1, 3, 11), s \in \mathbf{R}$,

$\vec{r}_2 = (5, -1, 4) + t(2, 0, 9), t \in \mathbf{R}$

L_1: $x = 4 + s, y = 2 + 3s, z = 6 + 11s$

L_2: $x = 5 + 2t, y = -1, z = 4 + 9t$

$y = -1 = 2 + 3s$

$s = -1$

L_1: $x = 4 + (-1), y = 2 + 3(-1),$
$z = 6 + 11(-1)$

$x = 3, y = -1, z = -5$

$(3, -1, -5)$

b. $x - y + z + 1 = 0$

$3 - (-1) + (-5) + 1 = 0$

$3 + 1 - 5 + 1 = 0$

$0 = 0$

2. Use the distance equation.

$d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$

a. $A(3, 2, 3)$

$8x - 8y + 4z - 7 = 0$

$$d = \frac{|8x_0 - 8y_0 + 4z_0 - 7|}{\sqrt{(8)^2 + (-8)^2 + (4)^2}}$$
$$= \frac{|8(3) - 8(2) + 4(3) - 7|}{\sqrt{(8)^2 + (-8)^2 + (4)^2}}$$
$$= \frac{13}{12} \text{ or } 1.08$$

b. First, find any point on one of the planes, then use the other plane equation with the distance formula.

$$2x - y + 2z - 16 = 0$$
$$2x - y + 2z + 24 = 0$$
$$2(8) - (0) + 2(0) - 16 = 0$$
$$A(8, 0, 0)$$
$$d = \frac{|2x_0 - 1y_0 + 2z_0 + 24|}{\sqrt{(2)^2 + (-1)^2 + (2)^2}}$$
$$= \frac{|2(8) - 1(0) + 2(0) + 24|}{\sqrt{(2)^2 + (-1)^2 + (2)^2}}$$
$$= \frac{40}{3} \text{ or } 13.33$$

3. a. $L_1: 2x + 3y - z = 3$

$L_2: -x + y + z = 1$

$L_1 + 2L_2: 5y + z = 5$

$$z = t,$$
$$5y + (t) = 5$$
$$y = 1 - \frac{t}{5}$$
$$-x + y + z = 1$$
$$-x + \left(1 - \frac{t}{5}\right) + (t) = 1$$
$$x = \frac{4t}{5}$$
$$x = \frac{4t}{5},\ y = 1 - \frac{t}{5},\ z = t,\ t \in \mathbf{R}$$

b. To determine the point of intersection with the xz-plane, set the above y parametric equation equal to 0 and solve for the t. This t corresponds to the point of intersection.

$$x = \frac{4t}{5},\ y = 1 - \frac{t}{5},\ z = t,\ t \in \mathbf{R}$$
$$0 = 1 - \frac{t}{5}$$
$$t = 5$$
$$x = \frac{4(5)}{5},\ y = 1 - \frac{(5)}{5},\ z = (5),\ t \in \mathbf{R}$$
$$(4, 0, 5)$$

4. a. ① $x - y + z = 10$

② $2x + 3y - 2z = -21$

③ $\frac{1}{2}x + \frac{2}{5}y + \frac{1}{4}z = -\frac{1}{2}$

Equation ② + (2 × equation ①) =

$4x + y = -1$

Equation ② + (8 × equation ③) =

$$6x + \frac{31}{5}y = -25$$
$$-\frac{31}{5}(4x + y = -1)$$
$$+\left(6x + \frac{31}{5}y = -25\right)$$
$$-18.8x = -18.8$$
$$x = 1$$
$$4(1) + y = -1$$
$$y = -5$$
$$(1) - (-5) + z = 10$$
$$z = 4$$
$$(1, -5, 4)$$

b. The three planes intersect at this point.

5. a. ① $x - y + z = -1$

② $2x + 2y - z = 0$

③ $x - 5y + 4z = -3$

Equation ② + (2 × equation ①) =

$4x + z = -2$

$$4x + z = -2$$
$$z = t$$
$$4x + (t) = -2$$
$$x = -\frac{1}{2} - \frac{t}{4}$$
$$x - y + z = -1$$
$$\left(-\frac{1}{2} - \frac{t}{4}\right) - y + (t) = -1$$
$$y = \frac{3t}{4} + \frac{1}{2}$$
$$x = -\frac{1}{2} - \frac{t}{4},\ y = \frac{3t}{4} + \frac{1}{2},\ z = t,\ t \in \mathbf{R}$$

b. The three planes intersect at this line.

6. a. $L_1: x + y + z = 0$

$L_2: x + 2y + 2z = 1$

$L_3: 2x - y + mz = n$

$L_2 + 2L_3: 5x + 0y + (2m + 2)z = 2n + 1$

$L_1 + L_3: 3x + 0y + (m + 1)z = n$

$$\frac{5}{3}(3x + 0y + (m + 1)z = n)$$
$$= 5x + 0y + \frac{5}{3}(m + 1)z = \frac{5}{3}n$$

Then set the two new equations to each other and solve for a m and n value that would give equivalent equations.

So we get that
$|\vec{x}| = 200$ and $|\vec{y}| = 200\sqrt{3}$. This means that the component of the weight of the mass parallel to the inclined plane is
$9.8 \times |\vec{x}| = 9.8 \times 200$
$= 1960$ N,
and the component of the weight of the mass perpendicular to the inclined plane is
$9.8 \times |\vec{y}| = 9.8 \times 200\sqrt{3}$
$\doteq 3394.82$ N.

35. a. True; all non-parallel pairs of lines intersect in exactly one point in R^2. However, this is not the case for lines in R^3 (skew lines provide a counterexample).

b. True; all non-parallel pairs of planes intersect in a line in R^3.

c. True; the line $x = y = z$ has direction vector $(1, 1, 1)$, which is not perpendicular to the normal vector $(1, -2, 2)$ to the plane $x - 2y + 2z = k$, k any constant. Since these vectors are not perpendicular, the line is not parallel to the plane, and so they will intersect in exactly one point.

d. False; a direction vector for the line $\frac{x}{2} = y - 1 = \frac{z + 1}{2}$ is $(2, 1, 2)$. A direction vector for the line $\frac{x - 1}{-4} = \frac{y - 1}{-2} = \frac{z + 1}{-2}$ is $(-4, -2, -2)$, or $(2, 1, 1)$ (which is parallel to $(-4, -2, -2)$). Since $(2, 1, 2)$ and $(2, 1, 1)$ are obviously not parallel, these two lines are not parallel.

