Day 3: 2.3 The Product Rule

Rule 5 The Product Rule
The product of two differentiable functions f(x) and g(x) is differentiable:
$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$

Proof of Rule 5 $P(x) = f(x)g(x)$

$$P'(x) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} g(x+h)\left[\frac{f(x+h)-f(x)}{h}\right] + \lim_{h\to 0} f(x)\left[\frac{g(x+h)-g(x)}{h}\right]$$

$$= g(x)f'(x) + f(x)g'(x)$$

Example of Rule 5
What is the derivative of $f(x) = (x^2+3)(x^3+2)$?

Using Rule 5

$$f'(x) = (x^2+3)'(x^3+2) + (x^2+3)(x^3+2)'$$

$$= 2x(x^3+2) + 3x^2(x^2+3)$$

$$= 2x^4 + 4x + 3x^4 + 9x^2$$

$$= 5x^4 + 9x^2 + 4x$$

Expanding, then Rule 4

$$f(x) = (x^2+3)(x^3+2)$$

FOIL

$$= x^5 + 2x^2 + 3x^3 + 6$$

$$f'(x) = 5x^4 + 9x^2 + 4x$$

Example 1: Find y′.

a. $y = (x+2)(3x^2 - x)$

$y' = 1\,(3x^2 - x) + (x+2)(6x-1)$

b. $f(x) = (x^4 - 3x^2 + 2x)(x^3 - 2x + 3)$

$f'(x) = (4x^3 - 6x + 2)(x^3 - 2x + 3) + (x^4 - 3x^2 + 2x)(3x^2 - 2)$

c. $y = (2x^4 - 3x + 5)\left(x^2 - \sqrt{x} + 2x\right)$

$y' = (8x^3 - 3)(x^2 - \sqrt{x} + 2x) + (2x^4 - 3x + 5)\left(2x - \frac{1}{2\sqrt{x}} + 2\right)$

Rule 5b The Extended Product Rule
The product of two differentiable functions u and v is differentiable:

$$\frac{d}{dx}(uvw) = \frac{du}{dx}vw + u\frac{dv}{dx}w + uv\frac{dw}{dx}$$

Example 2: Find the g′(x) if g(x) = (x + 3)(x² – 1)(x³ + 2)

$g'(x) = 1\,(x^2-1)(x^3+2) + (x+3)(2x)(x^3+2) + (x+3)(x^2-1)(3x^2)$

INVESTIGATION: Find y′ using the product rule for (a) and extended product rule for (b)

a. $y = [g(x)]^2 = g(x)g(x)$

$y' = g'(x)g(x) + g(x)g'(x)$

$= 2\,g(x)\,g'(x)$

b. $y = [g(x)]^3 = g(x)g(x)\,g(x)$

$y' = g'gg + gg'g + ggg'$

$= 3[g(x)]^2 \cdot g'(x)$

If $y = [g(x)]^n$, then the derivative will be $n[g(x)]^{n-1} \cdot g'(x)$

Example 3: Find y′.

a. $y = (3x + 5)^{10}$

$y' = 10(3x+5)^9(3)$

$= 30(3x+5)^9$

b. $y = (3x^2 - 2x + \sqrt{x})^7$

$y' = 7(3x^2 - 2x + \sqrt{x})^6\left(6x - 2 + \frac{1}{2\sqrt{x}}\right)$