Chapter 3 Test

1. Determine the second derivative of each of the following:

   a. $y = 7x^2 - 9x + 22$

   b. $f(x) = -9x^5 - 4x^3 + 6x - 12$

   c. $y = 5x^{-3} + 10x^3$

   d. $f(x) = (4x - 8)^3$

2. For each of the following displacement functions, calculate the velocity and acceleration at the indicated time:

   a. $s(t) = -3t^3 + 5t^2 - 6t,\ t = 3$

   b. $s(t) = (2t - 5)^3,\ t = 2$

3. The position function of an object moving horizontally along a straight line as a function of time is $s(t) = t^2 - 3t + 2,\ t \geq 0$, in metres, at time t, in seconds.

   a. Determine the velocity and acceleration of the object.

   b. Determine the position of the object when the velocity is 0.

   c. Determine the speed of the object when the position is 0.

   d. When does the object move to the left?

   e. Determine the average velocity from $t = 2$ to $t = 5$.

4. Determine the maximum and minimum of each function on the given interval.

   a. $f(x) = x^3 - 12x + 2,\ -5 \leq x \leq 5$

   b. $f(x) = x + \frac{9}{x},\ x \in [1, 6]$

5. After a football is punted, its height, h, in metres above the ground at t seconds, can be modelled by $h(t) = -4.9t^2 + 21t + 0.45,\ t \geq 0$.

   a. When does the football reach its maximum height?

   b. What is the football's maximum height?

6. A man purchased 2000 m of used wire fencing at an auction. He and his wife want to use the fencing to create three adjacent rectangular paddocks. Find the dimensions of the paddocks so that the fence encloses the largest possible area.

7. An engineer working on a new generation of computer called The Beaver is using compact VLSI circuits. The container design for the CPU is to be determined by marketing considerations and must be rectangular in shape. It must contain exactly 10 000 cm^3 of interior space, and the length must be twice the height. If the cost of the base is $\$0.02/cm^2$, the cost of the side walls is $\$0.05/cm^2$, and the cost of the upper face is $\$0.10/cm^2$, find the dimensions to the nearest millimetre that will keep the cost of the container to a minimum.

8. The landlord of a 50-unit apartment building is planning to increase the rent. Currently, residents pay \$850 per month, and all the units are occupied. A real estate agency advises that every \$100 increase in rent will result in 10 vacant units. What rent should the landlord charge to maximize revenue?

Chapter 4

CURVE SKETCHING

If you are having trouble figuring out a mathematical relationship, what do you do? Many people find that visualizing mathematical problems is the best way to understand them and to communicate them more meaningfully. Graphing calculators and computers are powerful tools for producing visual information about functions. Similarly, since the derivative of a function at a point is the slope of the tangent to the function at this point, the derivative is also a powerful tool for providing information about the graph of a function. It should come as no surprise, then, that the Cartesian coordinate system in which we graph functions and the calculus that we use to analyze functions were invented in close succession in the seventeenth century. In this chapter, you will see how to draw the graph of a function using the methods of calculus, including the first and second derivatives of the function.

CHAPTER EXPECTATIONS

In this chapter, you will

- determine properties of the graphs of polynomial and rational functions, **Sections 4.1, 4.3, 4.5**
- describe key features of a given graph of a function, **Sections 4.1, 4.2, 4.4**
- determine intercepts and positions of the asymptotes of a graph, **Section 4.3**
- determine the values of a function near its asymptotes, **Section 4.3**
- determine key features of the graph of a function, **Section 4.5, Career Link**
- sketch, by hand, the graph of the derivative of a given graph, **Section 4.2**
- determine, from the equation of a simple combination of polynomial or rational functions $\left(\text{such as } f(x) = x^2 + \frac{1}{x}\right)$, the key features of the graph of the function, using the techniques of differential calculus, and sketch the graph by hand, **Section 4.4**

Review of Prerequisite Skills

There are many features that we can analyze to help us sketch the graph of a function. For example, we can try to determine the x- and y-intercepts of the graph, we can test for horizontal and vertical asymptotes using limits, and we can use our knowledge of certain kinds of functions to help us determine domains, ranges, and possible symmetries.

In this chapter, we will use the derivatives of functions, in conjunction with the features mentioned above, to analyze functions and their graphs. Before you begin, you should

- be able to solve simple equations and inequalities
- know how to sketch graphs of parent functions and simple transformations of these graphs (including quadratic, cubic, and root functions)
- understand the intuitive concept of a limit of a function and be able to evaluate simple limits
- be able to determine the derivatives of functions using all known rules

Exercise

1. Solve each equation.

a. $2y^2 + y - 3 = 0$

b. $x^2 - 5x + 3 = 17$

c. $4x^2 + 20x + 25 = 0$

d. $y^3 + 4y^2 + y - 6 = 0$

2. Solve each inequality.

a. $3x + 9 < 2$

b. $5(3 - x) \geq 3x - 1$

c. $t^2 - 2t < 3$

d. $x^2 + 3x - 4 > 0$

3. Sketch the graph of each function.

a. $f(x) = (x + 1)^2 - 3$

b. $f(x) = x^2 - 5x - 6$

c. $f(x) = \dfrac{2x - 4}{x + 2}$

d. $f(x) = \sqrt{x - 2}$

4. Evaluate each limit.

a. $\lim\limits_{x \to 2^-} (x^2 - 4)$

b. $\lim\limits_{x \to 2} \dfrac{x^2 + 3x - 10}{x - 2}$

c. $\lim\limits_{x \to 3} \dfrac{x^3 - 27}{x - 3}$

d. $\lim\limits_{x \to 4^+} \sqrt{2x + 1}$

5. Determine the derivative of each function.

a. $f(x) = \frac{1}{4}x^4 + 2x^3 - \frac{1}{x}$

b. $f(x) = \frac{x + 1}{x^2 - 3}$

c. $f(x) = (3x^2 - 6x)^2$

d. $f(t) = \frac{2t}{\sqrt{t - 4}}$

6. Divide, and then write your answer in the form $ax + b + \frac{r}{q(x)}$. For example, $(x^2 + 4x - 5) \div (x - 2) = x + 6 + \frac{7}{x - 2}$.

a. $(x^2 - 5x + 4) \div (x + 3)$

b. $(x^2 + 6x - 9) \div (x - 1)$

7. Determine the points where the tangent is horizontal to $f(x) = x^3 + 0.5x^2 - 2x + 3$.

8. State each differentiation rule in your own words.

a. power rule

b. constant rule

c. product rule

d. quotient rule

e. chain rule

f. power of a function rule

9. Describe the end behaviour of each function as $x \to \infty$ and $x \to -\infty$.

a. $f(x) = 2x^2 - 3x + 4$

b. $f(x) = -2x^3 + 4x - 1$

c. $f(x) = -5x^4 + 2x^3 - 6x^2 + 7x - 1$

d. $f(x) = 6x^5 - 4x - 7$

10. For each function, determine the reciprocal, $y = \frac{1}{f(x)}$, and the equations of the vertical asymptotes of $y = \frac{1}{f(x)}$. Verify your results using graphing technology.

a. $f(x) = 2x$

b. $f(x) = -x + 3$

c. $f(x) = (x + 4)^2 + 1$

d. $f(x) = (x + 3)^2$

11. State the equation of the horizontal asymptote of each function.

a. $y = \frac{5}{x + 1}$

b. $y = \frac{4x}{x - 2}$

c. $y = \frac{3x - 5}{6x - 3}$

d. $y = \frac{10x - 4}{5x}$

12. For each function in question 11, determine the following:

a. the x- and y-intercepts

b. the domain and range

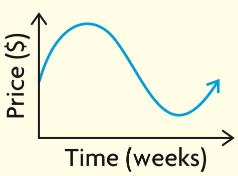
CAREER LINK Investigate

CHAPTER 4: PREDICTING STOCK VALUES

Stock-market analysts collect and interpret vast amounts of information and then predict trends in stock values. Stock analysts are classified into two main groups: the fundamentalists who predict stock values based on analysis of the companies' economic situations, and the technical analysts who predict stock values based on trends and patterns in the market. Technical analysts spend a significant amount of their time constructing and interpreting graphical models to find undervalued stocks that will give returns in excess of what the market predicts. In this chapter, your skills in producing and analyzing graphical models will be extended through the use of differential calculus.

Case Study: Technical Stock Analyst

To raise money for expansion, many privately owned companies give the public a chance to own part of their company through purchasing stock. Those who buy ownership expect to obtain a share in the future profits of the company. Some technical analysts believe that the greatest profits to be had in the stock market are through buying brand new stocks and selling them quickly. A technical analyst predicts that a stock's price over its first several weeks on the market will follow the pattern shown on the graph. The technical analyst is advising a person who purchased the stock the day it went on sale.



DISCUSSION QUESTIONS

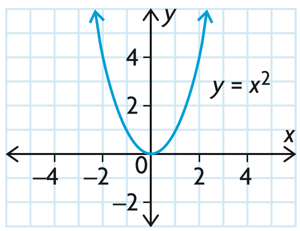
Make a rough sketch of the graph, and answer the following questions:

1. When would you recommend that the owner sell her shares? Label this point S on your graph. What do you notice about the slope, or instantaneous rate of change, of the graph at this point?
2. When would you recommend that the owner get back into the company and buy shares again? Label this point B on your graph. What do you notice about the slope, or instantaneous rate of change, of the graph at this point?
3. A concave-down section of a graph opens in a downward direction, and a concave-up section opens upward. On your graph, find the point where the concavity changes from concave down to concave up, and label this point C. Another analyst says that a change in concavity from concave down to concave up is a signal that a selling opportunity will soon occur. Do you agree with the analyst? Explain.