36. a. A direction vector for
$L_1: x = 2, \frac{y - 2}{3} = z$
is $(0, 3, 1)$, and a direction vector for
$L_2: x = y + k = \frac{z + 14}{k}$
is $(1, 1, k)$. But $(0, 3, 1)$ is not a nonzero scalar multiple of $(1, 1, k)$ for any k since the first component of $(0, 3, 1)$ is 0. This means that the direction vectors for L_1 and L_2 are never parallel, which means that these lines are never parallel for any k.

b. If L_1 and L_2 intersect, in particular their x-coordinates will be equal at this intersection point. But $x = 2$ always in L_1 so we get the equation
$2 = y + k$
$y = 2 - k$
Also, from L_1 we know that $z = \frac{y - 2}{3}$, so substituting this in for z in L_2 we get
$2k = z + 14$
$2k = \frac{y - 2}{3} + 14$
$3(2k - 14) = y - 2$
$y = 6k - 40$
So since we already know that $y = 2 - k$, we now get
$2 - k = 6k - 40$
$7k = 42$
$k = 6$
So these two lines intersect when $k = 6$. We have already found that $x = 2$ at this intersection point, but now we know that
$y = 6k - 40$
$= 6(6) - 40$
$= -4$
$z = \frac{y - 2}{3}$
$= \frac{-4 - 2}{3}$
$= -2$
So the point of intersection of these two lines is $(2, -4, -2)$, and this occurs when $k = 6$.

Vector Appendix

Gaussian Elimination, pp. 588–590

1. a. First write the system in matrix form (omitting the variables and using only coefficients of each equation). The coefficients of the unknowns are entered in columns on the left side of a matrix with a vertical line separating the coefficients from the numbers on the right side.

① $x + 2y - z = -1$
② $-x + 3y - 2z = -1$
③ $3y - 2z = -3$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ -1 & 3 & -2 & -1 \\ 0 & 3 & -2 & -3 \end{array}\right]$$

b. ① $2x - z = 1$
② $2y - z = 16$
③ $-3x + y = 10$

$$\left[\begin{array}{ccc|c} 2 & 0 & -1 & 1 \\ 0 & 2 & -1 & 16 \\ -3 & 1 & 0 & 10 \end{array}\right]$$

c. ① $2x - y - z = -2$
② $x - y + 4z = -1$
③ $-x - y = 13$

$$\left[\begin{array}{ccc|c} 2 & -1 & -1 & -2 \\ 1 & -1 & 4 & -1 \\ -1 & -1 & 0 & 13 \end{array}\right]$$

2. Answers may vary. For example:

$$\left[\begin{array}{cc|c} 2 & 3 & 0 \\ 3 & -1 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1.5 & 0 \\ 3 & -1 & 1 \end{array}\right] \begin{array}{l} \frac{1}{2}(\text{row 1}) \\ \\ \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 1.5 & 0 \\ 0 & -5.5 & 1 \end{array}\right] \begin{array}{l} \\ -3(\text{row 1}) + \text{row 2} \end{array}$$

$$\left[\begin{array}{cc|c} 2 & 3 & 0 \\ 0 & -5.5 & 1 \end{array}\right] \begin{array}{l} 2(\text{row 1}) \\ \\ \end{array}$$

The last two matrices are both in row-echelon form and are equivalent.

3. Answers may vary. For example:

$$\left[\begin{array}{ccc|c} 2 & 1 & 6 & 0 \\ 0 & -2 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0.5 & 3 & 0 \\ 0 & -2 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{array}\right] \begin{array}{l} \frac{1}{2}(\text{row 1}) \\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0.5 & 3 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -0.5 & -9 & 1 \end{array}\right] \begin{array}{l} \\ \\ -3(\text{row 1}) + \text{row 3} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0.5 & 3 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -9.25 & 1 \end{array}\right] \begin{array}{l} \\ \\ -\frac{1}{4}(\text{row 2}) + \text{row 3} \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 6 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -37 & 4 \end{array}\right] \begin{array}{l} 2(\text{row 1}) \\ \\ 4(\text{row 3}) \end{array}$$

4. a. Answers may vary. For example:

$$\left[\begin{array}{ccc|c} -1 & 0 & 1 & 2 \\ 0 & -1 & 2 & 0 \\ \frac{1}{2} & -\frac{3}{4} & -2 & \frac{1}{3} \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & -1 & 2 & 0 \\ 6 & -9 & -24 & 4 \end{array}\right] \begin{array}{l} -1(\text{row 1}) \\ \\ 12(\text{row 3}) \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & -1 & 2 & 0 \\ 0 & -9 & -18 & 16 \end{array}\right] \begin{array}{l} \\ \\ -6(\text{row 1}) + \text{row 3} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -36 & 16 \end{array}\right] \begin{array}{l} \\ \\ -9(\text{row 2}) + \text{row 3} \end{array}$$

b.
$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -36 & 16 \end{array}\right]$$

$-36z = 16 \text{ or } z = -\frac{4}{9}$

$-y + 2z = 0$

$y + 2\left(\frac{4}{9}\right) = 0 \text{ or } y = \frac{8}{9}$

$x - z = -2$

$x - \left(-\frac{4}{9}\right) = -2 \text{ or } x = -\frac{22}{9}$

$x = -\frac{22}{9}, y = -\frac{8}{9}, z = -\frac{4}{9}$

5. a. Each row of an augmented matrix corresponds to an equation in a system. The numbers on the left of the vertical line correspond to the coefficients of the unknowns, while the number to the right of the vertical line correspond to numerical answer to the equation.

