The **cost function C(x)** is the total cost of making x items. If the cost per item is fixed, it is equal to the cost per item (c) times the number of items produced (x), or

$$C(x) = c \bullet x$$

The **price function p(x)** – also called the demand function – describes how price affects the number of items sold. Normally, when the price increases, customers will not demand as many items, and so x will decrease. To sell more items, the price usually has to decrease.

The **revenue function R(x)** is the income from sales. It is equal to the price times the number sold, or
$$R(x) = x \bullet p(x)$$

The **profit function P(x)** is the money that is left over from the revenue (income) after the costs (expenses) have been subtracted. In other words,

$$P(x) = R(x) - C(x) \quad = x \, p(x) - C(x)$$

BREAK EVEN: Revenue = Cost

The **break-even point** occurs when the total revenue equals the total cost - or, in other words, when the profit is zero. To solve for a break-even quantity, set P(x) = 0 and solve for x using factored form or the quadratic formula.

Warm Up: You are the sole owner of a denim store downtown, Toronto. Last week, you sold 200 pairs of jeans priced at $36. You buy the pants from a local manufacturer located in Montreal, Quebec. To operate your business, it costs you average $20 per pants.
a) Calculate your **total revenue** for the last week.
b) Determine the total profit.

a) $R = (200)(36) = 7200$

b) $Cost = (200)(20) = 4000$

$Profit = 7200 - 4000 = 3200$

Ever since you took grade 11 math course, you have been wondering if you can apply **maximizing total revenue and profit** concept in your business to make more money. When you set your price for $36, you sell average of 200 pairs a week. After doing a mini survey, you find out that for each $2 increase in price, you sell 5 fewer pants. To operate your business, it costs you average $20 per pants.

Initial $p = 36$ items $= 200$

a) What price will **maximize** your total **revenue**? Calculate the max revenue.

Let x represent number of times price increases

$R = (36 + 2x)(200 - 5x)$
$\quad$ (Price) (# of jeans)

$36 + 2x = 0$
$2x = -36$
$x = -18$

$200 - 5x = 0$
$5x = 200$
$x = 40$

# price	p	R
x=0 → 36	200	7200
x=1 → 38	195	7410
40	190	7600

$x_v = \dfrac{-18 + 40}{2} = \dfrac{22}{2} = 11 \longrightarrow$ number of times price increase

$R_v = (36 + 2(11))(200 - 5(11))$
$\quad = \# 8410 \longrightarrow$ max revenue

1

b) What price will **maximize** your total **profit**? What is the **total profit** at this price?

$$P(x) = (36+2x)(200-5x) - 20(200-5x)$$
$$= 7200 - 180x + 400x$$
$$\quad - 10x^2 - 4000 + 100x$$

$$P(x) = -10x^2 + 320x + 3200.$$

$$= -10(x^2 - 32x) + 3200$$
$$= -10(x^2 - 32x + 256 - 256) + 3200$$
$$= -10(x-16)^2 + 5760$$

$$\underline{\underline{x=16}} \qquad Price = 36 + 2x$$
$$= 36 + 2(16)$$
$$= \$68.$$

c) Calculate break-even point? What is the **price** at these points?

$$P(x) = 0$$
$$-10x^2 + 320x + 3200 = 0$$

$$x^2 - 32x - 320 = 0 \qquad \longrightarrow \quad Use\ QF$$

$$(x-40)(x+8) = 0$$

$$\qquad \qquad \longrightarrow x = -8$$

$$x = 40$$

$$Price = 36 + 2x \qquad\qquad Price = 36 - 16$$
$$= 36 + 80 \qquad\qquad\qquad = \$20.$$
$$= 116$$
$$\quad \$$$

Maximizing the Area

A barn is to be formed by enclosing a rectangular fenced area in an open field. The barn must also be split in half by a fence, in order to separate the cattle from the chicken. Given that you only have 300 m of fence to enclose the barn, what are the dimensions of the barn that give the maximum area?

$$Area = xy.$$
$$= x(150 - 1.5x)$$

$$= -1.5x^2 + 150x$$
$$= -1.5(x^2 - 100x)$$
$$= -1.5(x^2 - 100x + 2500 - 2500)$$
$$= -1.5(x-50)^2 + 3750$$
$$x = 50\ m$$
$$y = 75\ m$$

Zeros: 0, 100

$$\frac{x}{2} = 50$$
$$y = 150 - 1.5(50)$$
$$= 75m$$
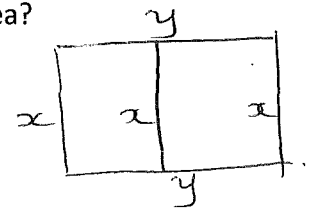
$$P = 300$$
$$3x + 2y = 300$$
$$2y = 300 - 3x$$
$$y = 150 - 1.5x$$

$$Area = 3750\ m^2$$

dimensions.

