Chapter 2

DERIVATIVES

Imagine a driver speeding down a highway, at 140 km/h. He hears a police siren and is quickly pulled over. The police officer tells him that he was speeding, but the driver argues that because he has travelled 200 km from home in two hours, his average speed is within the 100 km/h limit. The driver's argument fails because police officers charge speeders based on their instantaneous speed, not their average speed.

There are many other situations in which the instantaneous rate of change is more important than the average rate of change. In calculus, the derivative is a tool for finding instantaneous rates of change. This chapter shows how the derivative can be determined and applied in a great variety of situations.

CHAPTER EXPECTATIONS

In this chapter, you will

- understand and determine derivatives of polynomial and simple rational functions from first principles, **Section 2.1**
- identify examples of functions that are not differentiable, **Section 2.1**
- justify and use the rules for determining derivatives, **Sections 2.2, 2.3, 2.4, 2.5**
- identify composition as two functions applied in succession, **Section 2.5**
- determine the composition of two functions expressed in notation, and decompose a given composite function into its parts, **Section 2.5**
- use the derivative to solve problems involving instantaneous rates of change, **Sections 2.2, 2.3, 2.4, 2.5**

Review of Prerequisite Skills

Before beginning your study of derivatives, it may be helpful to review the following concepts from previous courses and the previous chapter:

- Working with the properties of exponents
- Simplifying radical expressions
- Finding the slopes of parallel and perpendicular lines
- Simplifying rational expressions
- Expanding and factoring algebraic expressions
- Evaluating expressions
- Working with the difference quotient

Exercise

1. Use the exponent laws to simplify each of the following expressions. Express your answers with positive exponents.

a. $a^5 \times a^3$

b. $(-2a^2)^3$

c. $\dfrac{4p^7 \times 6p^9}{12p^{15}}$

d. $(a^4b^{-5})(a^{-6}b^{-2})$

e. $(3e^6)(2e^3)^4$

f. $\dfrac{(3a^{-4})[2a^3(-b)^3]}{12a^5b^2}$

2. Simplify and write each expression in exponential form.

a. $(x^{\frac{1}{2}})(x^{\frac{2}{3}})$

b. $(8x^6)^{\frac{2}{3}}$

c. $\dfrac{\sqrt{a}\sqrt[3]{a}}{\sqrt{a}}$

3. Determine the slope of a line that is perpendicular to a line with each given slope.

a. $\dfrac{2}{3}$

b. $-\dfrac{1}{2}$

c. $\dfrac{5}{3}$

d. -1

4. Determine the equation of each of the following lines:

a. passing through points $A(-3, -4)$ and $B(9, -2)$

b. passing through point $A(-2, -5)$ and parallel to the line $3x - 2y = 5$

c. perpendicular to the line $y = \frac{3}{4}x - 6$ and passing through point $A(4, -3)$

5. Expand, and collect like terms.

a. $(x - 3y)(2x + y)$

b. $(x - 2)(x^2 - 3x + 4)$

c. $(6x - 3)(2x + 7)$

d. $2(x + y) - 5(3x - 8y)$

e. $(2x - 3y)^2 + (5x + y)^2$

f. $3x(2x - y)^2 - x(5x - y)(5x + y)$

6. Simplify each expression.

a. $\frac{3x(x + 2)}{x^2} \times \frac{5x^3}{2x(x + 2)}$

b. $\frac{y}{(y + 2)(y - 5)} \times \frac{(y - 5)^2}{4y^3}$

c. $\frac{4}{(h + k)} \div \frac{9}{2(h + k)}$

d. $\frac{(x + y)(x - y)}{5(x - y)} \div \frac{(x + y)^3}{10}$

e. $\frac{x - 7}{2x} + \frac{5x}{x - 1}$

f. $\frac{x + 1}{x - 2} - \frac{x + 2}{x + 3}$

7. Factor each expression completely.

a. $4k^2 - 9$

b. $x^2 + 4x - 32$

c. $3a^2 - 4a - 7$

d. $x^4 - 1$

e. $x^3 - y^3$

f. $r^4 - 5r^2 + 4$

8. Use the factor theorem to factor the following expressions:

a. $a^3 - b^3$ b. $a^5 - b^5$ c. $a^7 - b^7$ d. $a^n - b^n$

9. If $f(x) = -2x^4 + 3x^2 + 7 - 2x$, evaluate

a. $f(2)$ b. $f(-1)$ c. $f\left(\frac{1}{2}\right)$ d. $f(-0.25)$

10. Rationalize the denominator in each of the following expressions:

a. $\frac{3}{\sqrt{2}}$ b. $\frac{4 - \sqrt{2}}{\sqrt{3}}$ c. $\frac{2 + 3\sqrt{2}}{3 - 4\sqrt{2}}$ d. $\frac{3\sqrt{2} - 4\sqrt{3}}{3\sqrt{2} + 4\sqrt{3}}$

11. a. If $f(x) = 3x^2 - 2x$, determine the expression for the difference quotient $\frac{f(a + h) - f(a)}{h}$ when $a = 2$. Explain what this expression can be used for.

b. Evaluate the expression you found in part a. for a small value of h where $h = 0.01$.

c. Explain what the value you determined in part b. represents.

CAREER LINK Investigate

CHAPTER 2: THE ELASTICITY OF DEMAND

Have you ever wondered how businesses set prices for their goods and services? An important idea in marketing is *elasticity of demand*, or the response of consumers to a change in price. Consumers respond differently to a change in the price of a staple item, such as bread, than they do to a change in the price of a luxury item, such as jewellery. A family would probably still buy the same quantity of bread if the price increased by 20%. This is called *inelastic* demand. If the price of a gold chain, however, increased by 20%, sales would likely decrease 40% or more. This is called *elastic* demand. Mathematically, elasticity is defined as the negative of the relative (percent) change in the number demanded $\left(\frac{\Delta n}{n}\right)$ divided by the relative (percent) change in the price $\left(\frac{\Delta p}{p}\right)$:

$$E = -\left[\left(\frac{\Delta n}{n}\right) \div \left(\frac{\Delta p}{p}\right)\right]$$

For example, if a store increased the price of a CD from \$17.99 to \$19.99, and the number sold per week went from 120 to 80, the elasticity would be

$$E = -\left[\left(\frac{80 - 120}{120}\right) \div \left(\frac{19.99 - 17.99}{17.99}\right)\right] \doteq 3.00$$

An elasticity of about 3 means that the change in demand is three times as large, in percent terms, as the change in price. The CDs have an elastic demand because a small change in price can cause a large change in demand. In general, goods or services with elasticities greater than one ($E > 1$) are considered elastic (e.g., new cars), and those with elasticities less than one ($E < 1$) are considered inelastic (e.g., milk). In our example, we calculated the average elasticity between two price levels, but, in reality, businesses want to know the elasticity at a specific, or *instantaneous*, price level. In this chapter, you will develop the rules of differentiation that will enable you to calculate the instantaneous rate of change for several classes of functions.

Case Study—Marketer: Product Pricing

In addition to developing advertising strategies, marketing departments also conduct research into, and make decisions on, pricing. Suppose that the demand–price relationship for weekly movie rentals at a convenience store is $n(p) = \frac{500}{p}$, where $n(p)$ is demand and p is price.

DISCUSSION QUESTIONS

1. Generate two lists, each with at least five goods and services that you have purchased recently, classifying each of the goods and services as having elastic or inelastic demand.
2. Calculate and discuss the elasticity if a movie rental fee increases from \$1.99 to \$2.99.

Section 2.1—The Derivative Function

In this chapter, we will extend the concepts of the slope of a tangent and the rate of change to introduce the **derivative**. We will examine the methods of differentiation, which we can use to determine the derivatives of polynomial and rational functions. These methods include the use of the power rule, sum and difference rules, and product and quotient rules, as well as the chain rule for the composition of functions.

The Derivative at a Point

In the previous chapter, we encountered limits of the form $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$.

This limit has two interpretations: the slope of the tangent to the graph $y = f(x)$ at the point $(a, f(a))$, and the instantaneous rate of change of $y = f(x)$ with respect to x at $x = a$. Since this limit plays a central role in calculus, it is given a name and a concise notation. It is called the **derivative of $f(x)$ at $x = a$**. It is denoted by $f'(a)$ and is read as "f prime of a."

The **derivative of f at the number a** is given by $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$, provided that this limit exists.

EXAMPLE 1 **Selecting a limit strategy to determine the derivative at a number**

Determine the derivative of $f(x) = x^2$ at $x = -3$.

Solution

Using the definition, the derivative at $x = -3$ is given by

$$\begin{aligned} f'(-3) &= \lim_{h \to 0} \frac{f(-3+h) - f(-3)}{h} \\ &= \lim_{h \to 0} \frac{(-3+h)^2 - (-3)^2}{h} && \text{(Expand)} \\ &= \lim_{h \to 0} \frac{9 - 6h + h^2 - 9}{h} && \text{(Simplify and factor)} \\ &= \lim_{h \to 0} \frac{h(-6+h)}{h} \\ &= \lim_{h \to 0} (-6 + h) \\ &= -6 \end{aligned}$$

Therefore, the derivative of $f(x) = x^2$ at $x = -3$ is -6.

An alternative way of writing the derivative of f at the number a is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

In applications where we are required to find the value of the derivative for a number of particular values of x, using the definition repeatedly for each value is tedious.

The next example illustrates the efficiency of calculating the derivative of $f(x)$ at an arbitrary value of x and using the result to determine the derivatives at a number of particular x-values.

EXAMPLE 2

Connecting the derivative of a function to an arbitrary value

a. Determine the derivative of $f(x) = x^2$ at an arbitrary value of x.
b. Determine the slopes of the tangents to the parabola $y = x^2$ at $x = -2$, 0, and 1.

Solution

a. Using the definition, $f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{(x + h)^2 - x^2}{h} \quad \text{(Expand)}$$

$$= \lim_{h \to 0} \frac{x^2 + 2hx + h^2 - x^2}{h} \quad \text{(Simplify and factor)}$$

$$= \lim_{h \to 0} \frac{h(2x + h)}{h}$$

$$= \lim_{h \to 0} (2x + h)$$

$$= 2x$$

The derivative of $f(x) = x^2$ at an arbitrary value of x is $f'(x) = 2x$.

b. The required slopes of the tangents to $y = x^2$ are obtained by evaluating the derivative $f'(x) = 2x$ at the given x-values. We obtain the slopes by substituting for x:

$f'(-2) = -4$ $\qquad$ $f'(0) = 0$ $\qquad$ $f'(1) = 2$

The slopes are -4, 0, and 2, respectively.

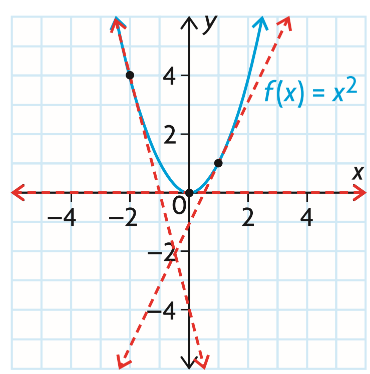
In fact, knowing the x-coordinate of a point on the parabola $y = x^2$, we can easily find the slope of the tangent at that point. For example, given the x-coordinates of points on the curve, we can produce the following table.

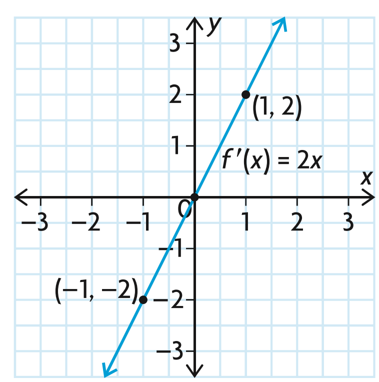
The slope of the tangent to the curve $f(x) = x^2$ at a point $P(x, y)$ is given by the derivative $f'(x) = 2x$. For each x-value, there is an associated value $2x$.

For the Parabola $f(x) = x^2$

$P(x, y)$	x-Coordinate of P	Slope of Tangent at P
$(-2, 4)$	-2	$2(-2) = -4$
$(-1, 1)$	-1	-2
$(0, 0)$	0	0
$(1, 1)$	1	2
$(2, 4)$	2	4
(a, a^2)	a	$2a$

The graphs of $f(x) = x^2$ and the derivative function $f'(x) = 2x$ are shown below. The tangents at $x = -2$, 0, and 1 are shown on the graph of $f(x) = x^2$.





Notice that the graph of the derivative function of the quadratic function (of degree two) is a linear function (of degree one).

INVESTIGATION

A. Determine the derivative with respect to x of each of the following functions:

a. $f(x) = x^3$ b. $f(x) = x^4$ c. $f(x) = x^5$

B. In Example 2, we showed that the derivative of $f(x) = x^2$ is $f'(x) = 2x$. Referring to step 1, what pattern do you see developing?

C. Use the pattern from step 2 to predict the derivative of $f(x) = x^{39}$.

D. What do you think $f'(x)$ would be for $f(x) = x^n$, where n is a positive integer?

The Derivative Function

The derivative of f at $x = a$ is a number $f'(a)$. If we let a be arbitrary and assume a general value in the domain of f, the derivative f' is a function. For example, if $f(x) = x^2, f'(x) = 2x$, which is itself a function.

The Definition of the Derivative Function

The derivative of $f(x)$ with respect to x is the function $f'(x)$, where

$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$, provided that this limit exists.

The $f'(x)$ notation for this limit was developed by Joseph Louis Lagrange (1736–1813), a French mathematician. When you use this limit to determine the derivative of a function, it is called determining the derivative from first principles.

In Chapter 1, we discussed velocity at a point. We can now define (instantaneous) velocity as the derivative of position with respect to time. If the position of a body at time t is $s(t)$, then the velocity of the body at time t is

$$v(t) = s'(t) = \lim_{h \to 0} \frac{s(t + h) - s(t)}{h}.$$

Likewise, the (instantaneous) rate of change of $f(x)$ with respect to x is the function $f'(x)$, whose value is $f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$.