$$\left[\begin{array}{cc|c} 1 & -2 & -1 \\ 2 & -3 & 1 \\ 2 & -1 & 0 \end{array}\right]$$

$x - 2y = -1$
$2x - 3y = 1$
$2x - y = 0$

b. $$\left[\begin{array}{ccc|c} -2 & 0 & -1 & 0 \\ 1 & -2 & 0 & 4 \\ 0 & 1 & 2 & -3 \end{array}\right]$$

$-2x - z = 0$
$x - 2y = 4$
$y + 2z = -3$

c. $$\left[\begin{array}{ccc|c} 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -2 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$-z = 0$
$x = -2$
$y + z = 0$

6. a. This system can be solved using back substitution.

$$\left[\begin{array}{cc|c} -2 & 1 & 6 \\ 0 & -5 & 15 \end{array}\right]$$

$-5y = 15 \text{ or } y = -3$
$-2x + y = 6$

$-2x - 3 = 6 \text{ or } x = -\frac{9}{2}$

$x = -\frac{9}{2}, y = -3$

b. This system can be solved using back substitution.

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 11 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 6 & -36 \end{array}\right]$$

$6z = -36 \text{ or } z = -6$
$2y + 3z = 0$
$2y + 3(-6) = 0 \text{ or } y = 9$
$2x - y + z = 11$
$2x - (9) + (-6) = 11 \text{ or } x = 13$
$x = 13, y = 9, z = -6$

c. A solution does not exist to this system, because the last row has no variables, but is still equal to a non-zero number, which is not possible.

$$\left[\begin{array}{ccc|c} -1 & 3 & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & -13 \end{array}\right]$$

d. This system can be solved using back substitution.

$$\left[\begin{array}{ccc|c} 4 & -1 & -1 & 0 \\ 0 & -1 & 0 & 4 \\ 0 & 0 & -1 & 5 \end{array}\right]$$

$-z = 5 \text{ or } z = -5$
$-y = 4 \text{ or } y = -4$
$4x - y - z = 0$

$4x - (-4) - (-5) = 0 \text{ or } x = -\frac{9}{4}$

$x = -\frac{9}{4}, y = -4, z = -5$

e. $$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x - y + 3z = 2$
Set $z = t$ and $y = s$,
$x - (s) + 3(t) = 2 \text{ or } x = 2 - 3t + s$
$x = 2 - 3t + s, y = s, z = t, s, t \in \mathbf{R}$

f. This system can be solved using back substitution.

$$\left[\begin{array}{ccc|c} -1 & 0 & 0 & -4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & -1 & 2 \end{array}\right]$$

$-z = 2 \text{ or } z = -2$
$y + 2z = 4$
$y + 2(-2) = 4 \text{ or } y = 8$
$-x = -4 \text{ or } x = 4$
$x = 4, y = 8, z = -2$

7. a. This matrix is in row-echelon form, because it satisfies both properties of a matrix in row-echelon form.

1. All rows that consist entirely of zeros must be written at the bottom of the matrix.
2. In any two successive rows not consisting entirely of zeros, the first nonzero number in the lower row must occur further to the right that the first nonzero number in the row directly above.

b. A solution does not exist to this system, because the second row has no variables, but is still equal to a nonzero number, which is not possible.

c. Answers may vary. For example:

$$\left[\begin{array}{ccc|c} -1 & 1 & 1 & 3 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} -1 & 1 & 1 & 3 \\ -2 & 2 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{ row 2 + 2(row 1)}$$

$$\left[\begin{array}{ccc|c} -1 & 1 & 1 & 3 \\ -2 & 2 & 2 & 3 \\ -1 & 1 & 1 & 3 \end{array}\right] \text{ row 1 + row 3}$$

8. a. This matrix is not in row-echelon form. Answers may vary.

$$\left[\begin{array}{ccc|c} -1 & 0 & 1 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} -1 & 0 & 1 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \end{array}\right] \text{ row 3 - row 2}$$

b. This matrix is not in row-echelon form. Answers may vary.

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -3 \\ 3 & 1 & -4 & 2 \\ 0 & 0 & 3 & 6 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -3 \\ 0 & 1 & -10 & 11 \\ 0 & 0 & 3 & 6 \end{array}\right] \text{ row 2 - 3(row 1)}$$

c. This matrix is not in row-echelon form. Answers may vary. Switch row 2 and row 3 to obtain row-echelon form.

$$\left[\begin{array}{ccc|c} -1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -6 \end{array}\right] \text{ Interchange row 2 \& row 3}$$

$$\left[\begin{array}{ccc|c} -1 & 2 & 1 & 0 \\ 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

d. This matrix is in row-echelon form, because it satisfies both conditions of a matrix in row-echelon form.

1. All rows that consist entirely of zeros must be written at the bottom of the matrix.
2. If any two successive rows not consisting entirely of zeros, the first nonzero number in the lower row must occur further to the right that the first nonzero number in the row directly above.

9. a. i. $\left[\begin{array}{ccc|c} -1 & 0 & 1 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \end{array}\right]$

$2z = 1$ or $z = \frac{1}{2}$

$y = 0$

Using back substitution,

$-x + z = 3$

$-x + \frac{1}{2} = 3$ or $x = -\frac{5}{2}$

$x = -\frac{5}{2}, y = 0, z = \frac{1}{2}$

ii. $\left[\begin{array}{ccc|c} 1 & 0 & 2 & -3 \\ 0 & 1 & -10 & 11 \\ 0 & 0 & 3 & 6 \end{array}\right]$

$3z = 6$ or $z = 2$

Using back substitution,

$y - 10z = 11$

$y - 10(2) = 11$ or $y = 31$

$x + 2z = -3$

$x + 2(2) = -3$ or $x = -7$

$x = -7, y = 31, z = 2$

iii. $\left[\begin{array}{ccc|c} -1 & 2 & 1 & 0 \\ 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$z = -6$

$-x + 2y + z = 0$

Set $y = t$ and $z = -6$.