1. When priced at $40 each, a toy company sells 5000 toys. The manufacturer estimates that each $1 increase in price will decrease sales by 100 units. Find the unit price of a toy that will maximize the total revenue.

Step 1 "Let "x" rep number of price change

Rev = Price × Amount Sold

$R(x) = (40 + 1x)(5000 - 100x)$

$40 + x = 0$
$\boxed{x = -40}$
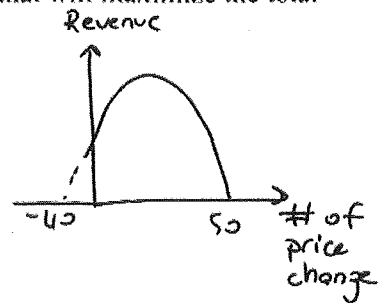
$5000 - 100x = 0$
$5000 = 100x$
$\boxed{50 = x}$

Revenue graph, # of price change axis, -40 and 50 marked

Step 2 Averaging zeros

$x = \dfrac{-40 + 50}{2}$
$\boxed{x = 5}$ → We just found the # of price change

Step 3 Price = 40 + x
            = 40 + 5
            = 45

∴ The price must be $45 to maximize the revenue.

2. The city transit system carries 24800 bus riders per day for a fare of $1.85. The city hopes to reduce car pollution by getting more people to ride the bus, while maximizing the transit system's revenue at the same time. A survey indicates that the number of riders will increase by 800 for every $0.05 decrease in the fare. What fare will produce the **greatest revenue**?

Revenue = Price · Riders   Let "x" rep # of change in price

$R(x) = (1.85 - 0.05x)(24800 + 800x)$

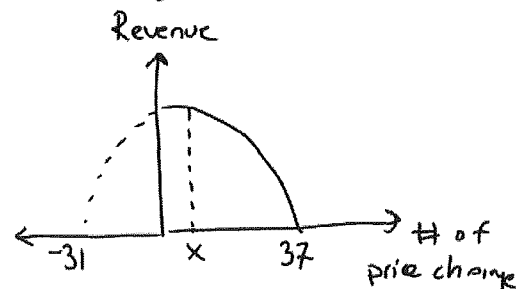We need to find the "x" coordinate of the vertex to calculate the price.

$1.85 - 0.05x = 0$
$1.85 = 0.05x$
$\boxed{x = 37}$

$24800 + 800x = 0$
$800x = -24800$
$\boxed{x = -31}$

Revenue graph, # of price change axis, -31, x, 37 marked

$\dfrac{\text{Average zeros}}{}$
$x = \dfrac{-31 + 37}{2}$
$\boxed{x = 3}$ → This tells us the # of price change

Price = 1.85 - 0.05(3)
      = 1.7

∴ When the fare is $1.7, it will produce the max revenue.

3. The Thunderbirds professional indoor soccer team has 900 season ticket holders. The management of the team wants to increase the current price of $400. A survey indicated that for every $20 increase in price, the team will lose 15 season ticket holders. What price would maximize revenue from season ticket holders? What is the **maximum revenue** the team could receive?

Revenue = Price × Tickets Sold   Let "x" rep # of price change

Step 1 $R(x) = (400 + 20x)(900 - 15x)$ NOTE: We need to find the vertex in this quest'
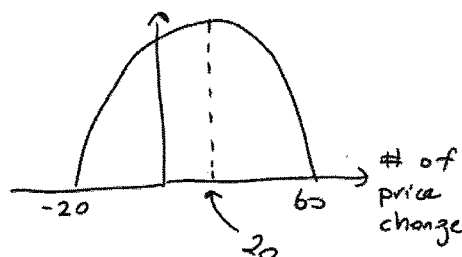
$400 + 20x = 0$ $900 - 15x = 0$
$20x = -400$ $900 = 15x$
$\boxed{x = -20}$ $\boxed{x = 60}$

Step 2
$x = \dfrac{-20 + 60}{2}$ $y = (400 + 20 \cdot 20)(900 - 15 \cdot 20)$
$\boxed{x = 20}$ Price $= (400 + 400)(900 - 300)$
                           $= (800)(600)$ ← tickets sold
(# of price change / Price) $= 480,000$ ← revenue max

∴ $800 will maximize the revenue which is $480,000.