EXAMPLE 3 **Determining the derivative from first principles**

Determine the derivative $f'(t)$ of the function $f(t) = \sqrt{t}, t \geq 0$.

Solution

Using the definition,

$$\begin{aligned} f'(t) &= \lim_{h \to 0} \frac{f(t + h) - f(t)}{h} \\ &= \lim_{h \to 0} \frac{\sqrt{t + h} - \sqrt{t}}{h} \\ &= \lim_{h \to 0} \frac{\sqrt{t + h} - \sqrt{t}}{h}\left(\frac{\sqrt{t + h} + \sqrt{t}}{\sqrt{t + h} + \sqrt{t}}\right) && \text{(Rationalize the numerator)} \\ &= \lim_{h \to 0} \frac{(t + h) - t}{h(\sqrt{t + h} + \sqrt{t})} \\ &= \lim_{h \to 0} \frac{h}{h(\sqrt{t + h} + \sqrt{t})} \\ &= \lim_{h \to 0} \frac{1}{\sqrt{t + h} + \sqrt{t}} \\ &= \frac{1}{2\sqrt{t}}, \text{ for } t > 0 \end{aligned}$$

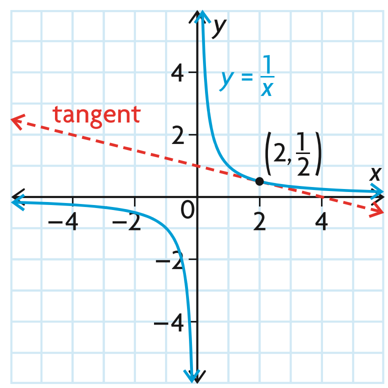
Note that $f(t) = \sqrt{t}$ is defined for all instances of $t \geq 0$, whereas its derivative $f'(t) = \dfrac{1}{2\sqrt{t}}$ is defined only for instances when $t > 0$. From this, we can see that a function need not have a derivative throughout its entire domain.

EXAMPLE 4 Selecting a strategy involving the derivative to determine the equation of a tangent

Determine an equation of the tangent to the graph of $f(x) = \frac{1}{x}$ at the point where $x = 2$.

Solution

When $x = 2$, $y = \frac{1}{2}$. The graph of $y = \frac{1}{x}$, the point $\left(2, \frac{1}{2}\right)$, and the tangent at the point are shown.



First find $f'(x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{x}{x(x+h)} - \frac{x+h}{x(x+h)}}{h}$$

(Simplify the fraction)

$$= \lim_{h \to 0} \frac{x - (x+h)}{h(x+h)x}$$

$$= \lim_{h \to 0} \frac{-1}{(x+h)x}$$

$$= -\frac{1}{x^2}$$

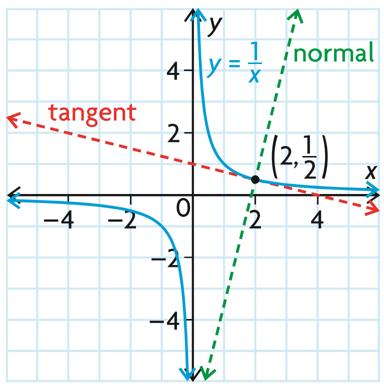
The slope of the tangent at $x = 2$ is $m = f'(2) = -\frac{1}{4}$. The equation of the tangent is $y - \frac{1}{2} = -\frac{1}{4}(x - 2)$ or, in standard form, $x + 4y - 4 = 0$.

EXAMPLE 5 Selecting a strategy involving the derivative to solve a problem

Determine an equation of the line that is perpendicular to the tangent to the graph of $f(x) = \frac{1}{x}$ at $x = 2$ and that intersects it at the point of tangency.

Solution

In Example 4, we found that the slope of the tangent at $x = 2$ is $f'(2) = -\frac{1}{4}$, and the point of tangency is $\left(2, \frac{1}{2}\right)$. The perpendicular line has slope 4, the negative reciprocal of $-\frac{1}{4}$. Therefore, the required equation is $y - \frac{1}{2} = 4(x - 2)$, or $8x - 2y - 15 = 0$.

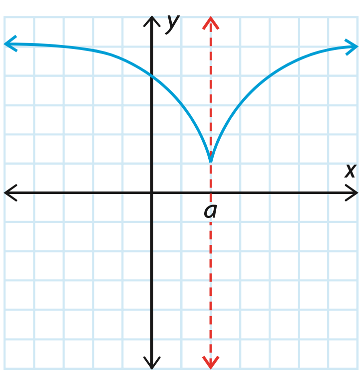


The line whose equation we found in Example 5 has a name.

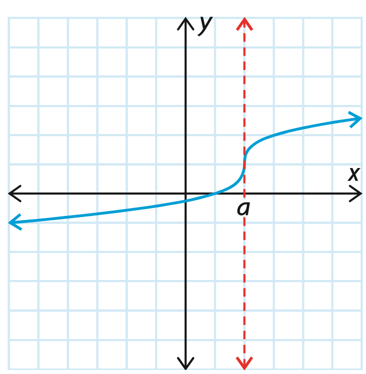
The **normal** to the graph of f at point P is the line that is perpendicular to the tangent at P.

The Existence of Derivatives

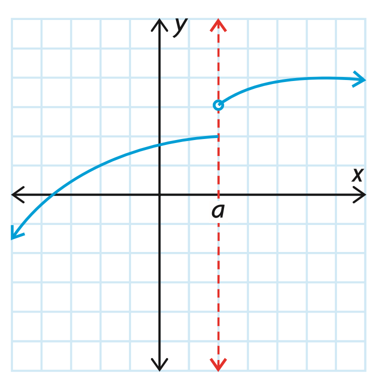
A function f is said to be **differentiable** at a if $f'(a)$ exists. At points where f is not differentiable, we say that the *derivative does not exist*. Three common ways for a derivative to fail to exist are shown.



Cusp



Vertical Tangent



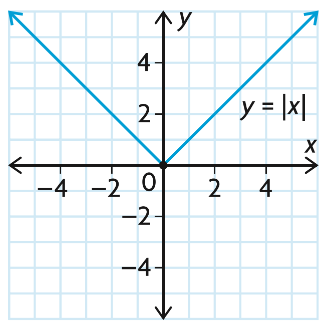
Discontinuity

EXAMPLE 6 Reasoning about differentiability at a point

Show that the absolute value function $f(x) = |x|$ is not differentiable at $x = 0$.

Solution

The graph of $f(x) = |x|$ is shown. Because the slope for $x < 0$ is -1, whereas the slope for $x > 0$ is $+1$, the graph has a "corner" at $(0, 0)$, which prevents a unique tangent from being drawn there. We can show this using the definition of a derivative.



$$f'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h}$$
$$= \lim_{h \to 0} \frac{f(h) - 0}{h}$$
$$= \lim_{h \to 0} \frac{|h|}{h}$$

Now, we will consider one-sided limits.

$|h| = h$ when $h > 0$ and $|h| = -h$ when $h < 0$.

$$\lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} (-1) = -1$$

$$\lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} (1) = 1$$

Since the left-hand limit and the right-hand limit are not the same, the derivative does not exist at $x = 0$.

From Example 6, we conclude that it is possible for a function to be **continuous** at a point and yet *not differentiable* at this point. However, if a function is differentiable at a point, then it is also continuous at this point.

Other Notation for Derivatives

Symbols other than $f'(x)$ are often used to denote the derivative. If $y = f(x)$, the symbols y' and $\frac{dy}{dx}$ are used instead of $f'(x)$. The notation $\frac{dy}{dx}$ was originally used by Leibniz and is read "dee y by dee x." For example, if $y = x^2$, the derivative is $y' = 2x$ or, in Leibniz notation, $\frac{dy}{dx} = 2x$. Similarly, in Example 4, we showed that if $y = \frac{1}{x}$, then $\frac{dy}{dx} = -\frac{1}{x^2}$. The Leibniz notation reminds us of the process by which the derivative is obtained—namely, as the limit of a difference quotient:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

By omitting y and f altogether, we can combine these notations and write $\frac{d}{dx}(x^2) = 2x$, which is read "the derivative of x^2 with respect to x is $2x$." It is important to note that $\frac{dy}{dx}$ is *not a fraction*.

IN SUMMARY

Key Ideas

- The derivative of a function f at a point $(a, f(a))$ is $f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$, or $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ if the limit exists.
- A function is said to be **differentiable** at a if $f'(a)$ exists. A function is differentiable on an interval if it is differentiable at every number in the interval.
- The derivative function for any function $f(x)$ is given by $f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$, if the limit exists.

Need to Know

- To find the derivative at a point $x = a$, you can use $\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$.
- The derivative $f'(a)$ can be interpreted as either
  - the slope of the tangent at $(a, f(a))$, or
  - the instantaneous rate of change of $f(x)$ with respect to x when $x = a$.
- Other notations for the derivative of the function $y = f(x)$ are $f'(x)$, y', and $\frac{dy}{dx}$.
- The normal to the graph of a function at point P, is a line that is perpendicular to the tangent line that passes through point P.

Exercise 2.1

PART A

1. State the domain on which f is differentiable.

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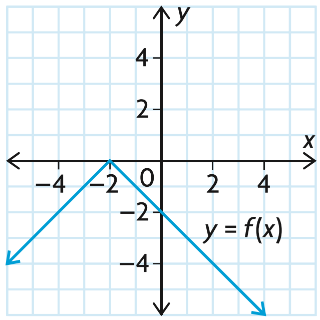

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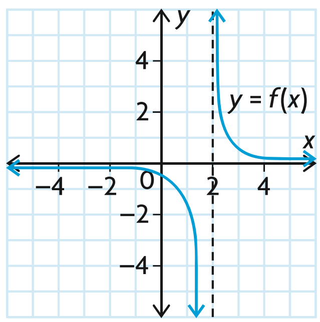

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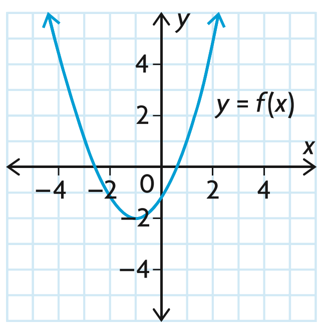

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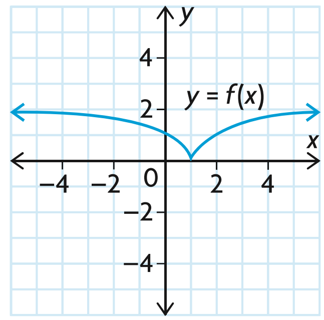

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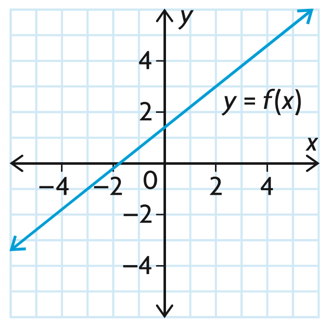

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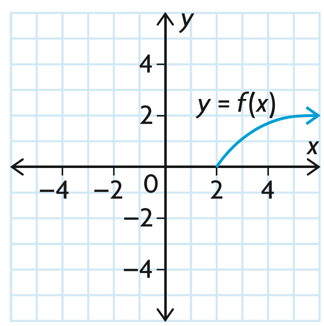


C 2. Explain what the derivative of a function represents.

3. Illustrate two situations in which a function does not have a derivative at $x = 1$.

4. For each function, find $f(a + h)$ and $f(a + h) - f(a)$.

a. $f(x) = 5x - 2$
b. $f(x) = x^2 + 3x - 1$
c. $f(x) = x^3 - 4x + 1$
d. $f(x) = x^2 + x - 6$
e. $f(x) = -7x + 4$
f. $f(x) = 4 - 2x - x^2$

PART B

K 5. For each function, find the value of the derivative $f'(a)$ for the given value of a.

a. $f(x) = x^2, a = 1$

b. $f(x) = x^2 + 3x + 1, a = 3$

c. $f(x) = \sqrt{x + 1}, a = 0$

d. $f(x) = \frac{5}{x}, a = -1$

6. Use the definition of the derivative to find $f'(x)$ for each function.

a. $f(x) = -5x - 8$

b. $f(x) = 2x^2 + 4x$

c. $f(x) = 6x^3 - 7x$

d. $f(x) = \sqrt{3x + 2}$

7. In each case, find the derivative $\frac{dy}{dx}$ from first principles.

a. $y = 6 - 7x$

b. $y = \frac{x + 1}{x - 1}$

c. $y = 3x^2$

8. Determine the slope of the tangents to $y = 2x^2 - 4x$ when $x = 0$, $x = 1$, and $x = 2$. Sketch the graph, showing these tangents.

9. a. Sketch the graph of $f(x) = x^3$.

b. Calculate the slopes of the tangents to $f(x) = x^3$ at points with x-coordinates $-2, -1, 0, 1, 2$.

c. Sketch the graph of the derivative function $f'(x)$.

d. Compare the graphs of $f(x)$ and $f'(x)$.

A 10. An object moves in a straight line with its position at time t seconds given by $s(t) = -t^2 + 8t$, where s is measured in metres. Find the velocity when $t = 0$, $t = 4$, and $t = 6$.

11. Determine an equation of the line that is tangent to the graph of $f(x) = \sqrt{x + 1}$ and parallel to $x - 6y + 4 = 0$.

12. For each function, use the definition of the derivative to determine $\frac{dy}{dx}$, where $a, b, c,$ and m are constants.

a. $y = c$

b. $y = x$

c. $y = mx + b$

d. $y = ax^2 + bx + c$

13. Does the function $f(x) = x^3$ ever have a negative slope? If so, where? Give reasons for your answer.

14. A football is kicked up into the air. Its height, h, above the ground, in metres, at t seconds can be modelled by $h(t) = 18t - 4.9t^2$.