$-x + 2(t) + (-6) = 0$

$x = 2t - 6$

$x = 2t - 6, y = t, z = -6, t \in \mathbf{R}$

iv. $\left[\begin{array}{rrr|r} 1 & -4 & 1 & 0 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$y + 2z = -3$

Set $z = t$,

$y + 2(t) = -3$ or $y = -3 - 2t$

$x - 4y + z = 0$

$x - 4(-3 - 2t) + t = 0$

$x = -12 - 9t$

$x = -12 - 9t, y = -3 - 2t, z = t, t \in \mathbf{R}$

b. i. The solution is the point at which the three planes meet.

ii. The solution is the point at which the three planes meet.

iii. The solution is the line at which the three planes meet. There is one parameter t.

iv. The solution is the line at which the three planes meet. There is one parameter t.

10. a. ① $-x + y + z = 9$

② $x - 2y + z = 15$

③ $2x - y - z = -12$

$$\left[\begin{array}{rrr|r} -1 & 1 & 1 & 9 \\ 1 & -2 & 1 & 15 \\ 2 & -1 & -1 & -12 \end{array}\right]$$

$$\left[\begin{array}{rrr|r} -1 & 1 & 1 & 9 \\ 0 & -1 & 2 & 24 \\ 0 & 1 & 1 & 6 \end{array}\right] \begin{array}{l} \\ \text{row 1 + row 2} \\ \text{row 3 + 2(row 1)} \end{array}$$

$$\left[\begin{array}{rrr|r} -1 & 1 & 1 & 9 \\ 0 & -1 & 2 & 24 \\ 0 & 0 & 3 & 30 \end{array}\right] \begin{array}{l} \\ \\ \text{row 2 + row 3} \end{array}$$

$3z = 30$ or $z = 10$,

$-y + 2z = 24$

$-y + 2(10) = 24$ or $y = -4$

$-x + y + z = 9$

$-x + (-4) + (10) = 9$ or $x = -3$

$x = -3, y = -4, z = 10$

These three planes meet at the point $(-3, -4, 10)$.

b. ① $x + y + z = 0$

② $2x + 3y + z = 0$

③ $-3x - 2y - 4z = 0$

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 0 \\ 2 & 3 & 1 & 0 \\ -3 & -2 & -4 & 0 \end{array}\right]$$

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \begin{array}{l} \\ \text{row 2 - 2(row 1)} \\ \text{row 3 + 3(row 1)} \end{array}$$

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ \text{row 2 + row 3} \end{array}$$

$y - z = 0$

Set $z = t$,

$y = t$

$x + y + z = 0$

$x + t + t = 0$

$x = -2t$

$x = -2t, y = t, z = t, t \in \mathbf{R}$

These three planes meet at this line.

c. ① $x - y + 3z = -1$

② $5x + y - 3z = -5$

③ $2x + y - 3z = -2$

$$\left[\begin{array}{rrr|r} 1 & -1 & 3 & -1 \\ 5 & 1 & -3 & -5 \\ 2 & 1 & -3 & -2 \end{array}\right]$$

$$\left[\begin{array}{rrr|r} 1 & -1 & 3 & -1 \\ 0 & 6 & -18 & 0 \\ 0 & 3 & -9 & 0 \end{array}\right] \begin{array}{l} \\ \text{row 2 - 5(row 1)} \\ \text{row 3 - 2(row 1)} \end{array}$$

$$\left[\begin{array}{rrr|r} 1 & -1 & 3 & -1 \\ 0 & 6 & 18 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ \text{row 3} - \frac{1}{2}\text{(row 2)} \end{array}$$

$6y - 18z = 0$

Set $z = t$,

$6y - 18t = 0$ or $y = 3t$

$x - y + 3z = -1$

Set $z = t$ and $y = 3t$,

$x - (3t) + 3t = -1$ or $x = -1$

$x = -1, y = 3t, z = t, t \in \mathbf{R}$

These three planes meet at this line.

d. ① $x + 3y + 4z = 4$

② $-x + 3y + 8z = -4$

③ $x - 3y - 4z = -4$

$$\left[\begin{array}{rrr|r} 1 & 3 & 4 & 4 \\ -1 & 3 & 8 & -4 \\ 1 & -3 & -4 & -4 \end{array}\right]$$

$$\left[\begin{array}{rrr|r} 1 & 3 & 4 & 4 \\ 0 & 6 & 12 & 0 \\ 0 & -6 & -8 & -8 \end{array}\right] \begin{array}{l} \\ \text{row 2 + row 1} \\ \text{row 3 - row 1} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 4 \\ 0 & 6 & 12 & 0 \\ 0 & 0 & 4 & -8 \end{array}\right] \text{ row 3 + row 2}$$

$4z = -8$ or $z = -2$

$6y + 12z = 0$

$6y + 12(-2) = 0$

$y = 4$

$x + 3y + 4z = 4$

$x + 3(4) + 4(-2) = 4$

$x = 0$

$x = 0,\ y = 4,\ z = -2$

The three planes meet at the point $(0, 4, -2)$.

e. ① $2x + y + z = 1$

② $4x + 2y + 2z = 2$

③ $-2x + y + z = 3$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 4 & 2 & 2 & 2 \\ -2 & 1 & 1 & 3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 4 \end{array}\right] \begin{array}{l} \\ \text{row 2} - 2(\text{row 1}) \\ \text{row 3 + row 1} \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{ Interchange row 2 \& row 3}$$

$2y + 2z = 4$

Set $z = t$,

$2y + 2t = 4$ or $y = 2 - t$

$2x + y + z = 1$

$2x + (2 - t) + t = 1$

$x = -\frac{1}{2}$

$x = -\frac{1}{2},\ y = 2 - t,\ z = t,\ t \in \mathbf{R}$

The three planes meet at this line.

f. ① $x - y = -500$

② $2y - z = 3500$

③ $x - z = 2000$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & -500 \\ 0 & 2 & -1 & 3500 \\ 1 & 0 & -1 & 2000 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & -500 \\ 0 & 2 & -1 & 3500 \\ 0 & 1 & -1 & 2500 \end{array}\right] \text{ row 3 − row 1}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & -500 \\ 0 & 2 & -1 & 3500 \\ 0 & 0 & -0.5 & 750 \end{array}\right] \text{ row 3} - \frac{1}{2}(\text{row 2})$$

$-0.5z = 750$ or $z = -1500$

$2y - z = 3500$

$2y - (-1500) = 3500$ or $y = 1000$

$x - y = -500$

$x - (1000) = -500$ or $x = 500$

$x = 500,\ y = 1000,\ z = -1500$

The three planes meet at the point $(500, 1000, -1500)$.

