2.2 – The Factor Theorem

Day 2 – Rational Zero Theorem

The Rational Zero Theorem *(leading coefficient isn't 1)*

If $ax - b$ is a factor of a polynomial $P(x)$ with integer coefficients, then

- **b is a factor of the constant term of $P(x)$**
- **a is a factor of the leading coefficient of $P(x)$**

* *Once one factor of a polynomial is found, division can be used to determine other factors.*

Example One– Determine possible factors of $h(x) = 3x^3 + 2x^2 - 7x + 2$

Factors of 'b' are: $\pm 1, \pm 2$

Factors of 'a' are $\pm 1, \pm 3$.

$\frac{b}{a}$: $\pm 1, \pm 2, \pm\frac{1}{3}, \pm\frac{2}{3}$

Example Two – Factor $f(x) = 15x^3 + 16x^2 - 25x + 6$

$\frac{b}{a}$: $\pm 1, \pm 2, \pm 3, \pm 6, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{1}{5}, \pm\frac{2}{5}, \pm\frac{3}{5}, \pm\frac{6}{5}, \pm\frac{1}{15}, \pm\frac{2}{15}, \pm\frac{6}{5}$

$P(1) \neq 0$ $\quad P(-1) = -15 + 16 + 25 + 6 \neq 0$

$P(2) \neq 0$ $\quad P(-2) = 0 \Rightarrow (x+2)$ is a factor.

$$\begin{array}{r|rrrr} & 15 & 16 & -25 & 6 \\ -2 & \downarrow & -30 & 28 & -6 \\ \hline & 15 & -14 & 3 & 0 \end{array}$$

$\therefore P(x) = (x+2)(15x^2 - 14x + 3)$

$= (x+2)(15x^2 - 9x - 5x + 3)$

$= (x+2)(3x(5x-3) - 1(5x-3))$

$= (x+2)(3x-1)(5x-3)$

Factoring a sum and difference of cubes

An expression that contains two perfect cubes that are added together is called a sum of cubes and can be factored as follows:

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

An expression that contains two perfect cubes where one is subtracted from the other is called a difference of cubes and can be factored as follows:

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

Example Three - Factor $27x^3 + 125$ $= (\underbrace{3x}_{a})^3 + (\underbrace{5}_{b})^3$

$\therefore \quad 27x^3 + 125 = \underbrace{(3x+5)}_{(a+b)}\underbrace{(9x^2 + 15x + 25)}_{(a^2 + ab + b^2)}$

Example Four – Factor $7x^3 - 448$

$= 7(x^3 - 64) \quad a = x \quad b = 4 \quad$ use $a^3 - b^3$

$= 7(x-4)(x^2 + 4x + 16)$

2.1 – 2.2 Remainder Theorem and Factor Theorem Practice

** Complete the following questions on a separate piece of paper*

1. Divide the following polynomials, and write each result in in ***quotient form***.
   a. $f(x) = x^3 - 4x^2 + 2x + 5$ by $g(x) = x - 2$
   b. $f(x) = 12x^3 - 11x^2 + 9x + 18$ by $g(x) = 4x + 3$
   c. $f(x) = 2x^3 + 4x^2 - 5$ by $g(x) = x + 3$

2. Determine whether each given value of x is a zero of the given function
   a. $x = 1, P(x) = x^3 - x^2 + x - 1$
   b. $x = -2, P(x) = -2x^3 - 5x^2 + 3x + 10$
   c. $x = -3, P(x) = -x^4 - 3x^3 - 2x^2 + 18$
   d. $x = 4, P(x) = x^4 - x^2 - 8x - 16$

3. Factor fully:
   a. $f(x) = x^3 + 2x^2 - 5x - 6$
   b. $f(x) = x^3 - 3x^2 - 3x - 4$
   c. $f(x) = 2x^3 + x^2 - 2x - 1$
   d. $f(x) = 3x^3 + 6x^2 + x + 2$

4. Factor fully:
   a. $x^3 + 27$
   b. $8x^3 - 125$

Answers:

1. a. $$\frac{x^3 - 4x^2 + 2x + 5}{x - 2} = x^2 - 2x - 2 + \frac{1}{x - 2}$$
   b. $$\frac{12x^3 - 11x^2 + 9x + 18}{4x + 3} = 3x^2 - 5x + 6$$
   c. $$\frac{2x^3 + 4x^2 - 5}{x + 3} = 2x^2 - 2x + 6 - \frac{23}{x + 3}$$

2. a. yes, $P(1) = 0$
   b. yes, $P(-2) = 0$
   c. yes, $P(-3) = 0$
   d. no, $P(4) \neq 0$

3. a. $(x+1)(x-2)(x+3)$
   b. $(x-4)(x^2 + x + 1)$
   c. $(x+1)(x-1)(2x+1)$
   d. $(x+2)(3x^2 + 1)$

