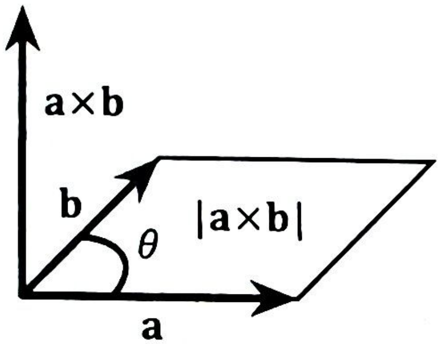
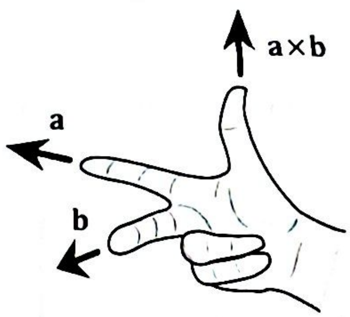
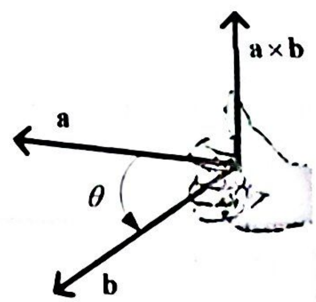
Day 6: 7.6 The Cross Product of 2 Vectors

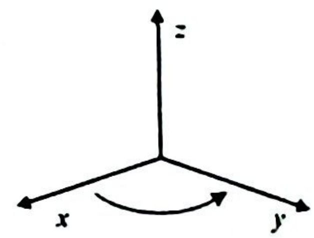


The **cross product** (vector product) is defined only in R^3 since the cross product of two vectors $\vec{a}$ *and* $\vec{b}$ is a vector that is perpendicular to both $\vec{a}$ *and* $\vec{b}$.

The cross product of the vectors $\vec{a}$ *and* $\vec{b}$ is the vector whose direction is perpendicular to $\vec{a}$ *and* $\vec{b}$, such that $\vec{a}$, $\vec{b}$, *and* $\vec{a} \times \vec{b}$ form a right-handed system







The vector $\vec{a} \times \vec{b}$ is the opposite of $\vec{b} \times \vec{a}$ and points in the **opposite direction**.

The magnitude of the cross product of two vectors $\vec{a}$ *and* $\vec{b}$ is:

$\left|\vec{a} \times \vec{b}\right| = |\vec{a}|\left|\vec{b}\right| \sin\theta$ where θ is the angle between $\vec{a}$ *and* $\vec{b}$, $0° \le \theta \le 180°$.

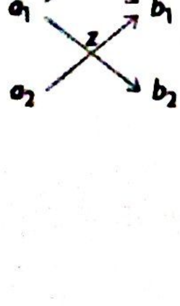
Algebraically:

Given: $\vec{a}(a_1, a_2, a_3)$ *and* $\vec{b}(b_1, b_2, b_3)$, $\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1)$

$$\vec{a} \times \vec{b} = \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right) \text{ where } \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$\vec{a}$ $\vec{b}$

$\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1)$



$y = a_3b_1 - a_1b_3$

$z = a_1b_2 - a_2b_1$

Another view of Cross Product:

$$\vec{a}\times\vec{b}=\vec{i}(a_y b_z-a_z b_y)+\vec{j}(a_z b_x-a_x b_z)+\vec{k}(a_x b_y-a_y b_x)$$

$$=\vec{i}\begin{vmatrix}a_y & a_z\\ b_y & b_z\end{vmatrix}+\vec{j}\begin{vmatrix}a_z & a_x\\ b_z & b_x\end{vmatrix}+\vec{k}\begin{vmatrix}a_x & a_y\\ b_x & b_y\end{vmatrix}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix}$$

$$\begin{array}{cccccc} & \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j}\\ \vec{a}\times\vec{b}= & a_x & a_y & a_z & a_x & a_y\\ & b_x & b_y & b_z & b_x & b_y\end{array}$$

Finding a vector perpendicular to two vectors:

If $\vec{a}$ *and* $\vec{b}$ are two non-collinear vectors in R³, then every vector perpendicular to both $\vec{a}$ *and* $\vec{b}$ is in the form $k(\vec{a}\times\vec{b})$, $k\in R$.

Properties of the Cross Product:

Let $\vec{a}, \vec{b}$ *and* $\vec{c}$ be vectors in R^3

- $\vec{a}\times\vec{b}=-\vec{b}\times\vec{a}$ *(not commutative)*
- $\vec{a}\times(\vec{b}+\vec{c})=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$ *(distributive law)*
- $(k\vec{a})\times\vec{b}=\vec{a}\times(k\vec{b})=k(\vec{a}\times\vec{b})$

Examples:

1. Find a $\underbrace{\text{vector perpendicular}}_{\vec{c}=\vec{a}\times\vec{b}}$ to both $\overset{\vec{a}}{(1, 2, 3)}$ and $\overset{\vec{b}}{(-1, 0, 4)}$.

2 0
3 4
1 -1
2 0

$\vec{c}=\left((2)(4)-(3)(0),\ 3(-1)-(4)(1),\ (1)(0)-(-1)(2)\right)$

$=(8, -7, 2)$

NOTE: We can check $\vec{a}\cdot\vec{c}=0$ and $\vec{b}\cdot\vec{c}=0$

2. If $|\vec{u}|=8$ *and* $|\vec{v}|=5$ and the angle θ between $\vec{u}$ *and* $\vec{v}$ is $30°$, find $|\vec{u}\times\vec{v}|$.

$|\vec{u}\times\vec{v}|=|\vec{u}||\vec{v}|\sin\theta$

$=(8)(5)(\sin 30)$

$=20$