11. $a = x + 2y - z$

$b = x - y + 2z$

$c = 3x + 3y + z$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & a \\ 1 & -1 & 2 & b \\ 3 & 3 & 1 & c \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & a \\ 0 & -3 & 3 & -a + b \\ 0 & -3 & 4 & -3a + c \end{array}\right] \begin{array}{l} \\ \text{row 2 − row 1} \\ \text{row 3} - 3(\text{row 1}) \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & a \\ 0 & -3 & 3 & -a + b \\ 0 & 0 & 1 & -2a - b + c \end{array}\right] \text{ row 3 − row 2}$$

$z = -2a - b + c$

$-3y + 3z = -a + b$

$-3y + 3(-2a - b + c) = -a + b$

$y = \frac{-5a - 4b + 3c}{3}$

$x + 2y - z = a$

$x + 2\left(\frac{-5a - 4b + 3c}{3}\right) - (-2a - b + c) = a$

$x = \frac{7a + 5b - 3c}{3}$

$x = \frac{7a + 5b - 3c}{3},\ y = \frac{-5a - 4b + 3c}{3},$

$z = -2a - b + c$

12. First write an equation corresponding to each point using the given equation $y = ax^2 + bx + c$. Then create a matrix corresponding to these three equations, and solve for a, b, and c.

$A(-1, -7): -7 = a(-1)^2 + b(-1) + c$

$-7 = a - b + c$

$B(2, 20): 20 = a(2)^2 + b(2) + c$

$20 = 4a + 2b + c$

$C(-3, -5): -5 = a(-3)^2 + b(-3) + c$

$-5 = 9a - 3b + c$

$$\left[\begin{array}{rrr|r} 1 & -1 & 1 & -7 \\ 4 & 2 & 1 & 20 \\ 9 & -3 & 1 & -5 \end{array}\right]$$

$$\left[\begin{array}{rrr|r} 1 & -1 & 1 & -7 \\ 0 & 6 & -3 & 48 \\ 0 & 6 & -8 & 58 \end{array}\right] \begin{array}{l} \\ \text{row 2} - 4(\text{row 1}) \\ \text{row 3} - 9(\text{row 1}) \end{array}$$

$$\left[\begin{array}{rrr|r} 1 & -1 & 1 & -7 \\ 0 & 6 & -3 & 48 \\ 0 & 0 & -5 & 10 \end{array}\right] \begin{array}{l} \\ \\ \text{row 3} - \text{row 2} \end{array}$$

$-5c = 10$ or $c = -2$

$6b - 3c = 48$

$6b - 3(-2) = 48$ or $b = 7$

$a - b + c = -7$ or $a = 2$

$a = 2, b = 7, c = -2$

$y = 2x^2 + 7x - 2$

13. First change the variables in the equations to easier variables to work with, and then use matrices to solve for the unknowns. Once you have figured out the unknowns you created, use back substitution to solve for p, q, and r.

① $pq - 2\sqrt{q} + 3rq = 8$

② $2pq - \sqrt{q} + 2rq = 7$

③ $-pq + \sqrt{q} + 2rq = 4$

Let $x = pq$, $y = \sqrt{q}$, and $z = rq$.

① $x - 2y + 3z = 8$

② $2x - y + 2z = 7$

③ $-x + y + 2z = 4$

$$\left[\begin{array}{rrr|r} 1 & -2 & 3 & 8 \\ 2 & -1 & 2 & 7 \\ -1 & 1 & 2 & 4 \end{array}\right]$$

$$\left[\begin{array}{rrr|r} 1 & -2 & 3 & 8 \\ 0 & 3 & -4 & -9 \\ 0 & -1 & 5 & 12 \end{array}\right] \begin{array}{l} \\ \text{row 2} - 2(\text{row 1}) \\ \text{row 3} + \text{row 1} \end{array}$$

$$\left[\begin{array}{rrr|r} 1 & -2 & 3 & 8 \\ 0 & 3 & -4 & -9 \\ 0 & 0 & \frac{11}{3} & 9 \end{array}\right] \begin{array}{l} \\ \\ \text{row 3} + \frac{1}{3}(\text{row 2}) \end{array}$$

$\frac{11}{3}z = 9$ or $z = \frac{27}{11}$

$3y - 4z = -9$

$3y - 4\left(\frac{27}{11}\right) = -9$

$y = \frac{3}{11}$

$x - 2y + 3z = 8$

$x - 2\left(\frac{3}{11}\right) + 3\left(\frac{27}{11}\right) = 8$

$x = \frac{13}{11}$

$x = \frac{13}{11}, y = \frac{3}{11}, z = \frac{27}{11}$

$y = \frac{3}{11} = \sqrt{q}$ or $q = \frac{9}{121}$

$x = \frac{13}{11} = pq = p\left(\frac{9}{121}\right)$

$p = \frac{143}{9}$

$z = \frac{27}{11} = rq = r\left(\frac{9}{121}\right)$

$r = 33$

$p = \frac{143}{9}, q = \frac{9}{121}, r = 33$

14. $$\left[\begin{array}{ccc|c} a & 1 & 1 & a \\ 1 & a & 1 & a \\ 1 & 1 & a & a \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & a & a \\ 1 & a & 1 & a \\ a & 1 & 1 & a \end{array}\right] \begin{array}{l} \text{Interchange row 1 \& row 3} \\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & a & a \\ 0 & a-1 & 1-a & 0 \\ 0 & 1-a & 1-a^2 & a-a^2 \end{array}\right] \begin{array}{l} \\ \text{row 2} - \text{row 1} \\ \text{row 3} - a(\text{row 1}) \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & a & a \\ 0 & a-1 & 1-a & 0 \\ 0 & 0 & (-a-2)(a-1) & a-a^2 \end{array}\right]$$

row 2 + row 3

Analyzing this matrix, you can tell that when $a = 1$, the matrix has a row of all zeros, which means that this matrix has an infinite number of solutions. If you substitute in $a = -2$, the matrix has a row of variables with the coefficient zero, but a non zero number, which means this system has no solution. Any other number that is substituted for a gives a unique solution.