At the end of this chapter, you will have an opportunity to apply the tools of curve sketching to create, evaluate, and apply a model that could be used to advise clients on when to buy, sell, and hold new stocks.

Section 4.1—Increasing and Decreasing Functions

The graph of the quadratic function $f(x) = x^2$ is a parabola. If we imagine a particle moving along this parabola from left to right, we can see that, while the x-coordinates of the ordered pairs steadily increase, the y-coordinates of the ordered pairs along the particle's path first decrease and then increase. Determining the intervals in which a function increases and decreases is extremely useful for understanding the behaviour of the function. The following statements give a clear picture:



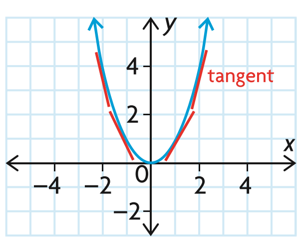
Intervals of Increase and Decrease

We say that a function f *is decreasing on an interval* if, for any value of $x_1 < x_2$ on the interval, $f(x_1) > f(x_2)$.

Similarly, we say that a function f *is increasing on an interval* if, for any value of $x_1 < x_2$ on the interval, $f(x_1) < f(x_2)$.

For the parabola with the equation $y = x^2$, the change from decreasing y-values to increasing y-values occurs at $(0, 0)$, the vertex of the parabola. The function $f(x) = x^2$ is decreasing on the interval $x < 0$ and is increasing on the interval $x > 0$.

If we examine tangents to the parabola anywhere on the interval where the y-values are decreasing (that is, on $x < 0$), we see that all of these tangents have negative slopes. Similarly, the slopes of tangents to the graph on the interval where the y-values are increasing are all positive.



For functions that are both continuous and differentiable, we can determine intervals of increasing and decreasing y-values using the derivative of the function. In the case of $y = x^2$, $\frac{dy}{dx} = 2x$. For $x < 0$, $\frac{dy}{dx} < 0$, and the slopes of the tangents are negative. The interval $x < 0$ corresponds to the decreasing portion of the graph of the parabola. For $x > 0$, $\frac{dy}{dx} > 0$, and the slopes of the tangents are positive on the interval where the graph is increasing.

We summarize this as follows: For a continuous and differentiable function, f, the function values (y-values) are increasing for all x-values where $f'(x) > 0$, and the function values (y-values) are decreasing for all x-values where $f'(x) < 0$.

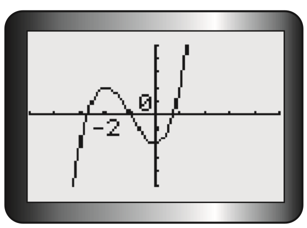
EXAMPLE 1 **Using the derivative to reason about intervals of increase and decrease**

Use your calculator to graph the following functions. Use the graph to estimate the values of x for which the function values (y-values) are increasing, and the values of x for which the y-values are decreasing. Verify your estimates with an algebraic solution.

a. $y = x^3 + 3x^2 - 2$ b. $y = \frac{x}{x^2 + 1}$

Solution

a. Using a calculator, we obtain the graph of $y = x^3 + 3x^2 - 2$. Using the TRACE key on the calculator, we estimate that the function values are increasing on $x < -2$, decreasing on $-2 < x < 0$, and increasing again on $x > 0$. To verify these estimates with an algebraic solution, we consider the slopes of the tangents.



The slope of a general tangent to the graph of $y = x^3 + 3x^2 - 2$ is given by $\frac{dy}{dx} = 3x^2 + 6x$. We first determine the values of x for which $\frac{dy}{dx} = 0$. These values tell us where the function has a **local maximum** or **local minimum** value. These are greatest and least values respectively of a function in relation to its neighbouring values.

Setting $\frac{dy}{dx} = 0$, we obtain $3x^2 + 6x = 0$

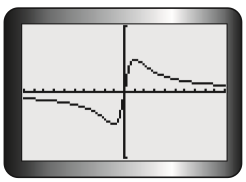
$$3x(x + 2) = 0$$
$$x = 0, x = -2$$

These values of x locate points on the graph where the slope of the tangent is zero (that is, where the tangent is horizontal).

Since this is a polynomial function it is continuous so $\frac{dy}{dx}$ is defined for all values of x. Because $\frac{dy}{dx} = 0$ only at $x = -2$ and $x = 0$, the derivative must be either positive or negative for all other values of x. We consider the intervals $x < -2$, $-2 < x < 0$, and $x > 0$.

Value of x	$x < -2$	$-2 < x < 0$	$x > 0$
Sign of $\frac{dy}{dx} = 3x(x + 2)$	$\frac{dy}{dx} > 0$	$\frac{dy}{dx} < 0$	$\frac{dy}{dx} > 0$
Slope of Tangents	positive	negative	positive
Values of y Increasing or Decreasing	increasing	decreasing	increasing

So $y = x^3 + 3x^2 - 2$ is increasing on the intervals $x < -2$ and $x > 0$ and is decreasing on the interval $-2 < x < 0$.



b. Using a calculator, we obtain the graph of $y = \frac{x}{x^2 + 1}$. Using the TRACE key on the calculator, we estimate that the function values (y-values) are decreasing on $x < -1$, increasing on $-1 < x < 1$, and decreasing again on $x > 1$.

We analyze the intervals of increasing/decreasing y-values for the function by determining where $\frac{dy}{dx}$ is positive and where it is negative.

$$y = \frac{x}{x^2 + 1}$$ (Express as a product)

$$= x(x^2 + 1)^{-1}$$

$$\frac{dy}{dx} = 1(x^2 + 1)^{-1} + x(-1)(x^2 + 1)^{-2}(2x)$$ (Product and chain rules)

$$= \frac{1}{x^2 + 1} - \frac{2x^2}{(x^2 + 1)^2}$$

$$= \frac{x^2 + 1 - 2x^2}{(x^2 + 1)^2}$$ (Simplify)

$$= \frac{-x^2 + 1}{(x^2 + 1)^2}$$

Setting $\frac{dy}{dx} = 0$, we obtain $\frac{-x^2 + 1}{(x^2 + 1)^2} = 0$ (Solve)

$$-x^2 + 1 = 0$$

$$x^2 = 1$$

$$x = 1 \text{ or } x = -1$$

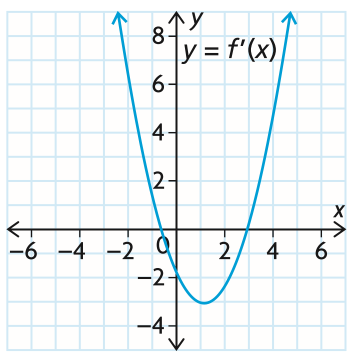
These values of x locate the points on the graph where the slope of the tangent is 0. Since the denominator of this rational function can never be 0, this function is continuous so $\frac{dy}{dx}$ is defined for all values of x. Because $\frac{dy}{dx} = 0$ at $x = -1$ and $x = 1$, we consider the intervals $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$.

Value of *x*	$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$
Sign of $\frac{dy}{dx} = \frac{-x^2 + 1}{(x^2 + 1)^2}$	$\frac{dy}{dx} < 0$	$\frac{dy}{dx} > 0$	$\frac{dy}{dx} < 0$
Slope of Tangents	negative	positive	negative
Values of *y* Increasing or Decreasing	decreasing	increasing	decreasing

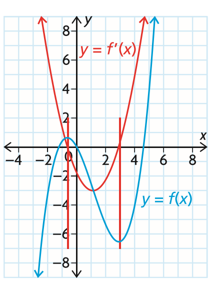
Then $y = \frac{x}{x^2 + 1}$ is increasing on the interval $(-1, 1)$ and is decreasing on the intervals $(-\infty, -1)$ and $(1, \infty)$.

EXAMPLE 2 Graphing a function given the graph of the derivative

Consider the graph of $f'(x)$. Graph $f(x)$.



Solution



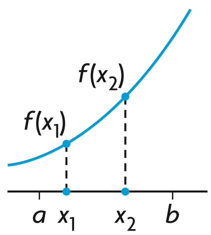
When the derivative, $f'(x)$, is positive, the graph of $f(x)$ is rising. When the derivative is negative, the graph is falling. In this example, the derivative changes sign from positive to negative at $x \doteq -0.6$. This indicates that the graph of $f(x)$ changes from increasing to decreasing, resulting in a local maximum for this value of x. The derivative changes sign from negative to positive at $x = 2.9$, indicating the graph of $f(x)$ changes from decreasing to increasing resulting in a local minimum for this value of x.

One possible graph of $f(x)$ is shown.

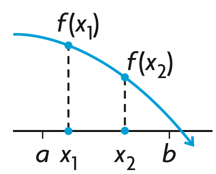
IN SUMMARY

Key Ideas

- A function f is **increasing** on an interval if, for any value of $x_1 < x_2$ in the interval, $f(x_1) < f(x_2)$.



- A function f is **decreasing** on an interval if, for any value of $x_1 < x_2$ in the interval, $f(x_1) > f(x_2)$.



- For a function f that is continuous and differentiable on an interval I
  - $f(x)$ is **increasing** on I if $f'(x) > 0$ for all values of x in I
  - $f(x)$ is **decreasing** on I if $f'(x) < 0$ for all values of x in I

Need to Know

- A function increases on an interval if the graph rises from left to right.
- A function decreases on an interval if the graph falls from left to right.
- The slope of the tangent at a point on a section of a curve that is increasing is always positive.
- The slope of the tangent at a point on a section of a curve that is decreasing is always negative.