4. The school council sells sweatshirts to raise funds. The students sell 500 sweatshirts a year at $45 each. They are planning to decrease the price to generate more sales. An informal survey was taken showing that for every $1 decrease in price they can expect to sell an additional 20 sweatshirts. If the survey results are correct, what price would maximize revenue from sweatshirt sales? How many sweatshirts must be sold? What would be the **maximum revenue** generated?

Let "x" rep # of price change.

Revenue = Price × Amount Sold

Step 1 $R(x) = (45 - 1x)(500 + 20x)$

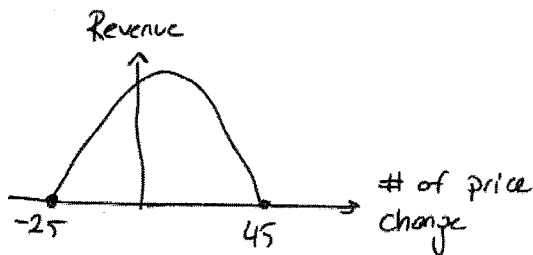$45 - x = 0$ $500 + 20x = 0$
$\boxed{45 = x}$ $20x = -500$
                      $\boxed{x = -25}$

Step 2 Find vertex   Price   Amount Sold
$x = \dfrac{-25 + 45}{2}$ $y = (45 - 10)(500 + 20 \cdot 10)$
$\boxed{x = 10}$ $= (35)(700)$
                             $= 24,500$

∴ Price is $35
Amount sold is 700
and max revenue is $24,500

5. An amusement park charges $8 admission and averages 2000 visitors per day. A survey shows that for each $1 increase in the admission price, 100 fewer people would visit the park.
   a) Determine what price the amusement park should charge to <u>maximize</u> **revenue**.
   b) At what price(s) will the revenue be equal to $0?
   c) Find the price(s) that would generate revenue of $11500.

a) Let "x" rep the number of change in price.

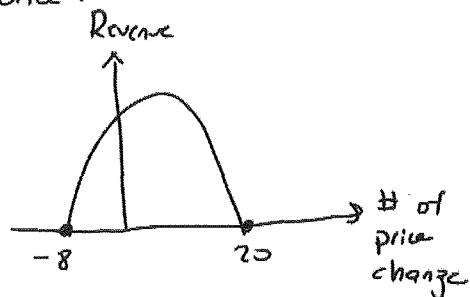Revenue = Price × Visitors

$$R(x) = (8 + 1 \cdot x)(2000 - 100x)$$

$8 + x = 0$
$$\boxed{x = -8}$$

$2000 - 100x = 0$
$2000 = 100x$
$$\boxed{20 = x}$$

$$x = \frac{-8 + 20}{2}$$
$$\boxed{x = 6}$$

Price = 8 + 1·6
= 14

∴ When the price is $14, the park generates max revenue.

b) The x-intercepts on the graph represents $0 revenue.

Price = 8 + 1(-8)   Price = 8 + 1·20
   = $0   = $28

∴ The revenue will be 0 when the price is either $0 or $28.

