Day 6 – Double Angled Formulas

$\sin 2A = 2\sin A\cos A$

$\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

$\cos 2A = \cos^2 A - \sin^2 A$

$\cos 2A = 1 - 2\sin^2 A$

$\cos 2A = 2\cos^2 A - 1$

Using the addition formulas for sine and cosine, we can derive the double angle formulae for sine and cosine.

Double Angle Formula for Sine

a. Start with the addition formula for sine, $\sin(A+B)=\sin A\cos B+\cos A\sin B$.
Let $B=A$ and simplify.

$\sin(A+A) = \sin A\cos A + \cos A\sin A$

$= 2\sin A\cos A$

Double Angle Formulae for Cosine

b. Start with the addition formula for cosine, $\cos(A+B)=\cos A\cos B-\sin A\sin B$.
Let $B=A$ and simplify.

$\cos(A+A) = \cos A\cos A - \sin A\sin A$

$\cos(2A) = \cos^2 A - \sin^2 A$

c. Using the Pythagorean identity $\sin^2 A+\cos^2 A=1$, we can derive two other forms of the double angle formulae for cosine.

Substitute $1-\cos^2 A$ for $\sin^2 A$ into the equation in part b and simplify.

$\cos(2A) = \cos^2 A - \sin^2 A$

$= \cos^2 A - (1-\cos^2 A) = 2\cos^2 A - 1$

Substitute $1-\sin^2 A$ for $\cos^2 A$ into the formula in part b and simplify.

$\cos 2A = \cos^2 A - \sin^2 A = (1-\sin^2 A) - \sin^2 A$

$= 1 - 2\sin^2 A$

The three versions of $\cos 2A$ are equivalent, but one form may be more convenient than the others in particular situations.

Ex 1: Simplify.

a) $(\sin x + \cos x)^2 - \sin 2x$

$= \sin^2 x + 2\sin x\cos x + \cos^2 x - 2\sin x\cos x$

$= \sin^2 x + \cos^2 x$

$= 1$

b) $\dfrac{\sin 2x}{1-\cos 2x}$

$= \dfrac{2\sin x\cos x}{1-(1-2\sin^2 x)}$

$= \dfrac{2\sin x\cos x}{2\sin^2 x} = \dfrac{\cos x}{\sin x} = \cot x$

Ex 2: Prove the identity $\sin 2x = 2\sin x\cos^3 x + 2\sin^3 x\cos x$

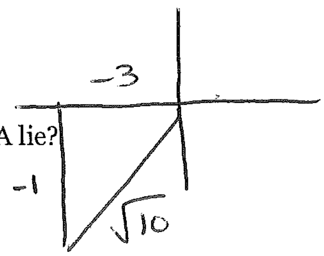
$RS = 2\sin x\cos^3 x + 2\sin^3 x\cos x$

$= 2\sin x\cos x\,[\cos^2 x + \sin^2 x]$

$= 2\sin x\cos x = \sin 2x$

$= LS$

Q.E.D



Ex 3: If $\tan A = \dfrac{1}{3}$ and $\pi < A < \dfrac{3\pi}{2}$, calculate each quantity. In what quadrant does the angle 2A lie?

Quadrant 1

a) sin2A

$= 2\sin A\cos A$

$= 2\left[\dfrac{-1}{\sqrt{10}}\right]\left[\dfrac{-3}{\sqrt{10}}\right]$

$= \dfrac{6}{10} = \dfrac{3}{5}$

b) cos2A

$= \cos^2 A - \sin^2 A$

$= \left[\dfrac{-3}{\sqrt{10}}\right]^2 - \left[\dfrac{-1}{\sqrt{10}}\right]^2$

$= \dfrac{9}{10} - \dfrac{1}{10} = \dfrac{8}{10} = \dfrac{4}{5}$

c) tan2A

$= \dfrac{\sin 2A}{\cos 2A} = \dfrac{3}{5} \div \dfrac{4}{5}$

$= \dfrac{3}{4}$

Ex 4: Use an appropriate double angle formula to simplify:

a) $2\sin \underbrace{3\theta}_{x} \cos \underbrace{3\theta}_{x}$

$= \sin 2x$

$= \sin(2\cdot 3\theta)$

$= \sin 6\theta$

b) $8\sin^2 2\theta - 4$

$= -4[-2\sin^2 2\theta + 1]$

$= -4[1 - 2\sin^2 2\theta]$

$= -4\cos(4\theta)$

c) $\dfrac{2\tan 4x}{1-\tan^2 4x}$

$= \tan(8x)$

HOMEWORK - DOUBLE ANGLED FORMULAS

1. Express each of the following as a single sine or cosine function:

a) 3 sin3Acos3A

$= \frac{3}{2}(2\sin 3A\cos 3A)$

$= \frac{3}{2}\sin 6A$

b) $2\sin\left(\frac{x}{4}\right)\cos\left(\frac{x}{4}\right)$

$= \sin\left(2\cdot\frac{x}{4}\right)$

$= \sin\left(\frac{x}{2}\right)$

c) $2-4\sin^2\left(\frac{\pi}{12}\right)$

$= 2\left[1-2\sin^2\left(\frac{\pi}{12}\right)\right]$

$= 2\left[\cos\left(2\cdot\frac{\pi}{12}\right)\right]$

$= 2\cos\frac{\pi}{6}$

$= 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$

d) $\cos^2 8x - \sin^2 8x$

$= \cos(2(8x))$

$= \cos(16x)$

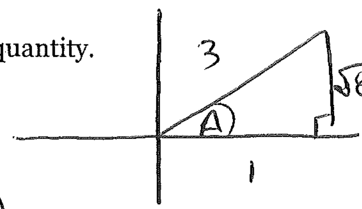
e) $\dfrac{2\tan\left(\frac{\pi}{6}\right)}{1-\tan^2\left(\frac{\pi}{6}\right)}$