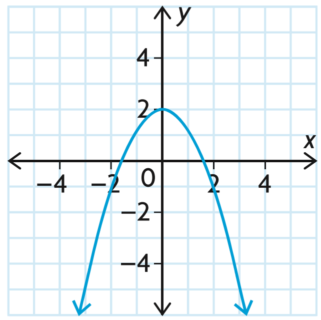
a. Determine $h'(2)$.

b. What does $h'(2)$ represent?

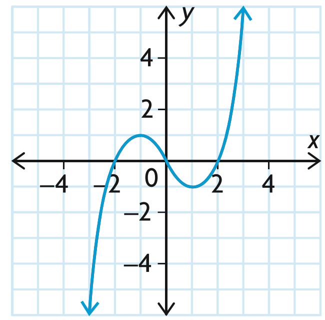
T

15. Match each function in graphs **a**, **b**, and **c** with its corresponding derivative, graphed in **d**, **e**, and **f**.

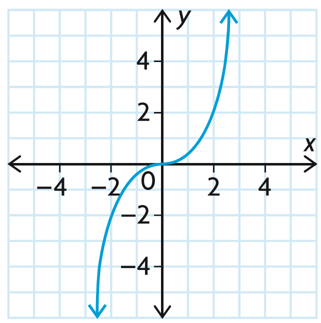
a.



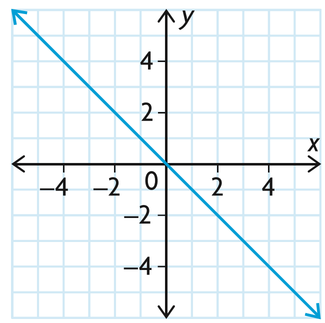
d.



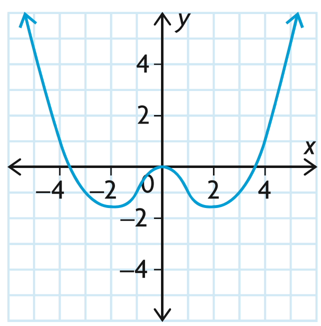
b.



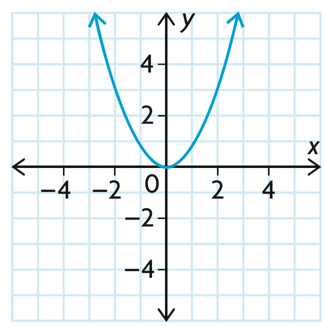
e.



c.



f.



PART C

16. For the function $f(x) = x|x|$, show that $f'(0)$ exists. What is the value?

17. If $f(a) = 0$ and $f'(a) = 6$, find $\lim_{h \to 0} \frac{f(a + h)}{2h}$.

18. Give an example of a function that is continuous on $-\infty < x < \infty$ but is not differentiable at $x = 3$.

19. At what point on the graph of $y = x^2 - 4x - 5$ is the tangent parallel to $2x - y = 1$?

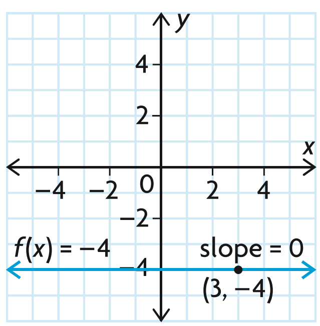
20. Determine the equations of both lines that are tangent to the graph of $f(x) = x^2$ and pass through point $(1, -3)$.

Section 2.2—The Derivatives of Polynomial Functions

We have seen that derivatives of functions are of practical use because they represent instantaneous rates of change.

Computing derivatives from the limit definition, as we did in Section 2.1, is tedious and time-consuming. In this section, we will develop some rules that simplify the process of differentiation.

We will begin developing the rules of differentiation by looking at the constant function, $f(x) = k$. Since the graph of any constant function is a horizontal line with slope zero at each point, the derivative should be zero. For example, if $f(x) = -4$, then $f'(3) = 0$. Alternatively, we can write $\frac{d}{dx}(-4) = 0$.



The Constant Function Rule

If $f(x) = k$, where k is a constant, then $f'(x) = 0$.

In Leibniz notation, $\frac{d}{dx}(k) = 0$.

Proof:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0}\frac{k - k}{h} \qquad \text{(Since } f(x) = k \text{ and } f(x+h) = k \text{ for all } h\text{)}$$

$$= \lim_{h\to 0} 0$$

$$= 0$$

EXAMPLE 1 Applying the constant function rule

a. If $f(x) = 5, f'(x) = 0$.

b. If $y = \frac{\pi}{2}, \frac{dy}{dx} = 0$.

A **power function** is a function of the form $f(x) = x^n$, where n is a real number. In the previous section, we observed that for $f(x) = x^2, f'(x) = 2x$; for $g(x) = \sqrt{x} = x^{\frac{1}{2}}, g'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$; and for $h(x) = \frac{1}{x} = x^{-1}, h'(x) = -x^{-2}$. As well, we hypothesized that $\frac{d}{dx}(x^n) = nx^{n-1}$. In fact, this is true and is called the **power rule**.

The Power Rule

If $f(x) = x^n$, where n is a real number, then $f'(x) = nx^{n-1}$.
In Leibniz notation, $\frac{d}{dx}(x^n) = nx^{n-1}$.

Proof:
(*Note:* n is a positive integer.)
Using the definition of the derivative,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}, \text{ where } f(x) = x^n$$

$$= \lim_{h\to 0}\frac{(x+h)^n - x^n}{h} \qquad \text{(Factor)}$$

$$= \lim_{h\to 0}\frac{(x+h-x)[(x+h)^{n-1} + (x+h)^{n-2}x + \dots + (x+h)x^{n-2} + x^{n-1}]}{h}$$

$$= \lim_{h\to 0}[(x+h)^{n-1} + (x+h)^{n-2}x + \dots + (x+h)x^{n-2} + x^{n-1}] \qquad \text{(Divide out } h\text{)}$$

$$= x^{n-1} + x^{n-2}(x) + \dots + (x)x^{n-2} + x^{n-1}$$

$$= x^{n-1} + x^{n-1} + \dots + x^{n-1} + x^{n-1} \qquad \text{(Since there are } n \text{ terms)}$$

$$= nx^{n-1}$$

EXAMPLE 2 Applying the power rule

a. If $f(x) = x^7$, then $f'(x) = 7x^6$.

b. If $g(x) = \frac{1}{x^3} = x^{-3}$, then $g'(x) = -3x^{-3-1} = -3x^{-4} = -\frac{3}{x^4}$.

c. If $s = t^{\frac{3}{2}}$, $\frac{ds}{dt} = \frac{3}{2}t^{\frac{1}{2}} = \frac{3}{2}\sqrt{t}$.

d. $\frac{d}{dx}(x) = 1x^{1-1} = x^0 = 1$

The Constant Multiple Rule

If $f(x) = kg(x)$, where k is a constant, then $f'(x) = kg'(x)$.
In Leibniz notation, $\frac{d}{dx}(ky) = k\frac{dy}{dx}$.

Proof:
Let $f(x) = kg(x)$. By the definition of the derivative,

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{kg(x+h)-kg(x)}{h} && \text{(Factor)} \\ &= \lim_{h\to 0}k\left[\frac{g(x+h)-g(x)}{h}\right] \\ &= k\lim_{h\to 0}\left[\frac{g(x+h)-g(x)}{h}\right] && \text{(Property of limits)} \\ &= kg'(x) \end{aligned}$$

EXAMPLE 3 Applying the constant multiple rule

Differentiate the following functions:

a. $f(x) = 7x^3$

b. $y = 12x^{\frac{4}{3}}$

Solution

a. $f(x) = 7x^3$

$$f'(x) = 7\frac{d}{dx}(x^3) = 7(3x^2) = 21x^2$$

b. $y = 12x^{\frac{4}{3}}$

$$\frac{dy}{dx} = 12\frac{d}{dx}(x^{\frac{4}{3}}) = 12\left(\frac{4}{3}x^{(\frac{4}{3}-1)}\right) = 16x^{\frac{1}{3}}$$

We conclude this section with the sum and difference rules.

The Sum Rule

If functions $p(x)$ and $q(x)$ are differentiable, and $f(x) = p(x) + q(x)$, then $f'(x) = p'(x) + q'(x)$.

In Leibniz notation, $\frac{d}{dx}(f(x)) = \frac{d}{dx}(p(x)) + \frac{d}{dx}(q(x))$.

Proof:
Let $f(x) = p(x) + q(x)$. By the definition of the derivative,

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{[p(x+h)+q(x+h)]-[p(x)+q(x)]}{h} \\ &= \lim_{h\to 0}\left\{\frac{[p(x+h)-p(x)]}{h}+\frac{[q(x+h)-q(x)]}{h}\right\} \\ &= \lim_{h\to 0}\left\{\frac{[p(x+h)-p(x)]}{h}\right\}+\lim_{h\to 0}\left\{\frac{[q(x+h)-q(x)]}{h}\right\} \\ &= p'(x)+q'(x) \end{aligned}$$

The Difference Rule

If functions $p(x)$ and $q(x)$ are differentiable, and $f(x) = p(x) - q(x)$, then $f'(x) = p'(x) - q'(x)$.

In Leibniz notation, $\frac{d}{dx}(f(x)) = \frac{d}{dx}(p(x)) - \frac{d}{dx}(q(x))$.

The proof for the difference rule is similar to the proof for the sum rule.

EXAMPLE 4 Selecting appropriate rules to determine the derivative

Differentiate the following functions:

a. $f(x) = 3x^2 - 5\sqrt{x}$

b. $y = (3x + 2)^2$

Solution

We apply the constant multiple, power, sum, and difference rules.

a. $f(x) = 3x^2 - 5\sqrt{x}$

$$f'(x) = \frac{d}{dx}(3x^2) - \frac{d}{dx}(5x^{\frac{1}{2}})$$
$$= 3\frac{d}{dx}(x^2) - 5\frac{d}{dx}(x^{\frac{1}{2}})$$
$$= 3(2x) - 5\left(\frac{1}{2}x^{-\frac{1}{2}}\right)$$
$$= 6x - \frac{5}{2}x^{-\frac{1}{2}}, \text{ or } 6x - \frac{5}{2\sqrt{x}}$$

b. We first expand $y = (3x + 2)^2$.

$$y = 9x^2 + 12x + 4$$
$$\frac{dy}{dx} = 9(2x) + 12(1) + 0$$
$$= 18x + 12$$

EXAMPLE 5 Selecting a strategy to determine the equation of a tangent

Determine the equation of the tangent to the graph of $f(x) = -x^3 + 3x^2 - 2$ at $x = 1$.

Solution A – Using the derivative

The slope of the tangent to the graph of f at any point is given by the derivative $f'(x)$.

For $\quad f(x) = -x^3 + 3x^2 - 2$

$$f'(x) = -3x^2 + 6x$$

Now, $f'(1) = -3(1)^2 + 6(1)$

$$= -3 + 6$$
$$= 3$$

The slope of the tangent at $x = 1$ is 3 and the point of tangency is $(1, f(1)) = (1, 0)$.

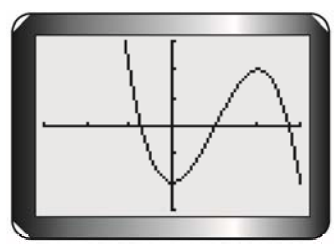
The equation of the tangent is $y - 0 = 3(x - 1)$ or $y = 3x - 3$.

Tech Support

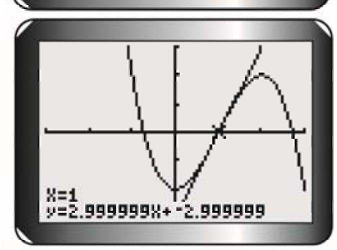
For help using the graphing calculator to graph functions and draw tangent lines See Technical Appendices p. 597 and p. 608.

Solution B – Using the graphing calculator

Draw the graph of the function using the graphing calculator.



Draw the tangent at the point on the function where $x = 1$. The calculator displays the equation of the tangent line.



The equation of the tangent line in this case is $y = 3x - 3$.

EXAMPLE 6 Connecting the derivative to horizontal tangents

Determine points on the graph in Example 5 where the tangents are horizontal.

Solution

Horizontal lines have slope zero. We need to find the values of x that satisfy $f'(x) = 0$.

$-3x^2 + 6x = 0$

$-3x(x - 2) = 0$

$x = 0$ or $x = 2$

The graph of $f(x) = -x^3 + 3x^2 - 2$ has horizontal tangents at $(0, -2)$ and $(2, 2)$.

IN SUMMARY

Key Ideas

The following table summarizes the derivative rules in this section.