c) $R(x) = (8 + x)(2000 - 100x)$

$11500 = (8 + x)(2000 - 100x)$

Find x" coordinates

$11500 = 16000 - 800x + 2000x - 100x^2$

$0 = -100x^2 + 1200x + 16000 - 11500$

$0 = -100x^2 + 1200x + 4500$

$0 = -100(x^2 - 12x - 45)$

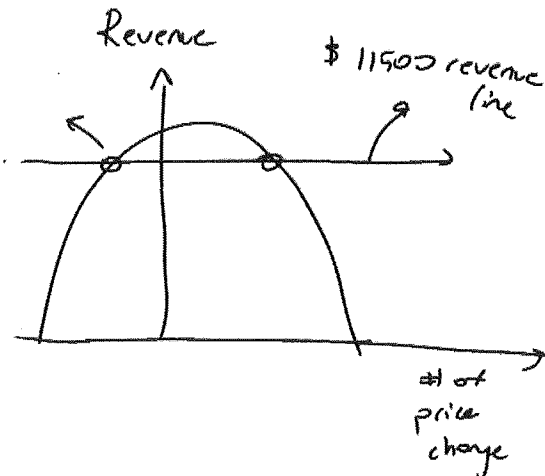$0 = -100(x + 3)(x - 15)$

$x + 3 = 0$
$$\boxed{x = -3}$$

$x - 15 = 0$
$$\boxed{x = 15}$$

Price = $5   Price change   Price = $23

When the price is $5 or $23, the revenue is $11500

Practice: Maximizing PROFIT – BREAK EVEN Problems

1. Research for a given orchard has shown that, if 100 pear trees are planted, then the annual revenue is $90 per tree. If more trees are planted, they have less room to grow and generate fewer pears per tree. As a result, the annual revenue per tree is reduced by $0.70 for each additional tree planted. No matter how many trees are planted, the cost of maintaining each tree is $7.40 per year. How many pear trees should be planted to maximize the **profit** from the orchard for one year?
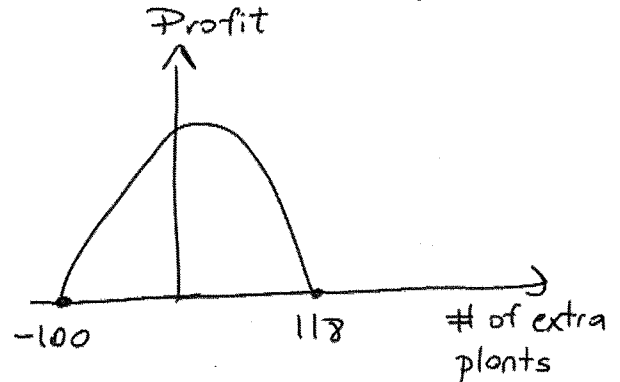
x of vertex

$R(x) = $ Price $\cdot$ Amount

$= (90 - 0.70x)(100 + x)$

$C(x) = $ Cost Price $\cdot$ Amount

$= 7.40(100 + x)$

$P(x) = (90 - 0.70x)(100 + x) - 7.40(100 + x)$

$= (100 + x)(90 - 0.70x - 7.40)$

$= (100 + x)(82.60 - 0.70x)$

GCF $= (100 + x)$

Profit

$100 + x = 0$

$\boxed{x = -100}$

$82.60 - 0.70x = 0$

$82.60 = 0.70x$

$\boxed{x = 118}$

Vertex (x, y)

$x = \dfrac{-100 + 118}{2}$

$\boxed{x = 9}$

∴ 109 pear trees must be planted to maximize the profit.

2. The demand function for a new product is $p(x) = -5x + 39$, where p represents the selling price of the product and x is the number sold in thousands. The cost function is $C(x) = 4x + 30$.
a) Find the value of x that will maximize the **profit**.
b) Find the **break-even** quantities.

a) Profit = Revenue - Cost

$\qquad$ amount $\qquad$ price

$$R(x) = x \cdot (-5x + 39)$$

$$P(x) = x(-5x+39) - (4x+30)$$

$$= -5x^2 + 39x - 4x - 30$$

$$P(x) = -5x^2 + 35x - 30 \quad \longrightarrow \quad \text{We need to find the "x" of vertex}$$

$$= -5(x^2 - 7x) - 30 \qquad -7 \div 2 = -3.5$$

$$\qquad\qquad\qquad (-3.5)^2 = 12.25 **$$

$$= -5(x - 3.5)^2 \quad \underline{\text{we don't need "y" of vertex}}$$

$$\therefore 3.5 \times 1000 = 3500 \text{ units will max profit.}$$

b) Break-even when your profit is ZERO (x-int)

$$P(x) = -5x^2 + 35x - 30$$

$$0 = -5x^2 + 35x - 30 \qquad \text{First GCF} = -5$$

$$0 = -5(x^2 - 7x + 6)$$

$$\begin{array}{c|c|c} M & 4 & N \\ \hline 6 & -7 & -1, -6 \end{array}$$

$$0 = -5(x - 1)(x - 6)$$

$$x - 1 = 0 \qquad\qquad x - 6 = 0$$

$$\boxed{x = 1} \qquad\qquad \boxed{x = 6}$$

$$\therefore \text{Break-even quantities are } 1000 \text{ and } 6000 \text{ units.}$$

3. A car rental agency has 150 cars. The owner finds that, at a price of $48 per day, he can rent all the cars. For each $2 increase in price, the demand is less and 4 fewer cars are rented. For each car that is rented, there are routine maintenance costs of $5 per day.