a. $a = -2$

b. $a = 1$

c. $a \neq -2$ or $a \neq 1$

Gauss-Jordan Method for Solving Systems of Equations, pp. 594–595

1. a. $\left[\begin{array}{cc|c} -1 & 3 & 1 \\ 0 & -1 & 2 \end{array}\right]$

$\left[\begin{array}{cc|c} -1 & 0 & 7 \\ 0 & -1 & 2 \end{array}\right]$ row 1 + 3 (row 2)

$\left[\begin{array}{cc|c} 1 & 0 & -7 \\ 0 & 1 & -2 \end{array}\right]$ -1 (row 1), -1 (row 2)

b. $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & -1 & 0 \end{array}\right]$

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -1 & 0 \end{array}\right]$ row 1 + 2 (row 3), row 2 + 2 (row 3)

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right]$ -1 (row 3)

c. $\left[\begin{array}{ccc|c} 1 & 1 & 4 & 2 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$

$\left[\begin{array}{ccc|c} 1 & 0 & 6 & 1 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$ row 1 + row 2

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$ row 1 − 6 (row 3), row 2 − 2 (row 3)

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$ -1 (row 2)

d. $\left[\begin{array}{ccc|c} 0 & 0 & -1 & 4 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & -1 \end{array}\right]$

$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 4 \end{array}\right]$ Rearrange rows

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -4 \end{array}\right]$ row 1 + 2 (row 3), -1 (row 3)

2. The solution to each reduced row-echelon matrix is the numbers corresponding to each leading 1.

a. $(-7, -2)$

b. $(3, 2, 0)$

c. $(1, 1, 0)$

d. $(8, -1, -4)$

3. a. ① $x - y + z = 0$

② $x + 2y - z = 8$

③ $2x - 2y + z = -11$

$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 1 & 2 & -1 & 8 \\ 2 & -2 & 1 & -11 \end{array}\right]$

$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 3 & -2 & 8 \\ 0 & 0 & -1 & -11 \end{array}\right]$ row 2 − row 1, row 3 − 2 (row 1)

$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -\frac{2}{3} & \frac{8}{3} \\ 0 & 0 & -1 & -11 \end{array}\right]$ $\frac{1}{3}$ (row 2)

$\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{3} & \frac{8}{3} \\ 0 & 1 & -\frac{2}{3} & \frac{8}{3} \\ 0 & 0 & -1 & -11 \end{array}\right]$ row 1 + row 2

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 11 \end{array}\right]$ row 1 + $\frac{1}{3}$ row 3, row 2 − $\frac{2}{3}$ row 3, -1 (row 3)

The solution to this system of equations is $x = -1$, $y = 10$, and $z = 11$.

b. ① $3x - 2y + z = 6$

② $x - 3y - 2z = -26$

③ $-x + y + z = 9$

$\left[\begin{array}{ccc|c} 3 & -2 & 1 & 6 \\ 1 & -3 & -2 & -26 \\ -1 & 1 & 1 & 9 \end{array}\right]$

$\left[\begin{array}{ccc|c} 1 & -3 & -2 & -26 \\ 3 & -2 & 1 & 6 \\ -1 & 1 & 1 & 9 \end{array}\right]$ Interchange rows 1 & 2

$\left[\begin{array}{ccc|c} 1 & -3 & -2 & -26 \\ 0 & 7 & 7 & 84 \\ 0 & -2 & -1 & -17 \end{array}\right]$ row 2 − 3(row 1), row 3 + row 1

$$\left[\begin{array}{ccc|c} 1 & -3 & -2 & -26 \\ 0 & 1 & 1 & 12 \\ 0 & -2 & -1 & -17 \end{array}\right] \frac{1}{7}(\text{row } 2)$$

$$\left[\begin{array}{ccc|c} 1 & -3 & -2 & -26 \\ 0 & 1 & 1 & 12 \\ 0 & 0 & 1 & 7 \end{array}\right] \text{row } 3 + 2(\text{row } 2)$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 10 \\ 0 & 1 & 1 & 12 \\ 0 & 0 & 1 & 7 \end{array}\right] \text{row } 1 + 3(\text{row } 2)$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 7 \end{array}\right] \begin{array}{l} \text{row } 1 - \text{row } 3 \\ \text{row } 2 - \text{row } 3 \\ \ \end{array}$$

The solution to this system of equations is $x = 3$, $y = 5$, and $z = 7$.

c. ① $2x + 2y + 5z = -14$
② $-x + z = -5$
③ $y - z = 6$

$$\left[\begin{array}{ccc|c} 2 & 2 & 5 & -14 \\ -1 & 0 & 1 & -5 \\ 0 & 1 & -1 & 6 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} -1 & 0 & 1 & -5 \\ 2 & 2 & 5 & -14 \\ 0 & 1 & -1 & 6 \end{array}\right] \text{Interchange rows 1 \& 2}$$

$$\left[\begin{array}{ccc|c} -1 & 0 & 1 & -5 \\ 0 & 2 & 7 & -24 \\ 0 & 1 & -1 & 6 \end{array}\right] \text{row } 2 + 2(\text{row } 1)$$

$$\left[\begin{array}{ccc|c} -1 & 0 & 1 & -5 \\ 0 & 2 & 7 & -24 \\ 0 & 0 & -\frac{9}{2} & 18 \end{array}\right] \text{row } 3 - \frac{1}{2}(\text{row } 2)$$

$$\left[\begin{array}{ccc|c} -1 & 0 & 1 & -5 \\ 0 & 2 & 7 & -24 \\ 0 & 0 & 1 & -4 \end{array}\right] -\frac{2}{9}(\text{row } 3)$$

$$\left[\begin{array}{ccc|c} -1 & 0 & 0 & -1 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & 1 & -4 \end{array}\right] \begin{array}{l} \text{row } 1 - \text{row } 3 \\ \text{row } 2 - 7(\text{row } 3) \\ \ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -4 \end{array}\right] \begin{array}{l} -1(\text{row } 1) \\ \frac{1}{2}(\text{row } 2) \\ \ \end{array}$$