Exercise 4.1

PART A

K 1. Determine the points at which $f'(x) = 0$ for each of the following functions:

a. $f(x) = x^3 + 6x^2 + 1$

b. $f(x) = \sqrt{x^2 + 4}$

c. $f(x) = (2x - 1)^2(x^2 - 9)$

d. $f(x) = \dfrac{5x}{x^2 + 1}$

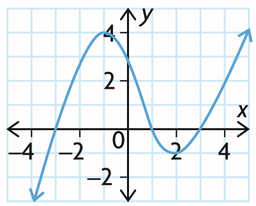
C 2. Explain how you would determine when a function is increasing or decreasing.

3. For each of the following graphs, state

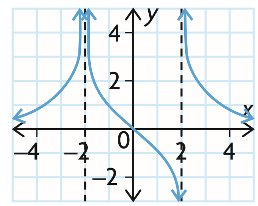
i. the intervals where the function is increasing

ii. the intervals where the function is decreasing

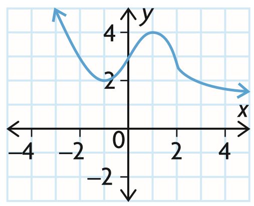
iii. the points where the tangent to the function is horizontal a.



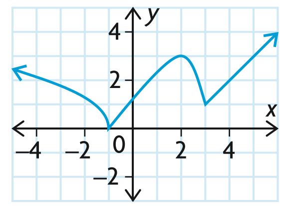
c.



b.



d.



4. Use a calculator to graph each of the following functions. Inspect the graph to estimate where the function is increasing and where it is decreasing. Verify your estimates with algebraic solutions.

a. $f(x) = x^3 + 3x^2 + 1$

b. $f(x) = x^5 - 5x^4 + 100$

c. $f(x) = x + \frac{1}{x}$

d. $f(x) = \frac{x - 1}{x^2 + 3}$

e. $f(x) = 3x^4 + 4x^3 - 12x^2$

f. $f(x) = x^4 + x^2 - 1$

PART B

5. Suppose that f is a differentiable function with the derivative $f'(x) = (x - 1)(x + 2)(x + 3)$. Determine the values of x for which the function f is increasing and the values of x for which the function is decreasing.

A

6. Sketch a graph of a function that is differentiable on the interval $-2 \leq x \leq 5$ and that satisfies the following conditions:

- The graph of f passes through the points $(-1, 0)$ and $(2, 5)$.
- The function f is decreasing on $-2 < x < -1$, increasing on $-1 < x < 2$, and decreasing again on $2 < x < 5$.

7. Find constants a, b, and c such that the graph of $f(x) = x^3 + ax^2 + bx + c$ will increase to the point $(-3, 18)$, decrease to the point $(1, -14)$, and then continue increasing.

8. Sketch a graph of a function f that is differentiable and that satisfies the following conditions:

- $f'(x) > 0$, when $x < -5$
- $f'(x) < 0$, when $-5 < x < 1$ and when $x > 1$
- $f'(-5) = 0$ and $f'(1) = 0$
- $f(-5) = 6$ and $f(1) = 2$

9. Each of the following graphs represents the derivative function $f'(x)$ of a function $f(x)$. Determine

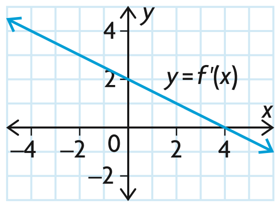
   i. the intervals where $f(x)$ is increasing

   ii. the intervals where $f(x)$ is decreasing

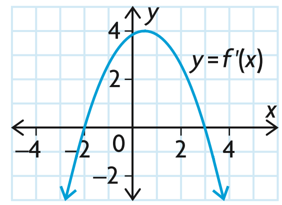
   iii. the x-coordinate for all local extrema of $f(x)$

   Assuming that $f(0) = 2$, make a rough sketch of the graph of each function.

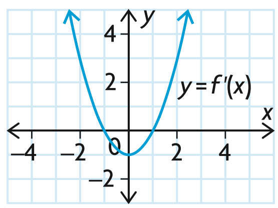
a.



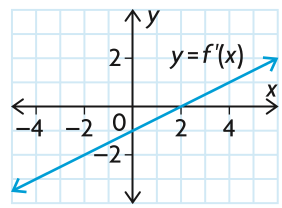
c.



b.



d.



10. Use the derivative to show that the graph of the quadratic function $f(x) = ax^2 + bx + c$, $a > 0$, is decreasing on the interval $x < -\frac{b}{2a}$ and increasing on the interval $x > -\frac{b}{2a}$.

11. For $f(x) = x^4 - 32x + 4$, find where $f'(x) = 0$, the intervals on which the function increases and decreases, and all the local extrema. Use graphing technology to verify your results.

12. Sketch a graph of the function g that is differentiable on the interval $-2 \leq x \leq 5$, decreases on $0 < x < 3$, and increases elsewhere on the domain. The absolute maximum of g is 7, and the absolute minimum is -3. The graph of g has local extrema at $(0, 4)$ and $(3, -1)$.

PART C

T

13. Let f and g be continuous and differentiable functions on the interval $a \leq x \leq b$. If f and g are both increasing on $a \leq x \leq b$, and if $f(x) > 0$ and $g(x) > 0$ on $a \leq x \leq b$, show that the product fg is also increasing on $a \leq x \leq b$.

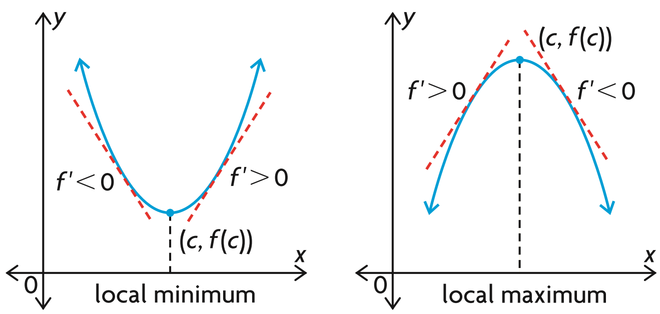
14. Let f and g be continuous and differentiable functions on the interval $a \leq x \leq b$. If f and g are both increasing on $a \leq x \leq b$, and if $f(x) < 0$ and $g(x) < 0$ on $a \leq x \leq b$, is the product fg increasing on $a \leq x \leq b$, decreasing, or neither?

Section 4.2—Critical Points, Local Maxima, and Local Minima

In Chapter 3, we learned that a maximum or minimum function value might occur at a point $(c, f(c))$ if $f'(c) = 0$. It is also possible that a maximum or minimum function value might occur at a point $(c, f(c))$ if $f'(c)$ is undefined. Since these points help to define the shape of the function's graph, they are called **critical points** and the values of c are called **critical numbers**. Combining this with the properties of increasing and decreasing functions, we have a **first derivative test** for local extrema.

The First Derivative Test

Test for local minimum and local maximum points. Let $f'(c) = 0$.



When moving left to right through x-values:

- if $f'(x)$ changes sign from negative to positive at $x = c$, then $f(x)$ has a local minimum at this point.
- if $f'(x)$ changes sign from positive to negative at $x = c$, then $f(x)$ has a local maximum at this point.

$f'(c) = 0$ may imply something other than the existence of a maximum or a minimum at $x = c$. There are also simple functions for which the derivative does not exist at certain points. In Chapter 2, we demonstrated three different ways that this could happen. For example, extrema could occur at points that correspond to cusps and corners on a function's graph and in these cases the derivative is undefined.

EXAMPLE 1 **Connecting the first derivative test to local extrema of a polynomial function**

For the function $y = x^4 - 8x^3 + 18x^2$, determine all the critical numbers. Determine whether each of these values of x gives a local maximum, a local minimum, or neither for the function.

Solution

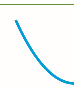
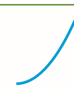
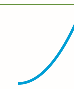
First determine $\frac{dy}{dx}$.

$$\frac{dy}{dx} = 4x^3 - 24x^2 + 36x$$
$$= 4x(x^2 - 6x + 9)$$
$$= 4x(x - 3)^2$$

For critical numbers, let $\frac{dy}{dx} = 0$.

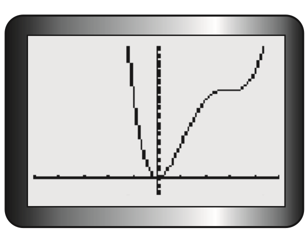
$$4x(x - 3)^2 = 0$$
$$x = 0 \text{ or } x = 3$$

Both values of x are in the domain of the function. There is a horizontal tangent at each of these values of x. To determine which of these values of x yield local maximum or minimum values of the function, we use a table to analyze the behaviour of $\frac{dy}{dx}$ and $y = x^4 - 8x^3 + 18x^2$.

Interval	$x < 0$	$0 < x < 3$	$x > 3$
$4x$	$-$	$+$	$+$
$(x - 3)^2$	$+$	$+$	$+$
$4x(x - 3)^2$	$(-)(+) = -$	$(+)(+) = +$	$(+)(+) = +$
$\frac{dy}{dx}$	< 0	> 0	> 0
$y = x^4 - 8x^3 + 18x^2$	decreasing	increasing	increasing
Shape of the Curve			

Using the information from the table, we see that there is a local minimum value of the function at $x = 0$, since the function values are decreasing before $x = 0$ and increasing after $x = 0$. We can also tell that there is neither a local maximum nor minimum value at $x = 3$, since the function values increase toward this point and increase away from it.

A calculator gives the following graph for $y = x^4 - 8x^3 + 18x^2$, which verifies our solution:



EXAMPLE 2 Reasoning about the significance of horizontal tangents

Determine whether or not the function $f(x) = x^3$ has a maximum or minimum at $(c, f(c))$, where $f'(c) = 0$.