4. a. $(x+3)(x^2 - 3x + 9)$
   b. $(2x-5)(4x^2 + 10x + 25)$

2.1 - 2.2 Remainder Theorem & Factor Theorem Practice

1) a)

$$
\begin{array}{r|l}
 & x^2 - 2x - 2 \\
\hline
x-2 & x^3 - 4x^2 + 2x + 5 \\
 & x^3 - 2x^2 \\
\hline
 & -2x^2 + 2x \\
 & -2x^2 + 4x \\
\hline
 & -2x + 5 \\
 & -2x + 4 \\
\hline
 & 1
\end{array}
$$

or

$$
\begin{array}{r|rrrr}
2 & 1 & -4 & 2 & 5 \\
 & \downarrow & 2 & -4 & -4 \\
\hline
 & 1 & -2 & -2 & 1
\end{array}
$$

Remainder

$\therefore \dfrac{f(x)}{x-2} = x^2 - 2x - 2 + \dfrac{1}{x-2}$

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b)

$$
\begin{array}{r|l}
 & 3x^2 - 5x + 6 \\
\hline
4x+3 & 12x^3 - 11x^2 + 9x + 18 \\
 & 12x^3 + 9x^2 \\
\hline
 & -20x^2 + 9x \\
 & -20x^2 - 15x \\
\hline
 & 24x + 18 \\
 & 24x + 18 \\
\hline
 & 0
\end{array}
$$

$$
\begin{array}{r|rrrr}
-\frac{3}{4} & 12 & -11 & 9 & 18 \\
 & \downarrow & -9 & 15 & -18 \\
\hline
 & \underbrace{12 \quad -20 \quad 24}_{\text{Quotient divided by 4}} & & & 0
\end{array}
$$

$\therefore \dfrac{f(x)}{4x+3} = 3x^2 - 5x + 6$

1 c)

$$\begin{array}{r|rrrr} -3 & 2 & 4 & 0 & -5 \\ & \downarrow & -6 & 6 & -18 \\ \hline & 2 & -2 & 6 & -23 \end{array}$$

$$\therefore \quad \frac{f(x)}{x+3} = 2x^2 - 2x + 6 - \frac{23}{x+3}$$

2) a) $P(x) = x^3 - x^2 + x - 1$

$P(1) = 1 - 1 + 1 - 1$

$= 0 \quad \therefore \; x - 1$ is a factor.

b) $P(-2) = 0 \quad \therefore \; x + 2$ is a factor

c) $P(-3) = 0 \Rightarrow x + 3$ is a factor

d) $p(4) = 4^4 - 4^2 - 8(4) - 16 \neq 0$

$\therefore x - 4$ is not a factor.

3) a) $f(x) = x^3 + 2x^2 - 5x - 6$

$\frac{b}{a}: \pm 1, \pm 2, \pm 3, \pm 6$

$f(1) = 1 + 2 - 5 - 6 \neq 0$ $\quad$ $f(-1) = -1 + 2 + 5 - 6 = 0$

$\therefore \; x+1$ is a factor

$$\begin{array}{r|rrrr} -1 & 1 & 2 & -5 & -6 \\ & \downarrow & -1 & -1 & 6 \\ \hline & 1 & 1 & -6 & 0 \end{array}$$

$\therefore \; f(x) = (x+1)(x^2 + x - 6)$

$= (x+1)(x+3)(x-2)$

3b) $f(x) = x^3 - 3x^2 - 3x - 4$

$\frac{b}{a}$: $\pm 1, \pm 2, \pm 4$

$f(1) = 1 - 3 - 3 - 4 \neq 0$

$f(-1) = -1 - 3 + 3 - 4 \neq 0$

$f(2) = 8 - 3(4) - 3(2) - 4 \neq 0$

$f(-2) = -8 - 3(4) + 6 - 4$

$f(4) = 4^3 - 3(4)^2 - 3(4) - 4$

$= 64 - 48 - 12 - 4 = 0 \Rightarrow$ $x - 4$ is a factor

$$
\begin{array}{r|rrrr}
 & 1 & -3 & -3 & -4 \\
4 & \downarrow & 4 & 4 & 4 \\
\hline
 & 1 & 1 & 1 & 0
\end{array}
$$

$\therefore f(x) = (x-4)(x^2 + x + 1) \longrightarrow$ does not factor

3c) $f(x) = 2x^3 + x^2 - 2x - 1$ & grouping

$= (2x^3 + x^2) - (2x + 1)$

$= x^2(2x+1) - 1(2x+1)$

$= (2x+1)(x^2 - 1)$

$= (2x+1)(x-1)(x+1)$

3d)

$f(x) = (3x^3 + 6x^2) + (x+2)$

$= 3x^2(x+2) + 1(x+2)$

$= (3x^2 + 1)(x+2)$

4a) $x^3 + 27 = x^3 + 3^3$

$= (x+3)(x^2 - 3x + 9)$

4b) $8x^3 - 125 = (2x)^3 - 5^3$

$= (2x-5)(4x^2 + 10x + 25)$

NOTE: $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$, $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$