The solution to this system of equations is $x = 1$, $y = 2$, and $z = -4$.

d. ① $x - y - 3z = 3$
② $2x + 2y + z = -1$
③ $-x - y + z = -1$

$$\left[\begin{array}{ccc|c} 1 & -1 & -3 & 3 \\ 2 & 2 & 1 & -1 \\ -1 & -1 & 1 & -1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -3 & 3 \\ 0 & 4 & 7 & -7 \\ 0 & -2 & -2 & 2 \end{array}\right] \begin{array}{l} \ \\ \text{row } 2 - 2(\text{row } 1) \\ \text{row } 3 + \text{row } 1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -3 & 3 \\ 0 & 4 & 7 & -7 \\ 0 & 0 & \frac{3}{2} & -\frac{3}{2} \end{array}\right] \text{row } 3 + \frac{1}{2}(\text{row } 2)$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -3 & 3 \\ 0 & 4 & 7 & -7 \\ 0 & 0 & 1 & -1 \end{array}\right] \frac{2}{3}(\text{row } 3)$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -3 & 3 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right] \text{row } 2 - 7(\text{row } 3)$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -3 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right] \frac{1}{4}(\text{row } 2)$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right] \text{row } 1 + \text{row } 2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right] \text{row } 1 + 3(\text{row } 3)$$

The solution to this system of equations is $x = 0$, $y = 0$, and $z = -1$.

e. ① $\frac{1}{2}x + 9y - z = 1$
② $x - 6y + z = -6$
③ $2x + 3y - z = -7$

$$\left[\begin{array}{ccc|c} \frac{1}{2} & 9 & -1 & 1 \\ 1 & -6 & 1 & -6 \\ 2 & 3 & -1 & -7 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -6 & 1 & -6 \\ \frac{1}{2} & 9 & -1 & 1 \\ 2 & 3 & -1 & -7 \end{array}\right] \text{Interchange rows 1 \& 2}$$

$$\left[\begin{array}{ccc|c} 1 & -6 & 1 & -6 \\ 1 & 18 & -2 & 2 \\ 2 & 3 & -1 & -7 \end{array}\right] \begin{array}{l} \\ 2(\text{row } 2) \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -6 & 1 & -6 \\ 0 & 24 & -3 & 8 \\ 0 & 15 & -3 & 5 \end{array}\right] \begin{array}{l} \\ \text{row } 2 - \text{row } 1 \\ \text{row } 3 - 2(\text{row } 1) \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{4} & -4 \\ 0 & 24 & -3 & 8 \\ 0 & 0 & -\frac{9}{8} & 0 \end{array}\right] \begin{array}{l} \text{row } 1 + \frac{1}{4}(\text{row } 2) \\ \\ \text{row } 3 - \frac{15}{24}(\text{row } 2) \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{4} & -4 \\ 0 & -24 & -3 & 8 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} \\ \\ -\frac{8}{9}(\text{row } 3) \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 2 & 3 & \frac{2}{3} \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} \text{row } 1 - \frac{1}{4}(\text{row } 3) \\ \text{row } 2 + 3(\text{row } 3) \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 2 & 0 & \frac{2}{3} \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} \\ \text{row } 2 - 3(\text{row } 3) \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} \\ \frac{1}{2}(\text{row } 2) \\ \\ \end{array}$$

The solution to this system of equations is $x = -4$, $y = \frac{1}{3}$, and $z = 0$.

f. ① $2x + 3y + 6z = 3$
② $x - y - z = 0$
③ $4x + 3y - 6z = 2$

$$\left[\begin{array}{ccc|c} 2 & 3 & 6 & 3 \\ 1 & -1 & -1 & 0 \\ 4 & 3 & -6 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 2 & 3 & 6 & 3 \\ 4 & 3 & -6 & 2 \end{array}\right] \begin{array}{l} \text{Interchange row 1 \& row 2} \\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 5 & 8 & 3 \\ 0 & 7 & -2 & 2 \end{array}\right] \begin{array}{l} \\ \text{row } 2 - 2(\text{row } 1) \\ \text{row } 3 - 4(\text{row } 1) \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 5 & 8 & 3 \\ 0 & 0 & -\frac{66}{5} & -\frac{11}{5} \end{array}\right] \begin{array}{l} \\ \\ \text{row } 3 - \frac{7}{5}(\text{row } 2) \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{3}{5} & \frac{3}{5} \\ 0 & 5 & 8 & 3 \\ 0 & 0 & 1 & \frac{1}{6} \end{array}\right] \begin{array}{l} \text{row } 1 + \frac{1}{5}(\text{row } 2) \\ \\ -\frac{5}{66}(\text{row } 3) \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 5 & 0 & \frac{5}{3} \\ 0 & 0 & 1 & \frac{1}{6} \end{array}\right] \begin{array}{l} \text{row } 1 - \frac{3}{5}(\text{row } 3) \\ \text{row } 2 - 8(\text{row } 3) \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{6} \end{array}\right] \begin{array}{l} \\ \frac{1}{5}(\text{row } 2) \\ \\ \end{array}$$

The solution to this system of equations is $x = \frac{1}{2}$, $y = \frac{1}{3}$, and $z = \frac{1}{6}$.