Rule	Function Notation	Leibniz Notation
Constant Function Rule	If $f(x) = k$, where k is a constant, then $f'(x) = 0$.	$\frac{d}{dx}(k) = 0$
Power Rule	If $f(x) = x^n$, where n is a real number, then $f'(x) = nx^{n-1}$.	$\frac{d}{dx}(x^n) = nx^{n-1}$
Constant Multiple Rule	If $f(x) = kg(x)$, then $f'(x) = kg'(x)$.	$\frac{d}{dx}(ky) = k\frac{dy}{dx}$
Sum Rule	If $f(x) = p(x) + q(x)$, then $f'(x) = p'(x) + q'(x)$.	$\frac{d}{dx}(f(x)) = \frac{d}{dx}(p(x)) + \frac{d}{dx}(q(x))$
Difference Rule	If $f(x) = p(x) - q(x)$, then $f'(x) = p'(x) - q'(x)$.	$\frac{d}{dx}(f(x)) = \frac{d}{dx}(p(x)) - \frac{d}{dx}(q(x))$

Need to Know

- To determine the derivative of a simple rational function, such as $f(x) = \frac{4}{x^6}$, express the function as a power, then use the power rule.
  If $f(x) = 4x^{-6}$, then $f'(x) = 4(-6)x^{(-6-1)} = -24x^{-7}$
- If you have a radical function such as $g(x) = \sqrt[3]{x^5}$, rewrite the function as $g(x) = x^{\frac{5}{3}}$, then use the power rule.
  If $g(x) = x^{\frac{5}{3}}$, then $g'(x) = \frac{5}{3}x^{\frac{2}{3}} = \frac{5}{3}\sqrt[3]{x^2}$

Exercise 2.2

PART A

1. What rules do you know for calculating derivatives? Give examples of each rule.

2. Determine $f'(x)$ for each of the following functions:

a. $f(x) = 4x - 7$

b. $f(x) = x^3 - x^2$

c. $f(x) = -x^2 + 5x + 8$

d. $f(x) = \sqrt[3]{x}$

e. $f(x) = \left(\frac{x}{2}\right)^4$

f. $f(x) = x^{-3}$

K

3. Differentiate each function. Use either Leibniz notation or prime notation, depending on which is appropriate.

a. $h(x) = (2x + 3)(x + 4)$

b. $f(x) = 2x^3 + 5x^2 - 4x - 3.75$

c. $s = t^2(t^2 - 2t)$

d. $y = \frac{1}{5}x^5 + \frac{1}{3}x^3 - \frac{1}{2}x^2 + 1$

e. $g(x) = 5(x^2)^4$

f. $s(t) = \frac{t^5 - 3t^2}{2t}, t > 0$

4. Apply the differentiation rules you learned in this section to find the derivatives of the following functions:

a. $y = 3x^{\frac{5}{3}}$

b. $y = 4x^{-\frac{1}{2}} - \frac{6}{x}$

c. $y = \frac{6}{x^3} + \frac{2}{x^2} - 3$

d. $y = 9x^{-2} + 3\sqrt{x}$

e. $y = \sqrt{x} + 6\sqrt{x^3} + \sqrt{2}$

f. $y = \frac{1 + \sqrt{x}}{x}$

PART B

5. Let s represent the position of a moving object at time t. Find the velocity $v = \frac{ds}{dt}$ at time t.

a. $s = -2t^2 + 7t$

b. $s = 18 + 5t - \frac{1}{3}t^3$

c. $s = (t - 3)^2$

6. Determine $f'(a)$ for the given function $f(x)$ at the given value of a.

a. $f(x) = x^3 - \sqrt{x}, a = 4$

b. $f(x) = 7 - 6\sqrt{x} + 5x^{\frac{2}{3}}, a = 64$

7. Determine the slope of the tangent to each of the curves at the given point.

a. $y = 3x^4, (1, 3)$

b. $y = \frac{1}{x^{-5}}, (-1, -1)$

c. $y = \frac{2}{x}, (-2, -1)$

d. $y = \sqrt{16x^3}, (4, 32)$

8. Determine the slope of the tangent to the graph of each function at the point with the given x-coordinate.

a. $y = 2x^3 + 3x, x = 1$

b. $y = 2\sqrt{x} + 5, x = 4$

c. $y = \frac{16}{x^2}, x = -2$

d. $y = x^{-3}(x^{-1} + 1), x = 1$

9. Write an equation of the tangent to each curve at the given point.

a. $y = 2x - \frac{1}{x}, P(0.5, -1)$

b. $y = \frac{3}{x^2} - \frac{4}{x^3}, P(-1, 7)$

c. $y = \sqrt{3x^3}, P(3, 9)$

d. $y = \frac{1}{x}\left(x^2 + \frac{1}{x}\right), P(1, 2)$

e. $y = (\sqrt{x} - 2)(3\sqrt{x} + 8), P(4, 0)$

f. $y = \frac{\sqrt{x} - 2}{\sqrt[3]{x}}, P(1, -1)$

C 10. What is a normal to the graph of a function? Determine the equation of the normal to the graph of the function in question 9, part b., at the given point.

T 11. Determine the values of x so that the tangent to the function $y = \frac{3}{\sqrt[3]{x}}$ is parallel to the line $x + 16y + 3 = 0$.

12. Do the functions $y = \frac{1}{x}$ and $y = x^3$ ever have the same slope? If so, where?

13. Tangents are drawn to the parabola $y = x^2$ at $(2, 4)$ and $\left(-\frac{1}{8}, \frac{1}{64}\right)$. Prove that these lines are perpendicular. Illustrate with a sketch.

14. Determine the point on the parabola $y = -x^2 + 3x + 4$ where the slope of the tangent is 5. Illustrate your answer with a sketch.

15. Determine the coordinates of the points on the graph of $y = x^3 + 2$ at which the slope of the tangent is 12.

16. Show that there are two tangents to the curve $y = \frac{1}{5}x^5 - 10x$ that have a slope of 6.

17. Determine the equations of the tangents to the curve $y = 2x^2 + 3$ that pass through the following points:

a. point $(2, 3)$

b. point $(2, -7)$

18. Determine the value of a, given that the line $ax - 4y + 21 = 0$ is tangent to the graph of $y = \frac{a}{x^2}$ at $x = -2$.

A 19. It can be shown that, from a height of h metres, a person can see a distance of d kilometres to the horizon, where $d = 3.53\sqrt{h}$.

a. When the elevator of the CN Tower passes the 200 m height, how far can the passengers in the elevator see across Lake Ontario?

b. Find the rate of change of this distance with respect to height when the height of the elevator is 200 m.

20. An object drops from a cliff that is 150 m high. The distance, d, in metres, that the object has dropped at t seconds in modelled by $d(t) = 4.9t^2$.
    a. Find the average rate of change of distance with respect to time from 2 s to 5 s.
    b. Find the instantaneous rate of change of distance with respect to time at 4 s.
    c. Find the rate at which the object hits the ground to the nearest tenth.
21. A subway train travels from one station to the next in 2 min. Its distance, in kilometres, from the first station after t minutes is $s(t) = t^2 - \frac{1}{3}t^3$. At what times will the train have a velocity of 0.5 km/min?
22. While working on a high-rise building, a construction worker drops a bolt from 320 m above the ground. After t seconds, the bolt has fallen a distance of s metres, where $s(t) = 5t^2$, $0 \leq t \leq 8$. The function that gives the height of the bolt above ground at time t is $R(t) = 320 - 5t^2$. Use this function to determine the velocity of the bolt at $t = 2$.
23. Tangents are drawn from the point $(0, 3)$ to the parabola $y = -3x^2$. Find the coordinates of the points at which these tangents touch the curve. Illustrate your answer with a sketch.
24. The tangent to the cubic function that is defined by $y = x^3 - 6x^2 + 8x$ at point $A(3, -3)$ intersects the curve at another point, B. Find the coordinates of point B. Illustrate with a sketch.
25. a. Find the coordinates of the points, if any, where each function has a horizontal tangent line.
        i. $f(x) = 2x - 5x^2$
        ii. $f(x) = 4x^2 + 2x - 3$
        iii. $f(x) = x^3 - 8x^2 + 5x + 3$
    b. Suggest a graphical interpretation for each of these points.

PART C

26. Let $P(a, b)$ be a point on the curve $\sqrt{x} + \sqrt{y} = 1$. Show that the slope of the tangent at P is $-\sqrt{\frac{b}{a}}$.
27. For the power function $f(x) = x^n$, find the x-intercept of the tangent to its graph at point $(1, 1)$. What happens to the x-intercept as n increases without bound $(n \to +\infty)$? Explain the result geometrically.
28. For each function, sketch the graph of $y = f(x)$ and find an expression for $f'(x)$. Indicate any points at which $f'(x)$ does not exist.
    a. $f(x) = \begin{cases} x^2, x < 3 \\ x + 6, x \geq 3 \end{cases}$ b. $f(x) = |3x^2 - 6|$ c. $f(x) = ||x| - 1|$

Section 2.3—The Product Rule

In this section, we will develop a rule for differentiating the product of two functions, such as $f(x) = (3x^2 - 1)(x^3 + 8)$ and $g(x) = (x - 3)^3(x + 2)^2$, without first expanding the expressions.

You might suspect that the derivative of a product of two functions is simply the product of the separate derivatives. An example shows that this is not so.

EXAMPLE 1 **Reasoning about the derivative of a product of two functions**

Let $p(x) = f(x)g(x)$, where $f(x) = (x^2 + 2)$ and $g(x) = (x + 5)$.
Show that $p'(x) \neq f'(x)g'(x)$.

Solution

The expression $p(x)$ can be simplified.

$$p(x) = (x^2 + 2)(x + 5)$$
$$= x^3 + 5x^2 + 2x + 10$$
$$p'(x) = 3x^2 + 10x + 2$$

$f'(x) = 2x$ and $g'(x) = 1$, so $f'(x)g'(x) = (2x)(1) = 2x$.

Since $2x$ is not the derivative of $p(x)$, we have shown that $p'(x) \neq f'(x)g'(x)$.

The correct method for differentiating a product of two functions uses the following rule.

The Product Rule

If $p(x) = f(x)g(x)$, then $p'(x) = f'(x)g(x) + f(x)g'(x)$.
If u and v are functions of x, $\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}$.

In words, the product rule says, "the derivative of the product of two functions is equal to the derivative of the first function times the second function plus the first function times the derivative of the second function."

Proof:
$p(x) = f(x)g(x)$, so using the definition of the derivative,

$$p'(x) = \lim_{h \to 0} \frac{f(x + h)g(x + h) - f(x)g(x)}{h}.$$

To evaluate $p'(x)$, we subtract and add the same term in the numerator.

$$\text{Now, } p'(x) = \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\left\{\left[\frac{f(x+h)-f(x)}{h}\right]g(x+h) + f(x)\left[\frac{g(x+h)-g(x)}{h}\right]\right\}$$

$$= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right]\lim_{h\to 0}g(x+h) + \lim_{h\to 0}f(x)\lim_{h\to 0}\left[\frac{g(x+h)-g(x)}{h}\right]$$

$$= f'(x)g(x) + f(x)g'(x)$$

EXAMPLE 2

Applying the product rule

Differentiate $h(x) = (x^2 - 3x)(x^5 + 2)$ using the product rule.

Solution

$h(x) = (x^2 - 3x)(x^5 + 2)$

Using the product rule, we get

$$h'(x) = \frac{d}{dx}[x^2 - 3x]\cdot(x^5 + 2) + (x^2 - 3x)\frac{d}{dx}[x^5 + 2]$$

$$= (2x - 3)(x^5 + 2) + (x^2 - 3x)(5x^4)$$

$$= 2x^6 - 3x^5 + 4x - 6 + 5x^6 - 15x^5$$

$$= 7x^6 - 18x^5 + 4x - 6$$

We can, of course, differentiate the function after we first expand. The product rule will be essential, however, when we work with products of polynomials such as $f(x) = (x^2 + 9)(x^3 + 5)^4$ or non-polynomial functions such as $f(x) = (x^2 + 9)\sqrt{x^3 + 5}$.

It is not necessary to simplify an expression when you are asked to calculate the derivative at a particular value of x. Because many expressions obtained using differentiation rules are cumbersome, it is easier to substitute, then evaluate the derivative expression.

The next example could be solved by finding the product of the two polynomials and then calculating the derivative of the resulting polynomial at $x = -1$. Instead, we will apply the product rule.

EXAMPLE 3 **Selecting an efficient strategy to determine the value of the derivative**

Find the value $h'(-1)$ for the function $h(x) = (5x^3 + 7x^2 + 3)(2x^2 + x + 6)$.

Solution

$h(x) = (5x^3 + 7x^2 + 3)(2x^2 + x + 6)$

Using the product rule, we get

$$\begin{aligned} h'(x) &= (15x^2 + 14x)(2x^2 + x + 6) + (5x^3 + 7x^2 + 3)(4x + 1) \\ h'(-1) &= [15(-1)^2 + 14(-1)][2(-1)^2 + (-1) + 6] \\ &\quad + [5(-1)^3 + 7(-1)^2 + 3][4(-1) + 1] \\ &= (1)(7) + (5)(-3) \\ &= -8 \end{aligned}$$

The following example illustrates the extension of the product rule to more than two functions.

EXAMPLE 4 **Connecting the product rule to a more complex function**

Find an expression for $p'(x)$ if $p(x) = f(x)g(x)h(x)$.

Solution

We temporarily regard $f(x)g(x)$ as a single function.

$$p(x) = [f(x)g(x)]h(x)$$

By the product rule,

$$p'(x) = [f(x)g(x)]'h(x) + [f(x)g(x)]h'(x)$$

A second application of the product rule yields

$$\begin{aligned} p'(x) &= [f'(x)g(x) + f(x)g'(x)]h(x) + f(x)g(x)h'(x) \\ &= f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x) \end{aligned}$$

This expression gives us the **extended product rule** for the derivative of a product of three functions. Its symmetrical form makes it easy to extend to a product of four or more functions.

The Power of a Function Rule for Positive Integers

Suppose that we now wish to differentiate functions such as $y = (x^2 - 3)^4$ or $y = (x^2 + 3x + 5)^6$.

These functions are of the form $y = u^n$, where n is a positive integer and $u = g(x)$ is a function whose derivative we can find. Using the product rule, we can develop an efficient method for differentiating such functions.

For $n = 2$,

$h(x) = [g(x)]^2$

$h(x) = g(x)g(x)$

Using the product rule,

$$\begin{aligned} h'(x) &= g'(x)g(x) + g(x)g'(x) \\ &= 2g'(x)g(x) \end{aligned}$$

Similarly, for $n = 3$, we can use the extended product rule.

$$\begin{aligned} \text{Thus, } h(x) &= [g(x)]^3 \\ &= g(x)g(x)g(x) \\ h'(x) &= g'(x)g(x)g(x) + g(x)g'(x)g(x) + g(x)g(x)g'(x) \\ &= 3[g(x)]^2 g'(x) \end{aligned}$$

These results suggest a generalization of the power rule.

The Power of a Function Rule for Integers

If u is a function of x, and n is an integer, then $\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$.
In function notation, if $f(x) = [g(x)]^n$, then $f'(x) = n[g(x)]^{n-1}g'(x)$.