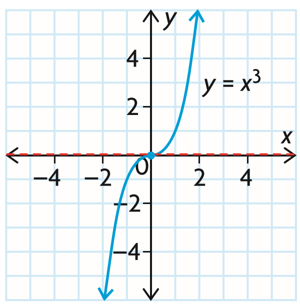
Solution

The derivative is $f'(x) = 3x^2$.

Setting $f'(x) = 0$ gives

$$3x^2 = 0$$
$$x = 0$$

$f(x)$ has a horizontal tangent at $(0, 0)$.



From the graph, it is clear that $(0, 0)$ is neither a maximum nor a minimum value since the values of this function are always increasing. Note that $f'(x) > 0$ for all values of x other than 0.

From this example, we can see that it is possible for a horizontal tangent to exist when $f'(c) = 0$, but that $(c, f(c))$ is neither a maximum nor a minimum. In the next example you will see that it is possible for a maximum or minimum to occur at a point at which the derivative does not exist.

EXAMPLE 3 Reasoning about the significance of a cusp

For the function $f(x) = (x + 2)^{\frac{2}{3}}$, determine the critical numbers. Use your calculator to sketch a graph of the function.

Solution

First determine $f'(x)$.

$$f'(x) = \frac{2}{3}(x + 2)^{-\frac{1}{3}}$$
$$= \frac{2}{3(x + 2)^{\frac{1}{3}}}$$

Note that there is no value of x for which $f'(x) = 0$ since the numerator is always positive. However, $f'(x)$ is undefined for $x = -2$.

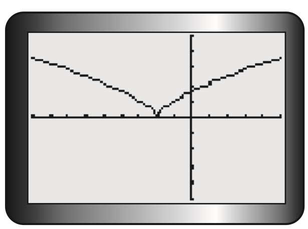
Since $f(-2) = (-2 + 2)^{\frac{2}{3}} = 0$, $x = -2$ is in the domain of $f(x) = (x + 2)^{\frac{2}{3}}$. We determine the slopes of tangents for x-values close to -2.

x	$f'(x) = \frac{2}{3(x + 2)^{\frac{1}{3}}}$	x	$f'(x) = \frac{2}{3(x + 2)^{\frac{1}{3}}}$
−2.1	−1.436 29	−1.9	1.436 29
−2.01	−3.094 39	−1.99	3.094 39
−2.001	−6.666 67	−1.999	6.666 67
−2.000 01	−30.943 9	−1.999 99	30.943 9

The slope of the tangent is undefined at the point $(-2, 0)$. Therefore, the function has one critical point, when $x = -2$.

In this example, the slopes of the tangents to the left of $x = -2$ are approaching $-\infty$, while the slopes to the right of $x = -2$ are approaching $+\infty$. Since the slopes on opposite sides of $x = -2$ are not approaching the same value, there is no tangent at $x = -2$ even though there is a point on the graph.

A calculator gives the following graph of $f(x) = (x + 2)^{\frac{2}{3}}$. There is a cusp at $(-2, 0)$.



If a value c is in the domain of a function $f(x)$, and if this value is such that $f'(c) = 0$ or $f'(c)$ is undefined, then $(c, f(c))$ is a critical point of the function f and c is called a critical number for f''.

In summary, critical points that occur when $\frac{dy}{dx} = 0$ give the locations of horizontal tangents on the graph of a function. Critical points that occur when $\frac{dy}{dx}$ is undefined give the locations of either vertical tangents or cusps (where we say that no tangent exists). Besides giving the location of interesting tangents (or lack thereof), critical points also determine other interesting features of the graph of a function.

Critical Numbers and Local Extrema

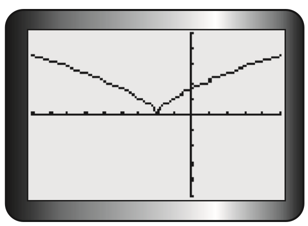
The critical number c determines the location of a local minimum value for a function f if $f(c) < f(x)$ for all values of x near c.

Similarly, the critical number c determines the location of a local maximum value for a function f if $f(c) > f(x)$ for all values of x near c.

Together, local maximum and minimum values of a function are called local extrema.

As mentioned earlier, a local minimum value of a function does not have to be the smallest value in the entire domain, just the smallest value in its neighbourhood. Similarly, a local maximum value of a function does not have to be the largest value in the entire domain, just the largest value in its neighbourhood. Local extrema occur graphically as peaks or valleys. The peaks and valleys can be either smooth or sharp.

To apply this reasoning, let's reconsider the graph of $f(x) = (x + 2)^{\frac{2}{3}}$.



The function $f(x) = (x + 2)^{\frac{2}{3}}$ has a local minimum value at $x = -2$, which also happens to be a critical value of the function.

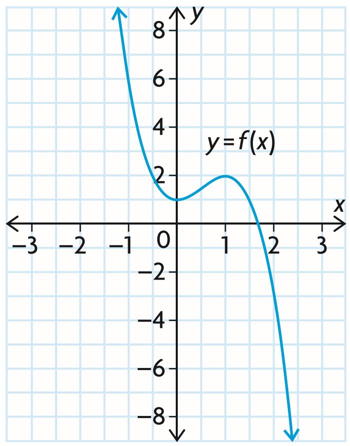
Every local maximum or minimum value of a function occurs at a critical point of the function.

In simple terms, peaks or valleys occur on the graph of a function at places where the tangent to the graph is horizontal, vertical, or does not exist.

How do we determine whether a critical point yields a local maximum or minimum value of a function without examining the graph of the function? We use the first derivative test to analyze whether the function is increasing or decreasing on either side of the critical point.

EXAMPLE 4 Graphing the derivative given the graph of a polynomial function

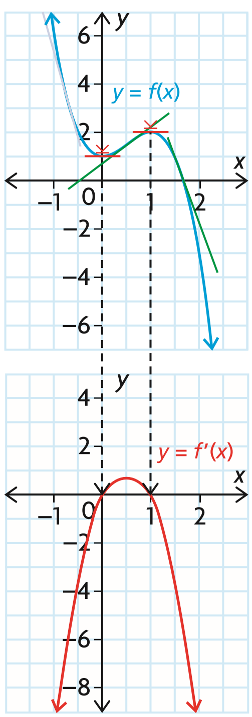
Given the graph of a polynomial function $y = f(x)$, graph $y = f'(x)$.



Solution

A polynomial function f is continuous for all values of x in the domain of f. The derivative of f, f', is also continuous for all values of x in the domain of f.

To graph $y = f'(x)$ using the graph of $y = f(x)$, first determine the slopes of the tangent lines, $f'(x_i)$, at certain x-values, x_i. These x-values include zeros, critical numbers, and numbers in each interval where f is increasing or decreasing. Then plot the corresponding ordered pairs on a graph. Draw a smooth curve through these points to complete the graph.



The given graph has a local minimum at (0, 1) and a local maximum at (1, 2). At these points, the tangents are horizontal. Therefore, $f'(0) = 0$ and $f'(1) = 0$.

At $x = \frac{1}{2}$, which is halfway between $x = 0$ and $x = 1$, the slope of the tangent is about $\frac{2}{3}$. So $f'\left(\frac{1}{2}\right) \doteq \frac{2}{3}$

The function $f(x)$ is decreasing when $f'(x) < 0$. The tangent lines show that $f'(x) < 0$ when $x < 0$ and when $x > 1$. Similarly, $f(x)$ is increasing when $f'(x) > 0$. The tangent lines show that $f'(x) > 0$ when $0 < x < 1$.

The shape of the graph of $f(x)$ suggests that $f(x)$ is a cubic polynomial with a negative leading coefficient. Assume that this is true. The derivative, $f'(x)$, may be a quadratic function with a negative leading coefficient. If it is, the graph of $f'(x)$ is a parabola that opens down.

Plot (0, 0), (1, 0), and $\left(\frac{1}{2}, \frac{2}{3}\right)$ on the graph of $f'(x)$. The graph of $f'(x)$ is a parabola that opens down and passes through these points.

IN SUMMARY

Key Idea

For a function $f(x)$, a **critical number** is a number, c, in the domain of $f(x)$ such that $f'(x) = 0$ or $f'(x)$ is undefined. As a result $(c, f(c))$ is called a critical point and usually corresponds to local or absolute extrema.

Need to Know

First Derivative Test

Let c be a critical number of a function f.

When moving through x-values from left to right:

- if $f'(x)$ changes from negative to positive at c, then $(c, f(c))$ is a **local minimum** of f.
- if $f'(x)$ changes from positive to negative at c, then $(c, f(c))$ is a **local maximum** of f.
- if $f'(x)$ does not change its sign at c, then $(c, f(c))$ is neither a local minimum or a local maximum.

Algorithm for Finding Local Maximum and Minimum Values of a Function *f*

1. Find critical numbers of the function (that is, determine where $f'(x) = 0$ and where $f'(x)$ is undefined) for all x-values in the domain of f.
2. Use the first derivative to analyze whether f is increasing or decreasing on either side of each critical number.
3. Based upon your findings in step 2., conclude whether each critical number locates a local maximum value of the function f, a local minimum value, or neither.

Exercise 4.2

PART A

C 1. Explain what it means to determine the critical points of the graph of a given function.

2. a. For the function $y = x^3 - 6x^2$, explain how you would find the critical points.

   b. Determine the critical points for $y = x^3 - 6x^2$, and then sketch the graph.