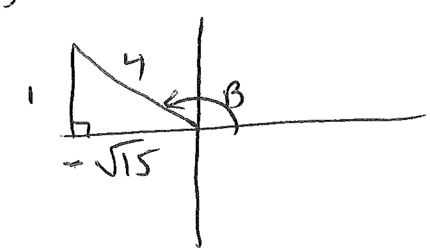
$= \tan\left(2\cdot\frac{\pi}{6}\right)$

$= \tan\left(\frac{\pi}{3}\right)$

$= \sqrt{3}$

2. If $\cos A = \frac{1}{3}$, with $0 < A < \frac{\pi}{2}$, and $\sin B = \frac{1}{4}$, with $\frac{\pi}{2} < B < \pi$, calculate each quantity.





a) $\cos(A+B)$

$= \cos A\cos B - \sin A\sin B$

$= \left(\frac{1}{3}\right)\left(\frac{-\sqrt{15}}{4}\right) - \left(\frac{\sqrt{8}}{3}\right)\left(\frac{1}{4}\right)$

$= \frac{-\sqrt{15}-\sqrt{8}}{12}$

b) $\sin(A+B)$

$= \sin A\cos B + \sin B\cos A$

$= \frac{\sqrt{8}}{3}\left(-\frac{\sqrt{15}}{4}\right) + \left(\frac{1}{4}\right)\left(\frac{1}{3}\right)$

$= \frac{-2\sqrt{30}+1}{12}$

c) cos2A

$= \cos^2 A - \sin^2 A$

$= \left(\frac{1}{3}\right)^2 - \left(\frac{\sqrt{8}}{3}\right)^2$

$= \frac{1}{9} - \frac{8}{9}$

$= -\frac{7}{9}$

d) sin2A

$= 2\sin A\cos A$

$= 2\left(\frac{\sqrt{8}}{3}\right)\left(\frac{1}{3}\right)$

$= \frac{2\sqrt{8}}{9} = \frac{4\sqrt{2}}{9}$

3. Prove the identities.

$\sin^2\theta + \cos^2\theta = 1$

$1 + \cot^2\theta = \csc^2\theta$

$\tan^2\theta + 1 = \sec^2\theta$

a) $(\cos^2 A - 1)(\tan^2 A + 1) = -\tan^2 A$

$LS = (-\sin^2 A)(\sec^2 A)$

$= -\dfrac{\sin^2 A}{\cos^2 A}$

$= -\tan^2 A$

b) $\dfrac{\cos A - \sin 2A}{\cos 2A + \sin A - 1} = \cot A$

$LS = \dfrac{\cos A - 2\sin A\cos A}{\cancel{1} - 2\sin^2 A + \sin A \cancel{-1}}$

$= \dfrac{\cos A(1 - 2\sin A)}{\sin A(1 - 2\sin A)}$

$= \dfrac{\cos A}{\sin A} = \cot A = RS.$

QED.

c) $\dfrac{\sin 2\theta}{1 - \cos 2\theta} = 2\csc 2\theta - \tan\theta$

$LS = \dfrac{2\sin\theta\cos\theta}{1 - (1 - 2\sin^2\theta)}$

$= \dfrac{2\sin\theta\cos\theta}{2\sin^2\theta} = \cot\theta$

$RS = \dfrac{2}{2\sin\theta\cos\theta} - \dfrac{\sin\theta}{\cos\theta}$

$= \dfrac{2 - 2\sin^2\theta}{2\sin\theta\cos\theta} = \dfrac{2\cos^2\theta}{2\sin\theta\cos\theta} = \cot\theta$

d) $\dfrac{\cos 2x}{1 + \sin 2x} = \dfrac{\cot x - 1}{\cot x + 1}$

$RS = \dfrac{\frac{\cos x}{\sin x} - 1}{\frac{\cos x}{\sin x} + 1} = \dfrac{\frac{\cos x - \sin x}{\sin x}}{\frac{\cos x + \sin x}{\sin x}}$

$= \dfrac{\cos x - \sin x}{\cos x + \sin x} \cdot \dfrac{\cos x + \sin x}{\cos x + \sin x}$

$= \dfrac{\cos^2 x - \sin^2 x}{\cos^2 x + 2\sin x\cos x + \sin^2 x} = \dfrac{\cos 2x}{1 + \sin 2x}$

$= LS$ QED.

e) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$LS = \sin(\theta + 2\theta)$

$= \sin\theta\cos 2\theta + \cos\theta\sin 2\theta$

$= \sin\theta(1 - 2\sin^2\theta) + \cos\theta(2\sin\theta\cos\theta)$

$= \sin\theta - 2\sin^3\theta + 2\sin\theta\cos^2\theta$

$= \sin\theta - 2\sin^3\theta + 2\sin\theta(1 - \sin^2\theta)$

$= 3\sin\theta - 4\sin^3\theta$

$= RS$ QED.

f) $\cos 3A = 4\cos^3 A - 3\cos A$

$LS = \cos(A + 2A)$

$= \cos A\cos 2A - \sin A\sin 2A$

$= \cos A(2\cos^2 A - 1) - \sin A(2\sin A\cos A)$

$= 2\cos^3 A - \cos A - 2\cos A(1 - \cos^2 A)$

$= 2\cos^3 A - \cos A - 2\cos A + 2\cos^3 A$

$= 4\cos^3 A - 3\cos A$

$= RS$ QED.