The power of a function rule is a *special case* of the chain rule, which we will discuss later in this chapter. We are now able to differentiate any polynomial, such as $h(x) = (x^2 + 3x + 5)^6$ or $h(x) = (1 - x^2)^4(2x + 6)^3$, without multiplying out the brackets. We can also differentiate rational functions, such as $f(x) = \frac{2x + 5}{3x - 1}$.

EXAMPLE 5 Applying the power of a function rule

For $h(x) = (x^2 + 3x + 5)^6$, find $h'(x)$.

Solution

Here $h(x)$ has the form $h(x) = [g(x)]^6$, where the "inner" function is $g(x) = x^2 + 3x + 5$.

By the power of a function rule, we get $h'(x) = 6(x^2 + 3x + 5)^5(2x + 3)$.

EXAMPLE 6 Selecting a strategy to determine the derivative of a rational function

Differentiate the rational function $f(x) = \frac{2x + 5}{3x - 1}$ by first expressing it as a product and then using the product rule.

Solution

$$f(x) = \frac{2x + 5}{3x - 1}$$

$$= (2x + 5)(3x - 1)^{-1} \qquad \text{(Express } f \text{ as a product)}$$

$$f'(x) = \frac{d}{dx}[(2x + 5)](3x - 1)^{-1} + (2x + 5)\frac{d}{dx}\left[(3x - 1)^{-1}\right] \qquad \text{(Product rule)}$$

$$= 2(3x - 1)^{-1} + (2x + 5)(-1)(3x - 1)^{-2}\frac{d}{dx}(3x - 1)$$

(Power of a function rule)

$$= 2(3x - 1)^{-1} - 3(2x + 5)(3x - 1)^{-2}$$

$$= \frac{2}{(3x - 1)} - \frac{3(2x + 5)}{(3x - 1)^2} \qquad \text{(Simplify)}$$

$$= \frac{2(3x - 1)}{(3x - 1)^2} - \frac{6x + 15}{(3x - 1)^2}$$

$$= \frac{6x - 2 - 6x - 15}{(3x - 1)^2}$$

$$= \frac{-17}{(3x - 1)^2}$$

EXAMPLE 7 Using the derivative to solve a problem

The position s, in centimetres, of an object moving in a straight line is given by $s = t(6 - 3t)^4$, $t \geq 0$, where t is the time in seconds. Determine the object's velocity at $t = 2$.

Solution

The velocity of the object at any time $t \geq 0$ is $v = \frac{ds}{dt}$.

$$v = \frac{d}{dt}[t(6 - 3t)^4]$$

$$= (1)(6 - 3t)^4 + (t)\frac{d}{dt}[(6 - 3t)^4] \qquad \text{(Product rule)}$$

$$= (6 - 3t)^4 + (t)[4(6 - 3t)^3(-3)] \qquad \text{(Power of a function rule)}$$

At $t = 2$, $v = 0 + (2)[4(0)(-3)]$

$$= 0$$

We conclude that the object is at rest at $t = 2$ s.

IN SUMMARY

Key Ideas

- The derivative of a product of differentiable functions is not the product of their derivatives.
- The **product rule** for differentiation:
  If $h(x) = f(x)g(x)$, then $h'(x) = f'(x)g(x) + f(x)g'(x)$.
- The **power of a function rule** for integers:
  If $f(x) = [g(x)]^n$, then $f'(x) = n[g(x)]^{n-1}g'(x)$.

Need to Know

- In some cases, it is easier to expand and simplify the product before differentiating, rather than using the product rule.
  $$\begin{aligned} \text{If } f(x) &= 3x^4(5x^3 - 7) \\ &= 15x^7 - 21x^4 \\ f'(x) &= 105x^6 - 84x^3 \end{aligned}$$
- If the derivative is needed at a particular value of the independent variable, it is not necessary to simplify before substituting.

Exercise 2.3

PART A

1. Use the product rule to differentiate each function. Simplify your answers.
   a. $h(x) = x(x - 4)$
   b. $h(x) = x^2(2x - 1)$
   c. $h(x) = (3x + 2)(2x - 7)$
   d. $h(x) = (5x^7 + 1)(x^2 - 2x)$
   e. $s(t) = (t^2 + 1)(3 - 2t^2)$
   f. $f(x) = \dfrac{x - 3}{x + 3}$

K 2. Use the product rule and the power of a function rule to differentiate the following functions. Do not simplify.
   a. $y = (5x + 1)^3(x - 4)$
   b. $y = (3x^2 + 4)(3 + x^3)^5$
   c. $y = (1 - x^2)^4(2x + 6)^3$
   d. $y = (x^2 - 9)^4(2x - 1)^3$

3. When is it not appropriate to use the product rule? Give examples.

4. Let $F(x) = [b(x)][c(x)]$. Express $F'(x)$ in terms of $b(x)$ and $c(x)$.

PART B

5. Determine the value of $\frac{dy}{dx}$ for the given value of x.

   a. $y = (2 + 7x)(x - 3),\ x = 2$

   b. $y = (1 - 2x)(1 + 2x),\ x = \frac{1}{2}$

   c. $y = (3 - 2x - x^2)(x^2 + x - 2),\ x = -2$

   d. $y = x^3(3x + 7)^2,\ x = -2$

   e. $y = (2x + 1)^5(3x + 2)^4,\ x = -1$

   f. $y = x(5x - 2)(5x + 2),\ x = 3$

6. Determine the equation of the tangent to the curve $y = (x^3 - 5x + 2)(3x^2 - 2x)$ at the point $(1, -2)$.

7. Determine the point(s) where the tangent to the curve is horizontal.

   a. $y = 2(x - 29)(x + 1)$

   b. $y = (x^2 + 2x + 1)(x^2 + 2x + 1)$

8. Use the extended product rule to differentiate the following functions. Do not simplify.

   a. $y = (x + 1)^3(x + 4)(x - 3)^2$

   b. $y = x^2(3x^2 + 4)^2(3 - x^3)^4$

A 9. A 75 L gas tank has a leak. After t hours, the remaining volume, V, in litres is $V(t) = 75\left(1 - \frac{t}{24}\right)^2,\ 0 \le t \le 24$. Use the product rule to determine how quickly the gas is leaking from the tank when the tank is 60% full of gas.

C 10. Determine the slope of the tangent to $h(x) = 2x(x + 1)^3(x^2 + 2x + 1)^2$ at $x = -2$. Explain how to find the equation of the normal at $x = -2$.

PART C

T 11. a. Determine an expression for $f'(x)$ if $f(x) = g_1(x)g_2(x)g_3(x)\ldots g_{n-1}(x)g_n(x)$.

   b. If $f(x) = (1 + x)(1 + 2x)(1 + 3x)\ldots(1 + nx)$, find $f'(0)$.

12. Determine a quadratic function $f(x) = ax^2 + bx + c$ if its graph passes through the point $(2, 19)$ and it has a horizontal tangent at $(-1, -8)$.

13. Sketch the graph of $f(x) = |x^2 - 1|$.

   a. For what values of x is f not differentiable?

   b. Find a formula for f', and sketch the graph of f'.

   c. Find $f'(x)$ at $x = -2$, 0, and 3.

14. Show that the line $4x - y + 11 = 0$ is tangent to the curve $y = \frac{16}{x^2} - 1$.

Mid-Chapter Review

1. a. Sketch the graph of $f(x) = x^2 - 5x$.
   b. Calculate the slopes of the tangents to $f(x) = x^2 - 5x$ at points with x-coordinates 0, 1, 2, ..., 5.
   c. Sketch the graph of the derivative function $f'(x)$.
   d. Compare the graphs of $f(x)$ and $f'(x)$.
2. Use the definition of the derivative to find $f'(x)$ for each function.
   a. $f(x) = 6x + 15$
   b. $f(x) = 2x^2 - 4$
   c. $f(x) = \frac{5}{x + 5}$
   d. $f(x) = \sqrt{x - 2}$
3. a. Determine the equation of the tangent to the curve $y = x^2 - 4x + 3$ at $x = 1$.
   b. Sketch the graph of the function and the tangent.
4. Differentiate each of the following functions:
   a. $y = 6x^4$
   b. $y = 10x^{\frac{1}{2}}$
   c. $g(x) = \frac{2}{x^3}$
   d. $y = 5x + \frac{3}{x^2}$
   e. $y = (11t + 1)^2$
   f. $y = \frac{x - 1}{x}$
5. Determine the equation of the tangent to the graph of $f(x) = 2x^4$ that has slope 1.
6. Determine $f'(x)$ for each of the following functions:
   a. $f(x) = 4x^2 - 7x + 8$
   b. $f(x) = -2x^3 + 4x^2 + 5x - 6$
   c. $f(x) = \frac{5}{x^2} - \frac{3}{x^3}$
   d. $f(x) = \sqrt{x} + \sqrt[3]{x}$
   e. $f(x) = 7x^{-2} - 3\sqrt{x}$
   f. $f(x) = -4x^{-1} + 5x - 1$
7. Determine the equation of the tangent to the graph of each function.
   a. $y = -3x^2 + 6x + 4$ when $x = 1$
   b. $y = 3 - 2\sqrt{x}$ when $x = 9$
   c. $f(x) = -2x^4 + 4x^3 - 2x^2 - 8x + 9$ when $x = 3$
8. Determine the derivative using the product rule.
   a. $f(x) = (4x^2 - 9x)(3x^2 + 5)$
   b. $f(t) = (-3t^2 - 7t + 8)(4t - 1)$
   c. $y = (3x^2 + 4x - 6)(2x^2 - 9)$
   d. $y = (3 - 2x^3)^3$

9. Determine the equation of the tangent to $y = (5x^2 + 9x - 2)(-x^2 + 2x + 3)$ at $(1, 48)$.

10. Determine the point(s) where the tangent to the curve $y = 2(x - 1)(5 - x)$ is horizontal.

11. If $y = 5x^2 - 8x + 4$, determine $\frac{dy}{dx}$ from first principles.

12. A tank holds 500 L of liquid, which takes 90 min to drain from a hole in the bottom of the tank. The volume, V, remaining in the tank after t minutes is

$$V(t) = 500\left(1 - \frac{t}{90}\right)^2, \text{ where } 0 \leq t \leq 90$$

   a. How much liquid remains in the tank at 1 h?

   b. What is the average rate of change of volume with respect to time from 0 min to 60 min?

   c. How fast is the liquid draining at 30 min?

13. The volume of a sphere is given by $V(r) = \frac{4}{3}\pi r^3$.

   a. Determine the average rate of change of volume with respect to radius as the radius changes from 10 cm to 15 cm.

   b. Determine the rate of change of volume when the radius is 8 cm.

14. A classmate says, "The derivative of a cubic polynomial function is a quadratic polynomial function." Is the statement always true, sometimes true, or never true? Defend your choice in words, and provide two examples to support your argument.

15. Show that $\frac{dy}{dx} = (a + 4b)x^{a+4b-1}$ if $y = \frac{x^{2a+3b}}{x^{a-b}}$ and a and b are integers.

16. a. Determine $f'(3)$, where $f(x) = -6x^3 + 4x - 5x^2 + 10$.

   b. Give two interpretations of the meaning of $f'(3)$.

17. The population, P, of a bacteria colony at t hours can be modelled by

$$P(t) = 100 + 120t + 10t^2 + 2t^3$$

   a. What is the initial population of the bacteria colony?

   b. What is the population of the colony at 5 h?

   c. What is the growth rate of the colony at 5 h?

18. The relative percent of carbon dioxide, C, in a carbonated soft drink at t minutes can be modelled by $C(t) = \frac{100}{t}$, where $t > 2$. Determine $C'(t)$ and interpret the results at 5 min, 50 min, and 100 min. Explain what is happening.

Section 2.4—The Quotient Rule

In the previous section, we found that the derivative of the product of two functions is not the product of their derivatives. The quotient rule gives the derivative of a function that is the quotient of two functions. It is derived from the product rule.

The Quotient Rule

If $h(x) = \frac{f(x)}{g(x)}$, then $h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$, $g(x) \neq 0$.

In Leibniz notation, $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2}$.

Proof:

We want to find $h'(x)$, given that $h(x) = \frac{f(x)}{g(x)}$, $g(x) \neq 0$.

We rewrite this as a product: $h(x)g(x) = f(x)$.

Using the product rule, $h'(x)g(x) + h(x)g'(x) = f'(x)$.

Solving for $h'(x)$, we get

$$h'(x) = \frac{f'(x) - h(x)g'(x)}{g(x)}$$

$$= \frac{f'(x) - \frac{f(x)}{g(x)}g'(x)}{g(x)}$$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

The quotient rule provides us with an alternative approach to differentiate rational functions, in addition to what we learned last section.

Memory Aid for the Product and Quotient Rules

It is worth noting that the quotient rule is similar to the product rule in that both have $f'(x)g(x)$ and $f(x)g'(x)$. For the product rule, we put an addition sign between the terms. For the quotient rule, we put a subtraction sign between the terms and then divide the result by the square of the original denominator.

Take note that in the quotient rule the $f'(x)g(x)$ term must come first. This isn't the case with the product rule.

EXAMPLE 1 Applying the quotient rule

Determine the derivative of $h(x) = \frac{3x - 4}{x^2 + 5}$.

Solution

Since $h(x) = \frac{f(x)}{g(x)}$, where $f(x) = 3x - 4$ and $g(x) = x^2 + 5$, use the quotient rule to find $h'(x)$.

Using the quotient rule, we get $h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

$$= \frac{(3)(x^2 + 5) - (3x - 4)(2x)}{(x^2 + 5)^2}$$

$$= \frac{3x^2 + 15 - 6x^2 + 8x}{(x^2 + 5)^2}$$

$$= \frac{-3x^2 + 8x + 15}{(x^2 + 5)^2}$$

EXAMPLE 2 Selecting a strategy to determine the equation of a line tangent to a rational function

Determine the equation of the tangent to $y = \frac{2x}{x^2 + 1}$ at $x = 0$.