3. Find the critical points for each function. Use the first derivative test to determine whether the critical point is a local maximum, local minimum, or neither.

   a. $y = x^4 - 8x^2$ b. $f(x) = \frac{2x}{x^2 + 9}$ c. $y = x^3 + 3x^2 + 1$

4. Find the x- and y-intercepts of each function in question 3, and then sketch the curve.

5. Determine the critical points for each function. Determine whether the critical point is a local maximum or minimum, and whether or not the tangent is parallel to the horizontal axis.

   a. $h(x) = -6x^3 + 18x^2 + 3$
   b. $g(t) = t^5 + t^3$
   c. $y = (x - 5)^{\frac{1}{3}}$
   d. $f(x) = (x^2 - 1)^{\frac{1}{3}}$

6. Use graphing technology to graph the functions in question 5 and verify your results.

PART B

K 7. Determine the critical points for each of the following functions, and determine whether the function has a local maximum value, a local minimum value, or neither at the critical points. Sketch the graph of each function.

   a. $f(x) = -2x^2 + 8x + 13$
   b. $f(x) = \frac{1}{3}x^3 - 9x + 2$
   c. $f(x) = 2x^3 + 9x^2 + 12x$
   d. $f(x) = -3x^3 - 5x$
   e. $f(x) = \sqrt{x^2 - 2x + 2}$
   f. $f(x) = 3x^4 - 4x^3$

A 8. Suppose that f is a differentiable function with the derivative $f'(x) = (x + 1)(x - 2)(x + 6)$. Find all the critical numbers of f, and determine whether each corresponds to a local maximum, a local minimum, or neither.

9. Sketch a graph of a function f that is differentiable on the interval $-3 \leq x \leq 4$ and that satisfies the following conditions:

   - The function f is decreasing on $-1 < x < 3$ and increasing elsewhere on $-3 \leq x \leq 4$.
   - The largest value of f is 6, and the smallest value is 0.
   - The graph of f has local extrema at $(-1, 6)$ and $(3, 1)$.

10. Determine values of a, b, and c such that the graph of $y = ax^2 + bx + c$ has a relative maximum at $(3, 12)$ and crosses the y-axis at $(0, 1)$.

11. For $f(x) = x^2 + px + q$, find the values of p and q such that $f(1) = 5$ is an extremum of f on the interval $0 \leq x \leq 2$. Is this extremum a maximum value or a minimum value? Explain.

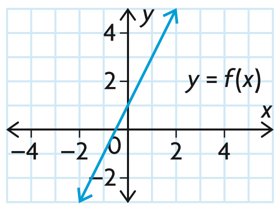
12. For $f(x) = x^3 - kx$, where $k \in \mathbf{R}$, find the values of k such that f has

    a. no critical numbers
    b. one critical number
    c. two critical numbers

T 13. Find values of a, b, c, and d such that $g(x) = ax^3 + bx^2 + cx + d$ has a local maximum at (2, 4) and a local minimum at (0, 0).

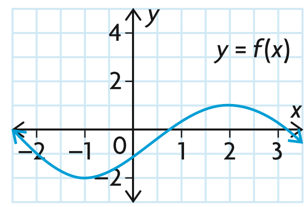
14. For each of the following graphs of the function $y = f(x)$, make a rough sketch of the derivative function $f'(x)$. By comparing the graphs of $f(x)$ and $f'(x)$, show that the intervals for which $f(x)$ is increasing correspond to the intervals where $f'(x)$ is positive. Also show that the intervals where $f(x)$ is decreasing correspond to the intervals for which $f'(x)$ is negative.

a.



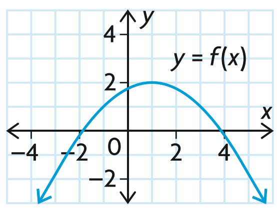
$f(x)$ is a linear function.

c.



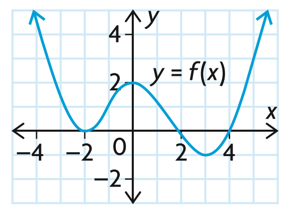
$f(x)$ is a cubic function.

b.



$f(x)$ is a quadratic function.

d.



$f(x)$ is a quartic function.

15. Consider the function $f(x) = 3x^4 + ax^3 + bx^2 + cx + d$.

a. Find constants a, b, c, and d such that the graph of f will have horizontal tangents at $(-2, -73)$ and $(0, -9)$.

b. There is a third point that has a horizontal tangent. Find this point.

c. For all three points, determine whether each corresponds to a local maximum, a local minimum, or neither.

PART C

16. For each of the following polynomials, find the local extrema and the direction that the curve is opening for $x = 100$. Use this information to make a quick sketch of the curve.

a. $y = 4 - 3x^2 - x^4$

b. $y = 3x^5 - 5x^3 - 30x$

17. Suppose that $f(x)$ and $g(x)$ are positive functions (functions where $f(x) > 0$ and $g(x) > 0$) such that $f(x)$ has a local maximum and $g(x)$ has a local minimum at $x = c$. Show that the function $h(x) = \frac{f(x)}{g(x)}$ has a local maximum at $x = c$.

Section 4.3—Vertical and Horizontal Asymptotes

Adding, subtracting, or multiplying two polynomial functions yields another polynomial function. Dividing two polynomial functions results in a function that is not a polynomial. The quotient is a **rational function**. Asymptotes are among the special features of rational functions, and they play a significant role in curve sketching. In this section, we will consider vertical and horizontal asymptotes of rational functions.

INVESTIGATION

The purpose of this investigation is to examine the occurrence of vertical asymptotes for rational functions.

A. Use your graphing calculator to obtain the graph of $f(x) = \frac{1}{x - k}$ and the table of values for each of the following: $k = 3, 1, 0, -2, -4,$ and -5.

B. Describe the behaviour of each graph as x approaches k from the right and from the left.

C. Repeat parts A and B for the function $f(x) = \frac{x + 3}{x - k}$ using the same values of k.

D. Repeat parts A and B for the function $f(x) = \frac{1}{x^2 + x - k}$ using the following values: $k = 2, 6,$ and 12.

E. Make a general statement about the existence of a vertical asymptote for a rational function of the form $y = \frac{p(x)}{q(x)}$ if there is a value c such that $q(c) = 0$ and $p(c) \neq 0$.

Vertical Asymptotes and Rational Functions

Recall that the notation $x \to c^+$ means that x approaches c from the right. Similarly, $x \to c^-$ means that x approaches c from the left.

You can see from this investigation that as $x \to c$ from either side, the function values get increasingly large and either positive or negative depending on the value of $p(c)$. We say that the function values approach $+\infty$ (positive infinity) or $-\infty$ (negative infinity). These are not numbers. They are symbols that represent the behaviour of a function that increases or decreases without limit.

Because the symbol ∞ is not a number, the limits $\lim_{x \to c^+} \frac{1}{x - c}$ and $\lim_{x \to c^-} \frac{1}{x - c}$ *do not exist*. For convenience, however, we use the notation $\lim_{x \to c^+} \frac{1}{x - c} = +\infty$ and $\lim_{x \to c^-} \frac{1}{x - c} = -\infty$.

These limits form the basis for determining the asymptotes of simple functions.

Vertical Asymptotes of Rational Functions

A rational function of the form $f(x) = \frac{p(x)}{q(x)}$ has a vertical asymptote $x = c$ if $q(c) = 0$ and $p(c) \neq 0$.

EXAMPLE 1 Reasoning about the behaviour of a rational function near its vertical asymptotes

Determine any vertical asymptotes of the function $f(x) = \frac{x}{x^2 + x - 2}$, and describe the behaviour of the graph of the function for values of x near the asymptotes.

Solution

First, determine the values of x for which $f(x)$ is undefined by solving the following:

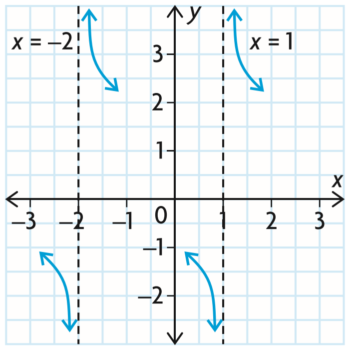
$$x^2 + x - 2 = 0$$
$$(x + 2)(x - 1) = 0$$
$$x = -2 \text{ or } x = 1$$

Neither of these values of x makes the numerator zero, so both of these values give vertical asymptotes. The equations of the asymptotes are $x = -2$ and $x = 1$.

To determine the behaviour of the graph near the asymptotes, it can be helpful to use a chart.

Values of x	x	$x + 2$	$x - 1$	$f(x) = \frac{x}{(x + 2)(x - 1)}$	$f(x) \rightarrow ?$
$x \rightarrow -2^-$	< 0	< 0	< 0	< 0	$-\infty$
$x \rightarrow -2^+$	< 0	> 0	< 0	> 0	$+\infty$
$x \rightarrow 1^-$	> 0	> 0	< 0	< 0	$-\infty$
$x \rightarrow 1^+$	> 0	> 0	> 0	> 0	$+\infty$

The behaviour of the graph can be illustrated as follows:



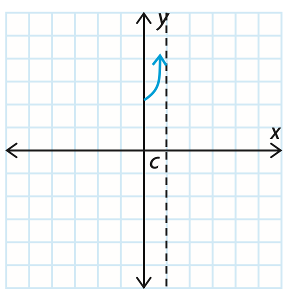
To proceed beyond this point, we require additional information.