4. a. ① $2x + y - z = -6$
② $x - y + 2z = 9$
③ $-x + y + z = 9$

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & -6 \\ 1 & -1 & 2 & 9 \\ -1 & 1 & 1 & 9 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 9 \\ 2 & 1 & -1 & -6 \\ -1 & 1 & 1 & 9 \end{array}\right] \begin{array}{l} \text{Interchange row 1 \& row 2} \\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 9 \\ 0 & 3 & -5 & -24 \\ 0 & 0 & 3 & 18 \end{array}\right] \begin{array}{l} \\ \text{row } 2 - 2(\text{row } 1) \\ \text{row } 3 + \text{row } 1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{3} & 1 \\ 0 & 3 & -5 & -24 \\ 0 & 0 & 1 & 6 \end{array}\right] \begin{array}{l} \text{row } 1 + \frac{1}{3}(\text{row } 2) \\ \\ \frac{1}{3}(\text{row } 3) \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 3 & 0 & 6 \\ 0 & 0 & 1 & 6 \end{array}\right] \begin{array}{l} \text{row } 1 - \frac{1}{3}(\text{row } 3) \\ \text{row } 2 + 5(\text{row } 3) \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 6 \end{array}\right] \begin{array}{l} \\ \frac{1}{3}(\text{row } 2) \\ \\ \end{array}$$

The solution to this system of equations is $x = -1$, $y = 2$, and $z = 6$.

b. ① $x - y + z = -30$

② $-2x + y + 6z = 90$

③ $2x - y - z = -20$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -30 \\ -2 & 1 & 6 & 90 \\ 2 & -1 & -1 & -20 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -30 \\ -2 & 1 & 6 & 90 \\ 0 & 0 & 5 & 70 \end{array}\right] \begin{array}{l} \\ \\ \text{row } 2 + \text{row } 3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -30 \\ 0 & -1 & 8 & 30 \\ 0 & 0 & 1 & 14 \end{array}\right] \begin{array}{l} \\ \text{row } 2 + 2(\text{row } 1) \\ \frac{1}{5}(\text{row } 3) \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -7 & -60 \\ 0 & -1 & 8 & 30 \\ 0 & 0 & 1 & 14 \end{array}\right] \begin{array}{l} \text{row } 1 - 1(\text{row } 2) \\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 38 \\ 0 & -1 & 0 & -82 \\ 0 & 0 & 1 & 14 \end{array}\right] \begin{array}{l} \text{row } 1 + 7(\text{row } 3) \\ \text{row } 2 - 8(\text{row } 3) \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 38 \\ 0 & 1 & 0 & 82 \\ 0 & 0 & 1 & 14 \end{array}\right] \begin{array}{l} \\ -1(\text{row } 2) \\ \\ \end{array}$$

The solution to this system of equations is $x = 38$, $y = 82$, and $z = 14$.

5. ① $x + y + z = -1$

② $x - y + z = 2$

③ $3x - y + 3z = k$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 2 \\ 3 & -1 & 3 & k \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & -2 & 0 & 3 \\ 0 & -4 & 0 & k+3 \end{array}\right] \begin{array}{l} \\ \text{row } 2 - \text{row } 1 \\ \text{row } 3 - 3(\text{row } 1) \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & -2 & 0 & 3 \\ 0 & 0 & 0 & k-3 \end{array}\right] \begin{array}{l} \\ \\ \text{row } 3 - 2(\text{row } 2) \end{array}$$

a. There is an infinite number of solutions for this system when $k = 3$, because this creates a zero row.

b. The system will have no solution when $k \neq 3$, $k \in \mathbf{R}$ because there will be a row of zeros equal to a nonzero number, which is not possible.

c. The system cannot have a unique solution, because the matrix cannot be put in reduced row-echelon form.

6. a. Every homogeneous system has at least one solution, because (0, 0, 0) satisfies each equation.

b. ① $2x - y + z = 0$

② $x + y + z = 0$

③ $5x - y + 3z = 0$

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 5 & -1 & 3 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & -1 & 1 & 0 \\ 5 & -1 & 3 & 0 \end{array}\right] \begin{array}{l} \text{Interchange row 1 and row 2} \\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -3 & -1 & 0 \\ 0 & -6 & -2 & 0 \end{array}\right] \begin{array}{l} \text{row } 2 - 2(\text{row } 1) \\ \text{row } 3 - 5(\text{row } 1) \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -3 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ \text{row } 3 - 2(\text{row } 2) \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{2}{3} & 0 \\ 0 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \text{row } 1 + \frac{1}{3}(\text{row } 2) \\ -\frac{1}{3}(\text{row } 2) \\ \\ \end{array}$$

The reduced row-echelon form shows that the intersection of these planes is a line that goes through the point (0, 0, 0). $x = -\frac{2}{3}t$, $y = -\frac{1}{3}t$, $z = t$, $t \in \mathbf{R}$

7. ① $\frac{1}{x} - \frac{2}{y} + \frac{6}{z} = \frac{5}{6}$

② $\frac{2}{x} - \frac{3}{y} + \frac{12}{z} = 2$

③ $\frac{3}{x} + \frac{6}{y} + \frac{2}{z} = \frac{23}{6}$

Let the variables be $\frac{1}{x}$, $\frac{1}{y}$, and $\frac{1}{z}$

$$\left[\begin{array}{ccc|c} 1 & -2 & 6 & \frac{5}{6} \\ 2 & -3 & 12 & 2 \\ 3 & 6 & 2 & \frac{23}{6} \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 6 & \frac{5}{6} \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 12 & -16 & \frac{4}{3} \end{array}\right] \begin{array}{l} \\ \text{row 2} - 2(\text{row 1}) \\ \text{row 3} - 3(\text{row 1}) \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 6 & \frac{3}{2} \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & -16 & -\frac{8}{3} \end{array}\right] \begin{array}{l} \text{row 1} + 2(\text{row 2}) \\ \\ \text{row 3} - 12(\text{row 2}) \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{6} \end{array}\right] \begin{array}{l} \text{row 1} + \frac{6}{16}(\text{row 3}) \\ \\ -\frac{1}{16}(\text{row 3}) \end{array}$$

$\frac{1}{x} = \frac{1}{2}, x = 2$

$\frac{1}{y} = \frac{1}{3}, y = 3$

$\frac{1}{z} = \frac{1}{6}, z = 6$

$(2, 3, 6)$