Solution A – Using the derivative

The slope of the tangent to the graph of f at any point is given by the derivative $\frac{dy}{dx}$.

By the quotient rule,

$$\frac{dy}{dx} = \frac{(2)(x^2 + 1) - (2x)(2x)}{(x^2 + 1)^2}$$

At $x = 0$,

$$\frac{dy}{dx} = \frac{(2)(0 + 1) - (0)(0)}{(0 + 1)^2} = 2$$

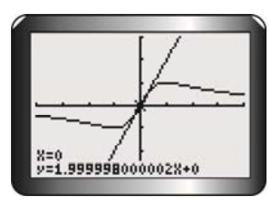
The slope of the tangent at $x = 0$ is 2 and the point of tangency is $(0, 0)$. The equation of the tangent is $y = 2x$.

Tech Support

For help using the graphing calculator to graph functions and draw tangent lines see Technical Appendices p. 597 and p. 608.

Solution B – Using the graphing calculator

Draw the graph of the function using the graphing calculator.



Draw the tangent at the point on the function where $x = 0$. The calculator displays the equation of the tangent line.

The equation of the tangent line in this case is $y = 2x$.

EXAMPLE 3 Using the quotient rule to solve a problem

Determine the coordinates of each point on the graph of $f(x) = \frac{2x + 8}{\sqrt{x}}$ where the tangent is horizontal.

Solution

The slope of the tangent at any point on the graph is given by $f'(x)$. Using the quotient rule,

$$f'(x) = \frac{(2)(\sqrt{x}) - (2x + 8)\left(\frac{1}{2}x^{-\frac{1}{2}}\right)}{(\sqrt{x})^2}$$

$$= \frac{2\sqrt{x} - \frac{2x + 8}{2\sqrt{x}}}{x}$$

$$= \frac{\frac{2x}{\sqrt{x}} - \frac{x + 4}{\sqrt{x}}}{x}$$

$$= \frac{\frac{2x - x - 4}{\sqrt{x}}}{x}$$

$$= \frac{x - 4}{x\sqrt{x}}$$

The tangent will be horizontal when $f'(x) = 0$; that is, when $x = 4$. The point on the graph where the tangent is horizontal is $(4, 8)$.

IN SUMMARY

Key Ideas

- The derivative of a quotient of two differentiable functions is not the quotient of their derivatives.
- The **quotient rule** for differentiation:

  If $h(x) = \frac{f(x)}{g(x)}$, then $h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$, $g(x) \neq 0$.

Need to Know

- To find the derivative of a rational function, you can use two methods:

Leave the function in fraction form, and use the quotient rule.	OR	Express the function as a product, and use the product and power of a function rules.
$f(x) = \frac{x - 2}{1 + x}$		$f(x) = (x - 2)(1 + x)^{-1}$

Exercise 2.4

PART A

1. What are the exponent rules? Give examples of each rule.

2. Copy the table, and complete it *without* using the quotient rule.

Function	Rewrite	Differentiate and Simplify, if Necessary
$f(x) = \frac{x^2 + 3x}{x}, x \neq 0$		
$g(x) = \frac{3x^{\frac{5}{3}}}{x}, x \neq 0$		
$h(x) = \frac{1}{10x^5}, x \neq 0$		
$y = \frac{8x^3 + 6x}{2x}, x \neq 0$		
$s = \frac{t^2 - 9}{t - 3}, t \neq 3$		

C 3. What are the different ways to find the derivative of a rational function? Give examples.

PART B

K 4. Use the quotient rule to differentiate each function. Simplify your answers.

a. $h(x) = \frac{x}{x + 1}$

b. $h(t) = \frac{2t - 3}{t + 5}$

c. $h(x) = \frac{x^3}{2x^2 - 1}$

d. $h(x) = \frac{1}{x^2 + 3}$

e. $y = \frac{x(3x + 5)}{1 - x^2}$

f. $y = \frac{x^2 - x + 1}{x^2 + 3}$

5. Determine $\frac{dy}{dx}$ at the given value of x.

a. $y = \frac{3x + 2}{x + 5}, x = -3$

b. $y = \frac{x^3}{x^2 + 9}, x = 1$

c. $y = \frac{x^2 - 25}{x^2 + 25}, x = 2$

d. $y = \frac{(x + 1)(x + 2)}{(x - 1)(x - 2)}, x = 4$

6. Determine the slope of the tangent to the curve $y = \frac{x^3}{x^2 - 6}$ at point $(3, 9)$.

7. Determine the points on the graph of $y = \frac{3x}{x - 4}$ where the slope of the tangent is $-\frac{12}{25}$.

T 8. Show that there are no tangents to the graph of $f(x) = \frac{5x + 2}{x + 2}$ that have a negative slope.

9. Find the point(s) at which the tangent to the curve is horizontal.

a. $y = \frac{2x^2}{x - 4}$

b. $y = \frac{x^2 - 1}{x^2 + x - 2}$

A

10. An initial population, p, of 1000 bacteria grows in number according to the equation $p(t) = 1000\left(1 + \frac{4t}{t^2 + 50}\right)$, where t is in hours. Find the rate at which the population is growing after 1 h and after 2 h.

11. Determine the equation of the tangent to the curve $y = \frac{x^2 - 1}{3x}$ at $x = 2$.

12. A motorboat coasts toward a dock with its engine off. Its distance s, in metres, from the dock t seconds after the engine is turned off is $s(t) = \frac{10(6 - t)}{t + 3}$ for $0 \le t \le 6$.

a. How far is the boat from the dock initially?

b. Find the velocity of the boat when it bumps into the dock.

13. a. The radius of a circular juice blot on a piece of paper towel t seconds after it was first seen is modelled by $r(t) = \frac{1 + 2t}{1 + t}$, where r is measured in centimetres. Calculate

i. the radius of the blot when it was first observed

ii. the time at which the radius of the blot was 1.5 cm

iii. the rate of increase of the area of the blot when the radius was 1.5 cm

b. According to this model, will the radius of the blot ever reach 2 cm? Explain your answer.

14. The graph of $f(x) = \frac{ax + b}{(x - 1)(x - 4)}$ has a horizontal tangent line at $(2, -1)$. Find a and b. Check using a graphing calculator.

15. The concentration, c, of a drug in the blood t hours after the drug is taken orally is given by $c(t) = \frac{5t}{2t^2 + 7}$. When does the concentration reach its maximum value?

16. The position from its starting point, s, of an object that moves in a straight line at time t seconds is given by $s(t) = \frac{t}{t^2 + 8}$. Determine when the object changes direction.

PART C

17. Consider the function $f(x) = \frac{ax + b}{cx + d}$, $x \neq -\frac{d}{c}$, where a, b, c, and d are nonzero constants. What condition on a, b, c, and d ensures that each tangent to the graph of f has a positive slope?

Section 2.5—The Derivatives of Composite Functions

Recall that one way of combining functions is through a process called **composition**. We start with a number x in the domain of g, find its image $g(x)$, and then take the value of f at $g(x)$, provided that $g(x)$ is in the domain of f. The result is the new function $h(x) = f(g(x))$, which is called the **composite function** of f and g, and is denoted $(f \circ g)$.

Definition of a composite function

Given two functions f and g, the **composite function** $(f \circ g)$ is defined by $(f \circ g)(x) = f(g(x))$.

EXAMPLE 1 **Reflecting on the process of composition**

If $f(x) = \sqrt{x}$ and $g(x) = x + 5$, find each of the following values:

a. $f(g(4))$ b. $g(f(4))$ c. $f(g(x))$ d. $g(f(x))$

Solution

a. Since $g(4) = 9$, we have $f(g(4)) = f(9) = 3$.

b. Since $f(4) = 2$, we have $g(f(4)) = g(2) = 7$. *Note:* $f(g(4)) \neq g(f(4))$.

c. $f(g(x)) = f(x + 5) = \sqrt{x + 5}$

d. $g(f(x)) = g(\sqrt{x}) = \sqrt{x} + 5$ *Note:* $f(g(x)) \neq g(f(x))$.

The chain rule states how to compute the derivative of the composite function $h(x) = f(g(x))$ in terms of the derivatives of f and g.

The Chain Rule

If f and g are functions that have derivatives, then the composite function $h(x) = f(g(x))$ has a derivative given by $h'(x) = f'(g(x))g'(x)$.

In words, the chain rule says, "the derivative of a composite function is the product of the derivative of the outer function evaluated at the inner function and the derivative of the inner function."

Proof:

By the definition of the derivative, $[f(g(x))]' = \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h}$.

Assuming that $g(x+h) - g(x) \neq 0$, we can write

$$[f(g(x))]' = \lim_{h \to 0}\left[\left(\frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)}\right)\left(\frac{g(x+h) - g(x)}{h}\right)\right]$$

$$= \lim_{h \to 0}\left[\frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)}\right]\lim_{h \to 0}\left[\frac{g(x+h) - g(x)}{h}\right]$$

(Property of limits)

Since $\lim_{h \to 0}[g(x+h) - g(x)] = 0$, let $g(x+h) - g(x) = k$ and $k \to 0$ as $h \to 0$. We obtain

$$[f(g(x))]' = \lim_{k \to 0}\left[\frac{f(g(x) + k) - f(g(x))}{k}\right]\lim_{h \to 0}\left[\frac{g(x+h) - g(x)}{h}\right]$$

Therefore, $[f(g(x))]' = f'(g(x))g'(x)$.

This proof is not valid for all circumstances. When dividing by $g(x+h) - g(x)$, we assume that $g(x+h) - g(x) \neq 0$. A proof that covers all cases can be found in advanced calculus textbooks.

EXAMPLE 2 Applying the chain rule

Differentiate $h(x) = (x^2 + x)^{\frac{3}{2}}$.

Solution

The inner function is $g(x) = x^2 + x$, and the outer function is $f(x) = x^{\frac{3}{2}}$.

The derivative of the inner function is $g'(x) = 2x + 1$.

The derivative of the outer function is $f'(x) = \frac{3}{2}x^{\frac{1}{2}}$.

The derivative of the outer function evaluated at the inner function $g(x)$ is $f'(x^2 + x) = \frac{3}{2}(x^2 + x)^{\frac{1}{2}}$.

By the chain rule, $h'(x) = \frac{3}{2}(x^2 + x)^{\frac{1}{2}}(2x + 1)$.

The Chain Rule in Leibniz Notation

If y is a function of u and u is a function of x (so that y is a composite function), then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$, provided that $\frac{dy}{du}$ and $\frac{du}{dx}$ exist.

If we interpret derivatives as rates of change, the chain rule states that if y is a function of x through the intermediate variable u, then the rate of change of y

with respect to x is equal to the product of the rate of change of y with respect to u and the rate of change of u with respect to x.

EXAMPLE 3

Applying the chain rule using Leibniz notation

If $y = u^3 - 2u + 1$, where $u = 2\sqrt{x}$, find $\frac{dy}{dx}$ at $x = 4$.

Solution

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (3u^2 - 2)\left[2\left(\frac{1}{2}x^{-\frac{1}{2}}\right)\right]$$
$$= (3u^2 - 2)\left(\frac{1}{\sqrt{x}}\right)$$

It is not necessary to write the derivative entirely in terms of x.

When $x = 4$, $u = 2\sqrt{4} = 4$ and $\frac{dy}{dx} = [3(4)^2 - 2]\left(\frac{1}{\sqrt{4}}\right) = (46)\left(\frac{1}{2}\right) = 23$.

EXAMPLE 4

Selecting a strategy involving the chain rule to solve a problem

An environmental study of a certain suburban community suggests that the average daily level of carbon monoxide in the air can be modelled by the function $C(p) = \sqrt{0.5p^2 + 17}$, where $C(p)$ is in parts per million and population p is expressed in thousands. It is estimated that t years from now, the population of the community will be $p(t) = 3.1 + 0.1t^2$ thousand. At what rate will the carbon monoxide level be changing with respect to time three years from now?

Solution

We are asked to find the value of $\frac{dC}{dt}$, when $t = 3$.

We can find the rate of change by using the chain rule.

Therefore, $\frac{dC}{dt} = \frac{dC}{dp}\frac{dp}{dt}$

$$= \frac{d(0.5p^2 + 17)^{\frac{1}{2}}}{dp}\,\frac{d(3.1 + 0.1t^2)}{dt}$$
$$= \left[\frac{1}{2}(0.5p^2 + 17)^{-\frac{1}{2}}(0.5)(2p)\right](0.2t)$$

When $t = 3$, $p(3) = 3.1 + 0.1(3)^2 = 4$.

$$\text{So, } \frac{dC}{dt} = \left[\frac{1}{2}(0.5(4)^2 + 17)^{-\frac{1}{2}}(0.5)(2(4))\right](0.2(3))$$
$$= 0.24$$

Since the sign of $\frac{dC}{dt}$ is positive, the carbon monoxide level will be increasing at the rate of 0.24 parts per million per year three years from now.

EXAMPLE 5 Using the chain rule to differentiate a power of a function

If $y = (x^2 - 5)^7$, find $\frac{dy}{dx}$.

Solution

The inner function is $g(x) = x^2 - 5$, and the outer function is $f(x) = x^7$.
By the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= 7(x^2 - 5)^6(2x)\\ &= 14x(x^2 - 5)^6\end{aligned}$$

Example 5 is a special case of the chain rule in which the outer function is a power function of the form $y = [g(x)]^n$. This leads to a generalization of the power rule seen earlier.

Power of a Function Rule

If n is a real number and $u = g(x)$, then $\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$,

or $\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1}g'(x)$.

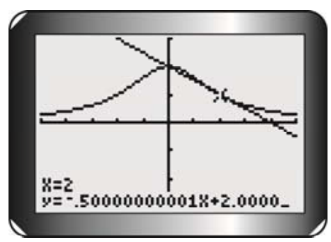
EXAMPLE 6 Connecting the derivative to the slope of a tangent

Using a graphing calculator, sketch the graph of the function $f(x) = \frac{8}{x^2 + 4}$.