Vertical Asymptotes and Infinite Limits

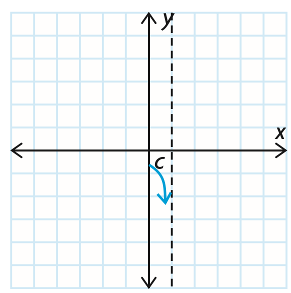
The graph of $f(x)$ has a vertical asymptote, $x = c$, if one of the following infinite limit statements is true:

$$\lim_{x \to c^-} f(x) = +\infty, \quad \lim_{x \to c^-} f(x) = -\infty, \quad \lim_{x \to c^+} f(x) = +\infty \quad \text{or} \quad \lim_{x \to c^+} f(x) = -\infty$$

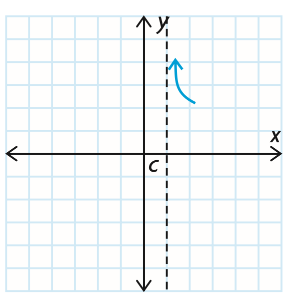
The following graphs correspond to each limit statement above:



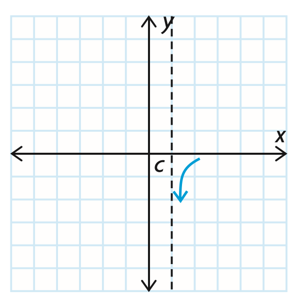
$\lim_{x \to c^-} f(x) = +\infty$



$\lim_{x \to c^-} f(x) = -\infty$



$\lim_{x \to c^+} f(x) = +\infty$



$\lim_{x \to c^+} f(x) = -\infty$

Horizontal Asymptotes and Rational Functions

Consider the behaviour of rational functions $f(x) = \frac{p(x)}{q(x)}$ as x increases without bound in both the positive and negative directions. The following notation is used to describe this behaviour:

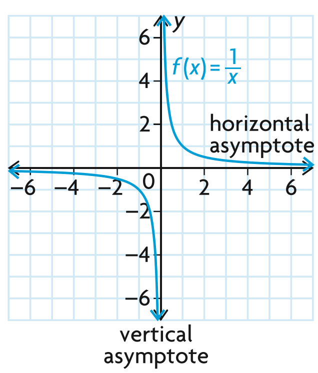
$$\lim_{x\to+\infty} f(x) \text{ and } \lim_{x\to-\infty} f(x)$$

The notation $x \to +\infty$ is read "x tends to positive infinity" and means that the values of x are positive and growing in magnitude without bound. Similarly, the notation $x \to -\infty$ is read "x tends to negative infinity" and means that the values of x are negative and growing in magnitude without bound.

The values of these limits can be determined by making two observations. The first observation is a list of simple limits, similar to those used for determining vertical asymptotes.

The Reciprocal Function and Limits at Infinity

$$\lim_{x\to+\infty} \frac{1}{x} = 0 \text{ and } \lim_{x\to-\infty} \frac{1}{x} = 0$$



The second observation is that a polynomial can always be written so the term of highest degree is a factor.

EXAMPLE 2 Expressing a polynomial function in an equivalent form

Write each function so the term of highest degree is a factor.

a. $p(x) = x^2 + 4x + 1$

b. $q(x) = 3x^2 - 4x + 5$

Solution

a.
$$\begin{aligned} p(x) &= x^2 + 4x + 1 \\ &= x^2\left(1 + \frac{4}{x} + \frac{1}{x^2}\right) \end{aligned}$$

b.
$$\begin{aligned} q(x) &= 3x^2 - 4x + 5 \\ &= 3x^2\left(1 - \frac{4}{3x} + \frac{5}{3x^2}\right) \end{aligned}$$

The value of writing a polynomial in this form is clear. It is easy to see that as x becomes large (either positive or negative), the value of the second factor always approaches 1.

We can now determine the limit of a rational function in which the degree of $p(x)$ is equal to or less than the degree of $q(x)$.

EXAMPLE 3 Selecting a strategy to evaluate limits at infinity

Determine the value of each of the following:

a. $\lim\limits_{x \to +\infty} \frac{2x - 3}{x + 1}$

b. $\lim\limits_{x \to -\infty} \frac{x}{x^2 + 1}$

c. $\lim\limits_{x \to +\infty} \frac{2x^2 + 3}{3x^2 - x + 4}$

Solution

a.
$$f(x) = \frac{2x - 3}{x + 1} = \frac{2x\left(1 - \frac{3}{2x}\right)}{x\left(1 + \frac{1}{x}\right)}$$
(Factor and simplify)

$$= \frac{2\left(1 - \frac{3}{2x}\right)}{1 + \frac{1}{x}}$$

$$\lim_{x \to +\infty} f(x) = \frac{2\left[\lim\limits_{x \to +\infty}\left(1 - \frac{3}{2x}\right)\right]}{\lim\limits_{x \to +\infty}\left(1 + \frac{1}{x}\right)}$$
(Apply limit properties)

$$= \frac{2(1 - 0)}{1 + 0}$$
(Evaluate)

$$= 2$$

b. $$g(x) = \frac{x}{x^2 + 1} \quad \text{(Factor)}$$

$$= \frac{x(1)}{x^2\left(1 + \frac{1}{x^2}\right)} \quad \text{(Simplify)}$$

$$= \frac{1}{x\left(1 + \frac{1}{x^2}\right)}$$

$$\lim_{x \to -\infty} g(x) = \frac{1}{\lim_{x \to -\infty} x \times \lim_{x \to -\infty}\left(1 + \frac{1}{x^2}\right)} \quad \text{(Apply limit properties)}$$

$$= \frac{1}{\lim_{x \to -\infty} x \times (1)}$$

$$= \lim_{x \to -\infty} \frac{1}{x} \quad \text{(Evaluate)}$$

$$= 0$$

c. To evaluate this limit, we can use the technique of dividing the numerator and denominator by the highest power of x in the denominator.

$$p(x) = \frac{2x^2 + 3}{3x^2 - x + 4} \quad \text{(Divide by } x^2\text{)}$$

$$= \frac{(2x^2 + 3) \div x^2}{(3x^2 - x + 4) \div x^2} \quad \text{(Simplify)}$$

$$= \frac{2 + \frac{3}{x^2}}{3 - \frac{1}{x} + \frac{4}{x^2}}$$

$$\lim_{x \to +\infty} p(x) = \frac{\lim_{x \to +\infty}\left(2 + \frac{3}{x^2}\right)}{\lim_{x \to +\infty}\left(3 - \frac{1}{x} + \frac{4}{x^2}\right)} \quad \text{(Apply limit properties)}$$

$$= \frac{2 + 0}{3 - 0 + 0} \quad \text{(Evaluate)}$$

$$= \frac{2}{3}$$

When $\lim_{x \to +\infty} f(x) = k$ or $\lim_{x \to -\infty} f(x) = k$, the graph of the function is approaching the line $y = k$. This line is a horizontal asymptote of the function. In Example 3, part a, $y = 2$ is a horizontal asymptote of $f(x) = \frac{2x - 3}{x + 1}$. Therefore, for large positive x-values, the y-values approach 2. This is also the case for large negative x-values.

To sketch the graph of the function, we need to know whether the curve approaches the horizontal asymptote from above or below. To find out, we need to consider $f(x) - k$, where k is the limit we just determined. This is illustrated in the following examples.

EXAMPLE 4 Reasoning about the end behaviours of a rational function

Determine the equations of any horizontal asymptotes of the function $f(x) = \frac{3x + 5}{2x - 1}$. State whether the graph approaches the asymptote from above or below.

Solution

$$f(x) = \frac{3x + 5}{2x - 1} = \frac{(3x + 5) \div x}{(2x - 1) \div x}$$ (Divide by x)

$$= \frac{3 + \frac{5}{x}}{2 - \frac{1}{x}}$$

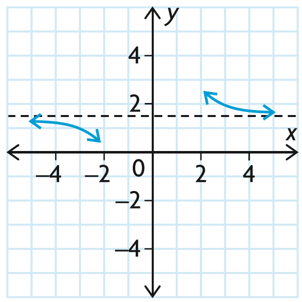
$$\lim_{x \to +\infty} f(x) = \frac{\lim_{x \to +\infty}\left(3 + \frac{5}{x}\right)}{\lim_{x \to +\infty}\left(2 - \frac{1}{x}\right)}$$ (Evaluate)

$$= \frac{3}{2}$$

Similarly, we can show that $\lim_{x \to -\infty} f(x) = \frac{3}{2}$. So, $y = \frac{3}{2}$ is a horizontal asymptote of the graph of $f(x)$ for both large positive and negative values of x. To determine whether the graph approaches the asymptote from above or below, we consider very large positive and negative values of x.

If x is large and positive (for example, if $x = 1000$), $f(x) = \frac{3005}{1999}$, which is greater than $\frac{3}{2}$. Therefore, the graph approaches the asymptote $y = \frac{3}{2}$ from above.

If x is large and negative (for example, if $x = -1000$), $f(x) = \frac{-2995}{-20\,001}$, which is less than $\frac{3}{2}$. This part of the graph approaches the asymptote $y = \frac{3}{2}$ from below, as illustrated in the diagram.



EXAMPLE 5 Selecting a limit strategy to analyze the behaviour of a rational function near its asymptotes

For the function $f(x) = \frac{3x}{x^2 - x - 6}$, determine the equations of all horizontal or vertical asymptotes. Illustrate the behaviour of the graph as it approaches the asymptotes.

Solution

For vertical asymptotes,

$$x^2 - x - 6 = 0$$

$$(x - 3)(x + 2) = 0$$

$$x = 3 \text{ or } x = -2$$

There are two vertical asymptotes, at $x = 3$ and $x = -2$.