"x" rep. # of change in price.

a) What rental charge will maximize **profit**?

b) Find the **break-even point**s for the profit as well as the price at these points.

$$\text{Profit} = \text{Revenue} - \text{Cost}$$

$$R(x) = \overset{\text{price}}{(48+2x)}\overset{\text{amount}}{(150-4x)}$$

$$\text{Cost} = 5(150-4x)$$

$$= (48+2x)(150-4x) - 5(150-4x) \qquad GCF = (150-4x)$$

$$= (48+2x - 5)(150-4x)$$

$$P(x) = (43+2x)(150-4x)$$

➤ The question is asking the price; therefore, you need to find x-int.
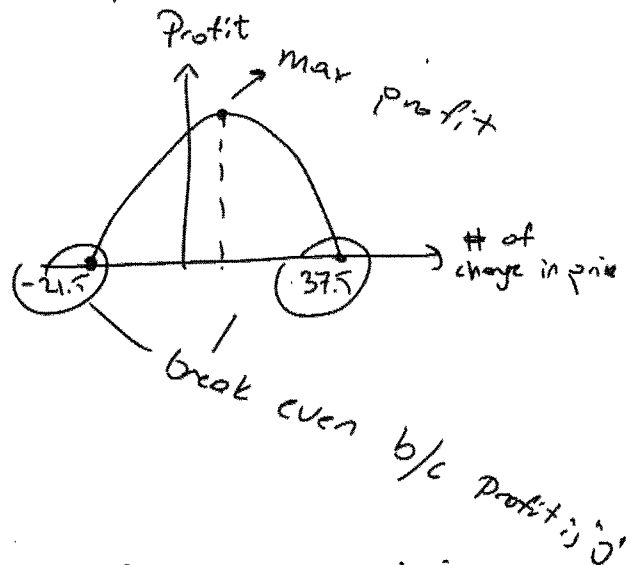
Profit is 0

$$0 = (43+2x)(150-4x)$$

$$43+2x = 0 \qquad\qquad 150-4x = 0$$
$$2x = -43 \qquad\qquad 150 = 4x$$
$$\boxed{x = -21.5} \qquad\qquad \boxed{37.5 = x}$$

Vertex

$$X = \frac{-21.5 + 37.5}{2}$$

$$\boxed{X = 8}$$

Price = 48 + 2x
$$= 48 + 2(8)$$
$$= \$64$$

∴ When the price is $64, the car company maximizes profit.

b) Break-even points are the X-int on the graph. In other words, profit equals ZERO.

$$\boxed{x = -21.5} \quad \text{and} \quad \boxed{x = 37.5}$$

Price $= 48 + 2(-21.5)$ Price $= 48 + 2(37.5)$
$\quad = 48 - 43$ $\quad = 48 + 75$
$\quad = \$5$ $\quad = \$123$

When the price is set to $5 or $123, the company break even.

4. An auditorium has seats for 1200 people. For the past several days, the auditorium has been filled to capacity for each show. Tickets currently cost $5.00 and the owner wants to increase the ticket prices. He estimates that for each $0.50 increase in price, 100 fewer people will attend. $3.5
a) What ticket price will maximize the **profit**?
b) Fin the **break-even points** for the profit as well as the price at these points.

Let "x" rep the number of price change

$$Profit = Revenue - Cost \longrightarrow R(x) = Price \cdot Amount$$

$$= (5 + 0.50x)(1200 - 100x)$$

$$C(x) = 35(1200 - 100x)$$

$$P(x) = (5 + 0.50x)(1200 - 100x) - 3.5(1200-100x)$$

$$P(x) = (1200 - 100x)(5 + 0.50x - 3.5)$$

$$P(x) = (1200 - 100x)(1.5 + 0.50x)$$

$$1200 - 100x = 0 \qquad\qquad 1.5 + 0.50x = 0$$

$$1200 = 100x \qquad\qquad -1.5 = 0.50x$$

$$\boxed{12 = x} \qquad\qquad \boxed{-3 = x}$$

X of vertex is the price change

$$X = \frac{-3 + 12}{2} = \boxed{4.5} \longleftarrow \text{when you change the price 4.5 times you'll max profit}$$

$$\left.\begin{array}{c}\end{array}\right\} \quad Price = 5 + 0.50(4.5)$$

$$= 5 + 2.25$$

$$= 7.75$$

∴ When the price is set for $7.75, company max profit.