Find the equation of the tangent at the point $(2, 1)$ on the graph.

Solution

Using a graphing calculator, the graph is



Tech Support

For help using the graphing calculator to graph functions and draw tangent lines see Technical Appendices p. 597 and p. 608.

The slope of the tangent at point $(2, 1)$ is given by $f'(2)$.
We first write the function as $f(x) = 8(x^2 + 4)^{-1}$.

By the power of a function rule, $f'(x) = -8(x^2 + 4)^{-2}(2x)$.

The slope at $(2, 1)$ is $f'(2) = -8(4 + 4)^{-2}(4)$

$$\begin{aligned}&= -\frac{32}{(8)^2}\\ &= -0.5\end{aligned}$$

The equation of the tangent is $y - 1 = -\frac{1}{2}(x - 2)$, or $x + 2y - 4 = 0$.

EXAMPLE 7 Combining derivative rules to differentiate a complex product

Differentiate $h(x) = (x^2 + 3)^4(4x - 5)^3$. Express your answer in a simplified factored form.

Solution

Here we use the product rule and the chain rule.

$$h'(x) = \frac{d}{dx}[(x^2 + 3)^4] \cdot (4x - 5)^3 + \frac{d}{dx}[(4x - 5)^3] \cdot (x^2 + 3)^4 \quad \text{(Product rule)}$$

$$= [4(x^2 + 3)^3(2x)] \cdot (4x - 5)^3 + [3(4x - 5)^2(4)] \cdot (x^2 + 3)^4 \quad \text{(Chain rule)}$$

$$= 8x(x^2 + 3)^3(4x - 5)^3 + 12(4x - 5)^2(x^2 + 3)^4 \quad \text{(Simplify)}$$

$$= 4(x^2 + 3)^3(4x - 5)^2[2x(4x - 5) + 3(x^2 + 3)] \quad \text{(Factor)}$$

$$= 4(x^2 + 3)^3(4x - 5)^2(11x^2 - 10x + 9)$$

EXAMPLE 8 Combining derivative rules to differentiate a complex quotient

Determine the derivative of $g(x) = \left(\frac{1 + x^2}{1 - x^2}\right)^{10}$.

Solution A – Using the product and chain rule

There are several approaches to this problem. You could keep the function as it is and use the chain rule and the quotient rule. You could also decompose the function and express it as $g(x) = \frac{(1 + x^2)^{10}}{(1 - x^2)^{10}}$, and then apply the quotient rule and the chain rule. Here we will express the function as the product $g(x) = (1 + x^2)^{10}(1 - x^2)^{-10}$ and apply the product rule and the chain rule.

$$g'(x) = \frac{d}{dx}\left[(1 + x^2)^{10}\right](1 - x^2)^{-10} + (1 + x^2)^{10}\frac{d}{dx}\left[(1 - x^2)^{-10}\right]$$

$$= 10(1 + x^2)^9(2x)(1 - x^2)^{-10} + (1 + x^2)^{10}(-10)(1 - x^2)^{-11}(-2x)$$

$$= 20x(1 + x^2)^9(1 - x^2)^{-10} + (20x)(1 + x^2)^{10}(1 - x^2)^{-11} \quad \text{(Simplify)}$$

$$= 20x(1 + x^2)^9(1 - x^2)^{-11}[(1 - x^2) + (1 + x^2)] \quad \text{(Factor)}$$

$$= 20x(1 + x^2)^9(1 - x^2)^{-11}(2)$$

$$= \frac{40x(1 + x^2)^9}{(1 - x^2)^{11}} \quad \text{(Rewrite using positive exponents)}$$

Solution B – Using the chain and quotient rule

In this solution, we will use the chain rule and the quotient rule, where $u = \frac{1 + x^2}{1 - x^2}$ is the inner function and u^{10} is the outer function.

$$g'(x) = \frac{dg}{du}\frac{du}{dx}$$

$$g'(x) = \frac{d\left[\left(\frac{1+x^2}{1-x^2}\right)^{10}\right]}{d\left(\frac{1+x^2}{1-x^2}\right)} \frac{d}{dx}\left(\frac{1+x^2}{1-x^2}\right) \quad \text{(Chain rule and quotient rule)}$$

$$= 10\left(\frac{1+x^2}{1-x^2}\right)^9 \frac{d}{dx}\left(\frac{1+x^2}{1-x^2}\right)$$

$$= 10\left(\frac{1+x^2}{1-x^2}\right)^9 \left[\frac{2x(1-x^2) - (-2x)(1+x^2)}{(1-x^2)^2}\right] \quad \text{(Expand)}$$

$$= 10\left(\frac{1+x^2}{1-x^2}\right)^9 \left[\frac{2x - 2x^3 + 2x + 2x^3}{(1-x^2)^2}\right] \quad \text{(Simplify)}$$

$$= 10\left(\frac{1+x^2}{1-x^2}\right)^9 \left[\frac{4x}{(1-x^2)^2}\right]$$

$$= \frac{10(1+x^2)^9}{(1-x^2)^9} \frac{4x}{(1-x^2)^2}$$

$$= \frac{40x(1+x^2)^9}{(1-x^2)^{11}}$$

IN SUMMARY

Key Idea

- The **chain rule**:
  If y is a function of u, and u is a function of x (i.e., y is a composite function), then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$, provided that $\frac{dy}{du}$ and $\frac{du}{dx}$ exist.
  Therefore, if $h(x) = (f \circ g)(x)$, then

  $$h'(x) = f'(g(x)) \cdot g'(x) \quad \text{(Function notation)}$$

  or

  $$\frac{d[h(x)]}{dx} = \frac{d[f(g(x))]}{d[g(x)]} \cdot \frac{d[g(x)]}{dx} \quad \text{(Leibniz notation)}$$

Need to Know

- When the outer function is a power function of the form $y = [g(x)]^n$, we have a special case of the chain rule, called the **power of a function rule**:

  $$\frac{d}{dx}[g(x)]^n = \frac{d[g(x)]^n}{d[g(x)]} \cdot \frac{d[g(x)]}{dx}$$
  $$= n[g(x)]^{n-1} \cdot g'(x)$$

Exercise 2.5

PART A

1. Given $f(x) = \sqrt{x}$ and $g(x) = x^2 - 1$, find the following value:
   a. $f(g(1))$
   b. $g(f(1))$
   c. $g(f(0))$
   d. $f(g(-4))$
   e. $f(g(x))$
   f. $g(f(x))$

2. For each of the following pairs of functions, find the composite functions $(f \circ g)$ and $(g \circ f)$. What is the domain of each composite function? Are the composite functions equal?
   a. $f(x) = x^2$, $g(x) = \sqrt{x}$
   b. $f(x) = \frac{1}{x}$, $g(x) = x^2 + 1$
   c. $f(x) = \frac{1}{x}$, $g(x) = \sqrt{x + 2}$

C 3. What is the rule for calculating the derivative of the composition of two differentiable functions? Give examples, and show how the derivative is determined.

4. Differentiate each function. Do not expand any expression before differentiating.
   a. $f(x) = (2x + 3)^4$
   b. $g(x) = (x^2 - 4)^3$
   c. $h(x) = (2x^2 + 3x - 5)^4$
   d. $f(x) = (\pi^2 - x^2)^3$
   e. $y = \sqrt{x^2 - 3}$
   f. $f(x) = \frac{1}{(x^2 - 16)^5}$

PART B

K 5. Rewrite each of the following in the form $y = u^n$ or $y = ku^n$, and then differentiate.
   a. $y = -\frac{2}{x^3}$
   b. $y = \frac{1}{x + 1}$
   c. $y = \frac{1}{x^2 - 4}$
   d. $y = \frac{3}{9 - x^2}$
   e. $y = \frac{1}{5x^2 + x}$
   f. $y = \frac{1}{(x^2 + x + 1)^4}$

6. Given $h = g \circ f$, where f and g are continuous functions, use the information in the table to evaluate $h(-1)$ and $h'(-1)$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
−1	1	18	−5	−15
0	−2	5	−1	−11
1	−1	−4	3	−7
2	4	−9	7	−3
3	13	−10	11	1

7. Given $f(x) = (x - 3)^2$, $g(x) = \frac{1}{x}$, and $h(x) = f(g(x))$, determine $h'(x)$.

8. Differentiate each function. Express your answer in a simplified factored form.

a. $f(x) = (x + 4)^3(x - 3)^6$

b. $y = (x^2 + 3)^3(x^3 + 3)^2$

c. $y = \dfrac{3x^2 + 2x}{x^2 + 1}$

d. $h(x) = x^3(3x - 5)^2$

e. $y = x^4(1 - 4x^2)^3$

f. $y = \left(\dfrac{x^2 - 3}{x^2 + 3}\right)^4$

9. Find the rate of change of each function at the given value of t. Leave your answers as rational numbers, or in terms of roots and the number π.

a. $s(t) = t^{\frac{1}{3}}(4t - 5)^{\frac{2}{3}}, t = 8$

b. $s(t) = \left(\dfrac{t - \pi}{t - 6\pi}\right)^{\frac{1}{3}}, t = 2\pi$

10. For what values of x do the curves $y = (1 + x^3)^2$ and $y = 2x^6$ have the same slope?

11. Find the slope of the tangent to the curve $y = (3x - x^2)^{-2}$ at $\left(2, \frac{1}{4}\right)$.

12. Find the equation of the tangent to the curve $y = (x^3 - 7)^5$ at $x = 2$.

13. Use the chain rule, in Leibniz notation, to find $\frac{dy}{dx}$ at the given value of x.

a. $y = 3u^2 - 5u + 2, u = x^2 - 1, x = 2$

b. $y = 2u^3 + 3u^2, u = x + x^{\frac{1}{2}}, x = 1$

c. $y = u(u^2 + 3)^3, u = (x + 3)^2, x = -2$

d. $y = u^3 - 5(u^3 - 7u)^2, u = \sqrt{x}, x = 4$

14. Find $h'(2)$, given $h(x) = f(g(x)), f(u) = u^2 - 1, g(2) = 3$, and $g'(2) = -1$.

A 15. A 50 000 L tank can be drained in 30 min. The volume of water remaining in the tank after t minutes is $V(t) = 50\,000\left(1 - \frac{t}{30}\right)^2, 0 \le t \le 30$. At what rate, to the nearest whole number, is the water flowing out of the tank when $t = 10$?

16. The function $s(t) = (t^3 + t^2)^{\frac{1}{2}}, t \ge 0$, represents the displacement s, in metres, of a particle moving along a straight line after t seconds. Determine the velocity when $t = 3$.

PART C

17. a. Write an expression for $h'(x)$ if $h(x) = p(x)q(x)r(x)$.

b. If $h(x) = x(2x + 7)^4(x - 1)^2$, find $h'(-3)$.

T 18. Show that the tangent to the curve $y = (x^2 + x - 2)^3 + 3$ at the point $(1, 3)$ is also the tangent to the curve at another point.

19. Differentiate $y = \dfrac{x^2(1 - x^3)}{(1 + x)^3}$.

Technology Extension: Derivatives on Graphing Calculators

Numerical derivatives can be approximated on a TI-83/84 Plus using **nDeriv(**.

To approximate $f'(0)$ for $f(x) = \frac{2x}{x^2 + 1}$ follow these steps:

Press MATH, and scroll down to **8:nDeriv(** under the MATH menu.

Press ENTER, and the display on the screen will be **nDeriv(**.

To find the derivative, key in the *expression*, the *variable*, the *value* at which we want the derivative, and a value for ε.

For this example, the display will be **nDeriv** $(2X/(X^2 + 1), X, 0, 0.01)$.

Press ENTER, and the value **1.99980002** will be returned.

Therefore, $f'(0)$is approximately 1.999 800 02.

A better approximation can be found by using a smaller value for ε, such as $\varepsilon = 0.0001$. The default value for ε is 0.001.

Try These:

a. Use the **nDeriv(** function on a graphing calculator to determine the value of the derivative of each of the following functions at the given point.

i. $f(x) = x^3, x = -1$

ii. $f(x) = x^4, x = 2$

iii. $f(x) = x^3 - 6x, x = -2$

iv. $f(x) = (x^2 + 1)(2x - 1)^4, x = 0$

v. $f(x) = x^2 + \frac{16}{x} - 4\sqrt{x}, x = 4$

vi. $f(x) = \frac{x^2 - 1}{x^2 + x - 2}, x = -1$

b. Determine the actual value of each derivative at the given point using the rules of differentiation.

The TI-89, TI-92, an TI-Nspire can find exact symbolic and numerical derivatives. If you have access to either model, try some of the functions above and compare your answers with those found using a TI-83/84 Plus. For example, on the TI-89 press DIFFERENTIATE under the CALCULATE menu, key $d(2x/(x^2 + 1), x)|x = 0$ and press ENTER.

CAREER LINK WRAP-UP Investigate and Apply

CHAPTER 2: THE ELASTICITY OF DEMAND

An electronics retailing chain has established the monthly price (p)–demand (n) relationship for an electronic game as

$$n(p) = 1000 - 10\,\frac{(p-1)^{\frac{4}{3}}}{\sqrt[3]{p}}$$

They are trying to set a price level that will provide maximum revenue (R). They know that when demand is *elastic* ($E > 1$), a drop in price will result in higher overall revenues ($R = np$), and that when demand is *inelastic* ($E < 1$), an increase in price will result in higher overall revenues. To complete the questions in this task, you will have to use the elasticity definition

$$E = -\left[\left(\frac{\Delta n}{n}\right) \div \left(\frac{\Delta p}{p}\right)\right]$$

converted into differential $\left(\frac{\Delta n}{\Delta p} = \frac{dn}{dp}\right)$ notation.

a. Determine the elasticity of demand at \$20 and \$80, classifying these price points as having elastic or inelastic demand. What does this say about where the optimum price is in terms of generating the maximum revenue? Explain. Also calculate the revenue at the \$20 and \$80 price points.

b. Approximate the demand curve as a linear function (tangent) at a price point of \$50. Plot the demand function and its linear approximation on the graphing calculator. What do you notice? Explain this by looking at the demand function.

c. Use your linear approximation to determine the price point that will generate the maximum revenue. (*Hint:* Think about the specific value of E where you will not want to increase or decrease the price to generate higher revenues.) What revenue is generated at this price point?

d. A second game has a price–demand relationship of

$$n(p) = \frac{12\,500}{p - 25}$$

The price is currently set at \$50. Should the company increase or decrease the price? Explain.