Values of x	x	$x - 3$	$x + 2$	$f(x)$	$f(x) \to ?$
$x \to 3^-$	> 0	< 0	> 0	< 0	$-\infty$
$x \to 3^+$	> 0	> 0	> 0	> 0	$+\infty$
$x \to -2^-$	< 0	< 0	< 0	< 0	$-\infty$
$x \to -2^+$	< 0	< 0	> 0	> 0	$+\infty$

For horizontal asymptotes,

$$f(x) = \frac{3x}{x^2 - x - 6} \qquad \text{(Factor)}$$

$$= \frac{3x}{x^2\left(1 - \frac{1}{x} - \frac{6}{x^2}\right)} \qquad \text{(Simplify)}$$

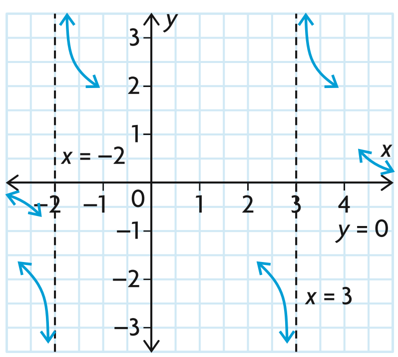
$$= \frac{3}{x\left(1 - \frac{1}{x} - \frac{6}{x^2}\right)}$$

$$\lim_{x \to +\infty} f(x) = \lim_{x \to \infty} \frac{3}{x} = 0$$

Similarly, we can show $\lim_{x \to -\infty} f(x) = 0$. Therefore, $y = 0$ is a horizontal asymptote of the graph of $f(x)$ for both large positive and negative values of x.

As x becomes large positively, $f(x) > 0$, so the graph is above the horizontal asymptote. As x becomes large negatively, $f(x) < 0$, so the graph is below the horizontal asymptote.

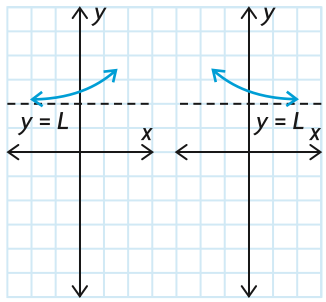
This diagram illustrates the behaviour of the graph as it nears the asymptotes:



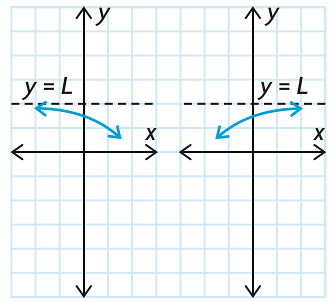
Horizontal Asymptotes and Limits at Infinity

If $\lim_{x \to +\infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$, we say that the line $y = L$ is a horizontal asymptote of the graph of $f(x)$.

The following graphs illustrate some typical situations:



$f(x) > L$, so the graph approaches from above.



$f(x) < L$, so the graph approaches from below.

In addition to vertical and horizontal asymptotes, it is possible for a graph to have **oblique asymptotes**. These are straight lines that are slanted and to which the curve becomes increasingly close. They occur with rational functions in which the degree of the numerator exceeds the degree of the denominator by exactly one. This is illustrated in the following example.

EXAMPLE 6 Reasoning about oblique asymptotes

Determine the equations of all asymptotes of the graph of $f(x) = \frac{2x^2 + 3x - 1}{x + 1}$.

Solution

Since $x + 1 = 0$ for $x = -1$, and $2x^2 + 3x - 1 \neq 0$ for $x = -1$, $x = -1$ is a vertical asymptote.

$$\text{Now } \lim_{x\to\infty} \frac{2x^2 + 3x - 1}{x + 1} = \lim_{x\to\infty} \frac{2x^2\left(1 + \frac{3}{2x} - \frac{1}{2x^2}\right)}{x\left(1 + \frac{1}{x}\right)}$$

$$= \lim_{x\to\infty} 2x$$

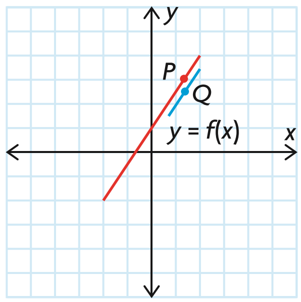
This limit does not exist, and, by a similar calculation, $\lim_{x\to-\infty} f(x)$ does not exist, so there is no horizontal asymptote.

Dividing the numerator by the denominator,

$$\begin{array}{r} 2x + 1 \\ x + 1\overline{)2x^2 + 3x - 1} \\ \underline{2x^2 + 2x} \\ x - 1 \\ \underline{x + 1} \\ -2 \end{array}$$

Thus, we can write $f(x)$ in the form $f(x) = 2x + 1 - \frac{2}{x + 1}$.

Now let's consider the straight line $y = 2x + 1$ and the graph of $y = f(x)$. For any value of x, we can determine point $P(x, 2x + 1)$ on the line and point $Q\left(x, 2x + 1 - \frac{2}{x + 1}\right)$ on the curve.



Then the vertical distance QP from the curve to the line is

$$QP = 2x + 1 - \left(2x + 1 - \frac{2}{x + 1}\right)$$
$$= \frac{2}{x + 1}$$

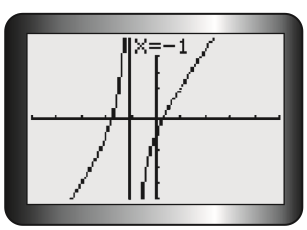
$$\lim_{x \to \infty} QP = \lim_{x \to \infty} \frac{2}{x + 1}$$
$$= 0$$

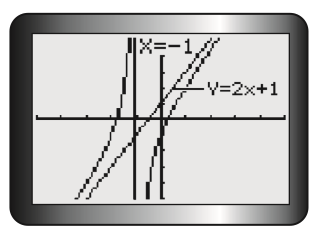
That is, as x gets very large, the curve approaches the line but never touches it. Therefore, the line $y = 2x + 1$ is an asymptote of the curve.

Since $\lim_{x \to -\infty} \frac{2}{x + 1} = 0$, the line is also an asymptote for large negative values of x.

In conclusion, there are two asymptotes of the graph of $f(x) = \frac{2x^2 + 3x - 1}{x + 1}$. They are $y = 2x + 1$ and $x = -1$.

Use a graphing calculator to obtain the graph of $f(x) = \frac{2x^2 + 3x - 1}{x + 1}$.





Note that the vertical asymptote $x = -1$ appears on the graph on the left, but the oblique asymptote $y = 2x + 1$ does not. Use the Y2 function to graph the oblique asymptote $y = 2x + 1$.

IN SUMMARY

Key Ideas

- The graph of $f(x)$ has a **vertical asymptote** $x = c$ if any of the following is true:
  $\lim_{x \to c^-} f(x) = +\infty$ $\qquad$ $\lim_{x \to c^-} f(x) = -\infty$
  $\lim_{x \to c^+} f(x) = +\infty$ $\qquad$ $\lim_{x \to c^+} f(x) = -\infty$
- The line $y = L$ is a **horizontal asymptote** of the graph of $f(x)$ if $\lim_{x \to +\infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$.
- In a rational function, an **oblique asymptote** occurs when the degree of the numerator is exactly one greater than the degree of the denominator**.**

Need to Know

The techniques for curve sketching developed to this point are described in the following algorithm. As we develop new ideas, the algorithm will be extended.

Algorithm for Curve Sketching (so far)

To sketch a curve, apply these steps in the order given.

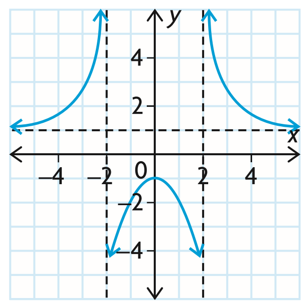
1. Check for any discontinuities in the domain. Determine if there are vertical asymptotes at these discontinuities, and determine the direction from which the curve approaches these asymptotes.
2. Find **both intercepts**.
3. Find any critical points.
4. Use the first derivative test to determine the type of critical points that may be present.
5. **Test end behaviour** by determining $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$.
6. Construct an interval of increase/decrease table and identify all local or absolute extrema.
7. Sketch the curve.

Exercise 4.3

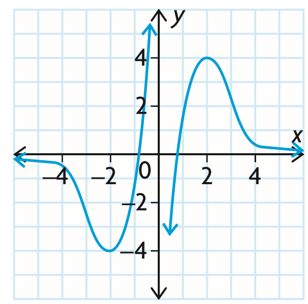
PART A

1. State the equations of the vertical and horizontal asymptotes of the curves shown.

a.



b.



C 2. Under what conditions does a rational function have vertical, horizontal, and oblique asymptotes?

3. Evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$, using the symbol "∞" when appropriate.

a. $f(x) = \dfrac{2x + 3}{x - 1}$

b. $f(x) = \dfrac{5x^2 - 3}{x^2 + 2}$

c. $f(x) = \dfrac{-5x^2 + 3x}{2x^2 - 5}$

d. $f(x) = \dfrac{2x^5 - 3x^2 + 5}{3x^4 + 5x - 4}$

4. For each of the following, check for discontinuities and state the equation of any vertical asymptotes. Conduct a limit test to determine the behaviour of the curve on either side of the asymptote.

a. $y = \dfrac{x}{x + 5}$

b. $f(x) = \dfrac{x + 2}{x - 2}$

c. $s = \dfrac{1}{(t - 3)^2}$

d. $y = \dfrac{x^2 - x - 6}{x - 3}$

e. $f(x) = \dfrac{6}{(x + 3)(x - 1)}$

f. $y = \dfrac{x^2}{x^2 - 1}$

5. For each of the following, determine the equations of any horizontal asymptotes. Then state whether the curve approaches the asymptote from above or below.

a. $y = \dfrac{x}{x + 4}$

b. $f(x) = \dfrac{2x}{x^2 - 1}$

c. $g(t) = \dfrac{3t^2 + 4}{t^2 - 1}$

d. $y = \dfrac{3x^2 - 8x - 7}{x - 4}$

PART B

K 6. For each of the following, check for discontinuities and then use at least two other tests to make a rough sketch of the curve. Verify using a calculator.

a. $y = \frac{x - 3}{x + 5}$

b. $f(x) = \frac{5}{(x + 2)^2}$

c. $g(t) = \frac{t^2 - 2t - 15}{t - 5}$

d. $y = \frac{(2 + x)(3 - 2x)}{(x^2 - 3x)}$

7. Determine the equation of the oblique asymptote for each of the following:

a. $f(x) = \frac{3x^2 - 2x - 17}{x - 3}$

b. $f(x) = \frac{2x^2 + 9x + 2}{2x + 3}$

c. $f(x) = \frac{x^3 - 1}{x^2 + 2x}$

d. $f(x) = \frac{x^3 - x^2 - 9x + 15}{x^2 - 4x + 3}$

8. a. For question 7, part a., determine whether the curve approaches the asymptote from above or below.

b. For question 7, part b., determine the direction from which the curve approaches the asymptote.

9. For each function, determine any vertical or horizontal asymptotes and describe its behaviour on each side of any vertical asymptote.

a. $f(x) = \frac{3x - 1}{x + 5}$

b. $g(x) = \frac{x^2 + 3x - 2}{(x - 1)^2}$

c. $h(x) = \frac{x^2 + x - 6}{x^2 - 4}$

d. $m(x) = \frac{5x^2 - 3x + 2}{x - 2}$

A 10. Use the algorithm for curve sketching to sketch the graph of each function.

a. $f(x) = \frac{3 - x}{2x + 5}$

b. $h(t) = 2t^3 - 15t^2 + 36t - 10$

c. $y = \frac{20}{x^2 + 4}$

d. $s(t) = t + \frac{1}{t}$

e. $g(x) = \frac{2x^2 + 5x + 2}{x + 3}$

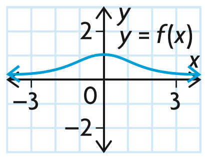
f. $s(t) = \frac{t^2 + 4t - 21}{t - 3}, t \geq -7$

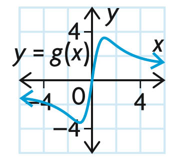
11. Consider the function $y = \frac{ax + b}{cx + d}$, where a, b, c, and d are constants, $a \neq 0, c \neq 0$.

a. Determine the horizontal asymptote of the graph.

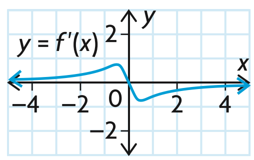
b. Determine the vertical asymptote of the graph.

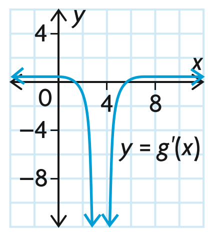
12. Use the features of each function's graph to sketch the graph of its first derivative.

a. 

b. 

13. A function's derivative is shown in each graph. Use the graph to sketch a possible graph for the original function.

a. 

b. 

14. Let $f(x) = \frac{-x - 3}{x^2 - 5x - 14}$, $g(x) = \frac{x - x^3}{x - 3}$, $h(x) = \frac{x^3 - 1}{x^2 + 4}$, and $r(x) = \frac{x^2 + x - 6}{x^2 - 16}$. How can you can tell from its equation which of these functions has

a. a horizontal asymptote?

b. an oblique asymptote?

c. no vertical asymptote?

Explain. Determine the equations of all asymptote(s) for each function. Describe the behaviour of each function close to its asymptotes.

PART C

T

15. Find constants a and b such that the graph of the function defined by $f(x) = \frac{ax + 5}{3 - bx}$ will have a vertical asymptote at $x = 5$ and a horizontal asymptote at $y = -3$.

16. To understand why we cannot work with the symbol ∞ as though it were a real number, consider the functions $f(x) = \frac{x^2 + 1}{x + 1}$ and $g(x) = \frac{x^2 + 2x + 1}{x + 1}$.

a. Show that $\lim_{x \to +\infty} f(x) = +\infty$ and $\lim_{x \to +\infty} g(x) = +\infty$.

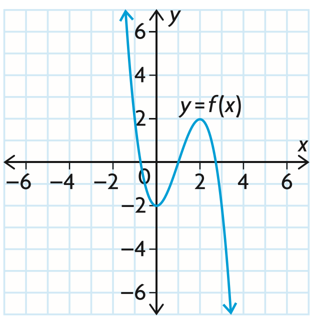
b. Evaluate $\lim_{x \to +\infty} [f(x) - g(x)]$, and show that the limit is not zero.

17. Use the algorithm for curve sketching to sketch the graph of the function $f(x) = \frac{2x^2 - 2x}{x^2 - 9}$.

Mid-Chapter Review

1. Use a graphing calculator or graphing software to graph each of the following functions. Inspect the graph to determine where the function is increasing and where it is decreasing.
   a. $y = 3x^2 - 12x + 7$
   b. $y = 4x^3 - 12x^2 + 8$
   c. $f(x) = \frac{x+2}{x+3}$
   d. $f(x) = \frac{x^2-1}{x^2+3}$
2. Determine where $g(x) = 2x^3 - 3x^2 - 12x + 15$ is increasing and where it is decreasing.
3. Graph $f(x)$ if $f'(x) < 0$ when $x < -2$ and $x > 3$, $f'(x) > 0$ when $-2 < x < 3$, $f(-2) = 0$, and $f(3) = 5$.
4. Find all the critical numbers of each function.
   a. $y = -2x^2 + 16x - 31$
   b. $y = x^3 - 27x$
   c. $y = x^4 - 4x^2$
   d. $y = 3x^5 - 25x^3 + 60x$
   e. $y = \frac{x^2-1}{x^2+1}$
   f. $y = \frac{x}{x^2+2}$
5. For each function, find the critical numbers. Use the first derivative test to identify the local maximum and minimum values.
   a. $g(x) = 2x^3 - 9x^2 + 12x$
   b. $g(x) = x^3 - 2x^2 - 4x$
6. Find a value of k that gives $f(x) = x^2 + kx + 2$ a local minimum value of 1.
7. For $f(x) = x^4 - 32x + 4$, find the critical numbers, the intervals on which the function increases and decreases, and all the local extrema. Use graphing technology to verify your results.
8. Find the vertical asymptote(s) of the graph of each function. Describe the behaviour of $f(x)$ to the left and right of each asymptote.
   a. $f(x) = \frac{x-1}{x+2}$
   b. $f(x) = \frac{1}{9-x^2}$
   c. $f(x) = \frac{x^2-4}{3x+9}$
   d. $f(x) = \frac{2-x}{3x^2-13x-10}$
9. For each of the following, determine the equations of any horizontal asymptotes. Then state whether the curve approaches the asymptote from above or below.
   a. $y = \frac{3x-1}{x+5}$
   b. $f(x) = \frac{x^2+3x-2}{(x-1)^2}$
10. For each of the following, check for discontinuities and state the equation of any vertical asymptotes. Conduct a limit test to determine the behaviour of the curve on either side of the asymptote.
    a. $f(x) = \frac{x}{(x-5)^2}$
    b. $f(x) = \frac{5}{x^2+9}$
    c. $f(x) = \frac{x-2}{x^2-12x+12}$

11. a. What does $f'(x) > 0$ imply about $f(x)$?
    b. What does $f'(x) < 0$ imply about $f(x)$?
12. A diver dives from the 3 m springboard. The diver's height above the water, in metres, at t seconds is $h(t) = -4.9t^2 + 9.5t + 2.2$.
    a. When is the height of the diver increasing? When is it decreasing?
    b. When is the velocity of the diver increasing? When is it decreasing?
13. The concentration, C, of a drug injected into the bloodstream t hours after injection can be modelled by $C(t) = \frac{t}{4} + 2t^{-2}$. Determine when the concentration of the drug is increasing and when it is decreasing.



14. Graph $y = f'(x)$ for the function shown at the left.
15. For each function $f(x)$,
    i. find the critical numbers
    ii. determine where the function increases and decreases
    iii. determine whether each critical number is at a local maximum, a local minimum, or neither
    iv. use all the information to sketch the graph
    a. $f(x) = x^2 - 7x - 18$
    b. $f(x) = -2x^3 + 9x^2 + 3$
    c. $f(x) = 2x^4 - 4x^2 + 2$
    d. $f(x) = x^5 - 5x$
16. Determine the equations of any vertical or horizontal asymptotes for each function. Describe the behaviour of the function on each side of any vertical or horizontal asymptote.
    a. $f(x) = \dfrac{x - 5}{2x + 1}$
    b. $g(x) = \dfrac{x^2 - 4x - 5}{(x + 2)^2}$
    c. $h(x) = \dfrac{x^2 + 2x - 15}{9 - x^2}$
    d. $m(x) = \dfrac{2x^2 + x + 1}{x + 4}$
17. Find each limit.
    a. $\lim\limits_{x \to \infty} \dfrac{3 - 2x}{3x}$
    b. $\lim\limits_{x \to \infty} \dfrac{x^2 - 2x + 5}{6x^2 + 2x - 1}$
    c. $\lim\limits_{x \to \infty} \dfrac{7 + 2x^2 - 3x^3}{x^3 - 4x^2 + 3x}$
    d. $\lim\limits_{x \to \infty} \dfrac{5 - 2x^3}{x^4 - 4x}$
    e. $\lim\limits_{x \to \infty} \dfrac{2x^5 - 1}{3x^4 - x^2 - 2}$
    f. $\lim\limits_{x \to \infty} \dfrac{x^2 + 3x - 18}{(x - 3)^2}$
    g. $\lim\limits_{x \to \infty} \dfrac{x^2 - 4x - 5}{x^2 - 1}$
    h. $\lim\limits_{x \to \infty} \left(5x + 4 - \dfrac{7}{x + 3}\right)$