Key Concepts Review

Now that you have completed your study of derivatives in Chapter 2, you should be familiar with such concepts as derivatives of polynomial functions, the product rule, the quotient rule, the power rule for rational exponents, and the chain rule. Consider the following summary to confirm your understanding of the key concepts.

- The derivative of f at a is given by $f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$ or, alternatively, by $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$.
- The derivative function of $f(x)$ with respect to x is $f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$.
- The derivative of a function at a point $(a, f(a))$ can be interpreted as
  - the slope of the tangent line at this point
  - the instantaneous rate of change at this point

Summary of Differentiation Techniques

Rule	Function Notation	Leibniz Notation
Constant	$f(x) = k$, $f'(x) = 0$	$\frac{d}{dx}(k) = 0$
Linear	$f(x) = x$, $f'(x) = 1$	$\frac{d}{dx}(x) = 1$
Power	$f(x) = x^n$, $f'(x) = nx^{n-1}$	$\frac{d}{dx}(x^n) = nx^{n-1}$
Constant Multiple	$f(x) = kg(x)$, $f'(x) = kg'(x)$	$\frac{d}{dx}(ky) = k\frac{dy}{dx}$
Sum or Difference	$f(x) = p(x) \pm q(x)$, $f'(x) = p'(x) \pm q'(x)$	$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$
Product	$h(x) = f(x)g(x)$ $h'(x) = f'(x)g(x) + f(x)g'(x)$	$\frac{d}{dx}[f(x)g(x)] = \left[\frac{d}{dx}f(x)\right]g(x) + f(x)\left[\frac{d}{dx}g(x)\right]$
Quotient	$h(x) = \frac{f(x)}{g(x)}$ $h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$	$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\left[\frac{d}{dx}f(x)\right]g(x) - f(x)\left[\frac{d}{dx}g(x)\right]}{[g(x)]^2}$
Chain	$h(x) = f(g(x))$, $h'(x) = f'(g(x))g'(x)$	$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$, where u is a function of x
Power of a Function	$f(x) = [g(x)]^n$, $f'(x) = n[g(x)]^{n-1}g'(x)$	$y = u^n$, $\frac{dy}{dx} = nu^{n-1}\frac{du}{dx}$, where u is a function of x

Review Exercise

1. Describe the process of finding a derivative using the definition of $f'(x)$.

2. Use the definition of the derivative to find $f'(x)$ for each of the following functions:

   a. $y = 2x^2 - 5x$
   b. $y = \sqrt{x - 6}$
   c. $y = \frac{x}{4 - x}$

3. Differentiate each of the following functions:

   a. $y = x^2 - 5x + 4$
   b. $f(x) = x^{\frac{3}{4}}$
   c. $y = \frac{7}{3x^4}$
   d. $y = \frac{1}{x^2 + 5}$
   e. $y = \frac{3}{(3 - x^2)^2}$
   f. $y = \sqrt{7x^2 + 4x + 1}$

4. Determine the derivative of the given function. In some cases, it will save time if you rearrange the function before differentiating.

   a. $f(x) = \frac{2x^3 - 1}{x^2}$
   b. $g(x) = \sqrt{x}(x^3 - x)$
   c. $y = \frac{x}{3x - 5}$
   d. $y = \sqrt{x - 1}(x + 1)$
   e. $f(x) = (\sqrt{x} + 2)^{-\frac{2}{3}}$
   f. $y = \frac{x^2 + 5x + 4}{x + 4}$

5. Determine the derivative, and give your answer in a simplified form.

   a. $y = x^4(2x - 5)^6$
   b. $y = x\sqrt{x^2 + 1}$
   c. $y = \frac{(2x - 5)^4}{(x + 1)^3}$
   d. $y = \left(\frac{10x - 1}{3x + 5}\right)^6$
   e. $y = (x - 2)^3(x^2 + 9)^4$
   f. $y = (1 - x^2)^3(6 + 2x)^{-3}$

6. If f is a differentiable function, find an expression for the derivative of each of the following functions:

   a. $g(x) = f(x^2)$
   b. $h(x) = 2xf(x)$

7. a. If $y = 5u^2 + 3u - 1$ and $u = \frac{18}{x^2 + 5}$, find $\frac{dy}{dx}$ when $x = 2$.

   b. If $y = \frac{u + 4}{u - 4}$ and $u = \frac{\sqrt{x} + x}{10}$, find $\frac{dy}{dx}$ when $x = 4$.

   c. If $y = f(\sqrt{x^2 + 9})$ and $f'(5) = -2$, find $\frac{dy}{dx}$ when $x = 4$.

8. Determine the slope of the tangent at point $(1, 4)$ on the graph of $f(x) = (9 - x^2)^{\frac{2}{3}}$.

9. For what values of x does the curve $y = -x^3 + 6x^2$ have a slope of -12? For what values of x does the curve $y = -x^3 + 6x^2$ have a slope of -15? Use a graphing calculator to graph the function and confirm your results.

10. a. Determine the values of x where the graph of each function has a horizontal tangent.
    i. $y = (x^2 - 4)^5$ ii. $y = (x^3 - x)^2$
    b. Use a graphing calculator to graph each function and its tangent at the values you found from part a. to confirm your result.

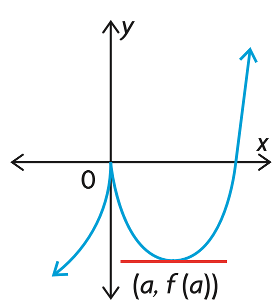
11. Determine the equation of the tangent to each function at the given point.
    a. $y = (x^2 + 5x + 2)^4$, $(0, 16)$ b. $y = (3x^{-2} - 2x^3)^5$, $(1, 1)$

12. A tangent to the parabola $y = 3x^2 - 7x + 5$ is perpendicular to $x + 5y - 10 = 0$. Determine the equation of the tangent.

13. The line $y = 8x + b$ is tangent to the curve $y = 2x^2$. Determine the point of tangency and the value of b.

14. a. Using a graphing calculator, graph the function $f(x) = \frac{x^3}{x^2 - 6}$.
    b. Using the draw function or an equivalent function on your calculator or graphing software, find the equations of the tangents where the slope is zero.
    c. Setting $f'(x) = 0$, find the coordinates of the points where the slope is zero.
    d. Determine the slope of the tangent to the graph at $(2, -4)$. Use the graph to verify that your answer is reasonable.

15. Consider the function $f(x) = 2x^{\frac{5}{3}} - 5x^{\frac{2}{3}}$.
    a. Determine the slope of the tangent at the point where the graph crosses the x-axis.
    b. Determine the value of a shown in the graph of $f(x)$ given below.



16. A rested student is able to memorize M words after t minutes, where $M = 0.1t^2 - 0.001t^3$, $0 \le t \le \frac{200}{3}$.
    a. How many words are memorized in the first 10 min? How many words are memorized in the first 15 min?
    b. What is the memory rate at $t = 10$? What is the memory rate at $t = 15$?

17. A grocery store determines that, after t hours on the job, a new cashier can scan $N(t) = 20 - \frac{30}{\sqrt{9 + t^2}}$ items per minute.

    a. Find $N'(t)$, the rate at which the cashier's productivity is changing.

    b. According to this model, does the cashier ever stop improving? Why?

18. An athletic-equipment supplier experiences weekly costs of $C(x) = \frac{1}{3}x^3 + 40x + 700$ in producing x baseball gloves per week.

    a. Find the marginal cost, $C'(x)$.

    b. Find the production level x at which the marginal cost is \$76 per glove.

19. A manufacturer of kitchen appliances experiences revenue of $R(x) = 750x - \frac{x^2}{6} - \frac{2}{3}x^3$ dollars from the sale of x refrigerators per month.

    a. Find the marginal revenue, $R'(x)$.

    b. Find the marginal revenue when 10 refrigerators per month are sold.

20. An economist has found that the demand function for a particular new product is given by $D(p) = \frac{20}{\sqrt{p - 1}}$, $p > 1$. Find the slope of the demand curve at the point $(5, 10)$.

21. Kathy has diabetes. Her blood sugar level, B, one hour after an insulin injection, depends on the amount of insulin, x, in milligrams injected.

$$B(x) = -0.2x^2 + 500, 0 \le x \le 40$$

    a. Find $B(0)$ and $B(30)$.

    b. Find $B'(0)$ and $B'(30)$.

    c. Interpret your results.

    d. Consider the values of $B'(50)$ and $B(50)$. Comment on the significance of these values. Why are restrictions given for the original function?

22. Determine which functions are differentiable at $x = 1$. Give reasons for your choices.

    a. $f(x) = \frac{3x}{1 - x^2}$

    b. $g(x) = \frac{x - 1}{x^2 + 5x - 6}$

    c. $h(x) = \sqrt[3]{(x - 2)^2}$

    d. $m(x) = |3x - 3| - 1$

23. At what x-values is each function *not* differentiable? Explain.

    a. $f(x) = \frac{3}{4x^2 - x}$

    b. $f(x) = \frac{x^2 - x - 6}{x^2 - 9}$

    c. $f(x) = \sqrt{x^2 - 7x + 6}$

24. At a manufacturing plant, productivity is measured by the number of items, p, produced per employee per day over the previous 10 years. Productivity is modelled by $p(t) = \frac{25t}{t + 1}$, where t is the number of years measured from 10 years ago. Determine the rate of change of p with respect to t.

25. Choose a simple polynomial function in the form $f(x) = ax + b$. Use the quotient rule to find the derivative of the reciprocal function $\frac{1}{ax + b}$. Repeat for other polynomial functions, and devise a rule for finding the derivative of $\frac{1}{f(x)}$. Confirm your rule using first principles.

26. Given $f(x) = \frac{(2x - 3)^2 + 5}{2x - 3}$,
    a. Express f as the composition of two simpler functions.
    b. Use this composition to determine $f'(x)$.

27. Given $g(x) = \sqrt{2x - 3} + 5(2x - 3)$,
    a. Express g as the composition of two simpler functions.
    b. Use this composition to determine $g'(x)$.

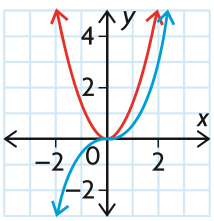
28. Determine the derivative of each function.
    a. $f(x) = (2x - 5)^3(3x^2 + 4)^5$
    b. $g(x) = (8x^3)(4x^2 + 2x - 3)^5$
    c. $y = (5 + x)^2(4 - 7x^3)^6$
    d. $h(x) = \frac{6x - 1}{(3x + 5)^4}$
    e. $y = \frac{(2x^2 - 5)^3}{(x + 8)^2}$
    f. $f(x) = \frac{-3x^4}{\sqrt{4x - 8}}$
    g. $g(x) = \left(\frac{2x + 5}{6 - x^2}\right)^4$
    h. $y = \left[\frac{1}{(4x + x^2)^3}\right]^3$

29. Find numbers a, b, and c so that the graph of $f(x) = ax^2 + bx + c$ has x-intercepts at $(0, 0)$ and $(8, 0)$, and a tangent with slope 16 where $x = 2$.

30. An ant colony was treated with an insecticide and the number of survivors, A, in hundreds at t hours is $A(t) = -t^3 + 5t + 750$.
    a. Find $A'(t)$.
    b. Find the rate of change of the number of living ants in the colony at 5 h.
    c. How many ants were in the colony before it was treated with the insecticide?
    d. How many hours after the insecticide was applied were no ants remaining in the colony?

Chapter 2 Test

1. Explain when you need to use the chain rule.



2. The graphs of a function and its derivative are shown at the left. Label the graphs f and f', and write a short paragraph stating the criteria you used to make your selection.
3. Use the definition of the derivative to find $\frac{d}{dx}(x - x^2)$.
4. Determine $\frac{dy}{dx}$ for each of the following functions:

   a. $y = \frac{1}{3}x^3 - 3x^{-5} + 4\pi$

   b. $y = 6(2x - 9)^5$

   c. $y = \frac{2}{\sqrt{x}} + \frac{x}{\sqrt{3}} + 6\sqrt[3]{x}$

   d. $y = \left(\frac{x^2 + 6}{3x + 4}\right)^5$ (Leave your answer in a simplified factored form.)

   e. $y = x^2\sqrt[3]{6x^2 - 7}$ (Simplify your answer.)

   f. $y = \frac{4x^5 - 5x^4 + 6x - 2}{x^4}$ (Simplify your answer.)

5. Determine the slope of the tangent to the graph of $y = (x^2 + 3x - 2)(7 - 3x)$ at $(1, 8)$.
6. Determine $\frac{dy}{dx}$ at $x = -2$ for $y = 3u^2 + 2u$ and $u = \sqrt{x^2 + 5}$.
7. Determine the equation of the tangent to $y = (3x^{-2} - 2x^3)^5$ at $(1, 1)$.
8. The amount of pollution in a certain lake is $P(t) = (t^{\frac{1}{4}} + 3)^3$, where t is measured in years and P is measured in parts per million (ppm). At what rate is the amount of pollution changing after 16 years?
9. At what point on the curve $y = x^4$ does the normal have a slope of 16?
10. Determine the points on the curve $y = x^3 - x^2 - x + 1$ where the tangent is horizontal.
11. For what values of a and b will the parabola $y = x^2 + ax + b$ be tangent to the curve $y = x^3$ at point $(1, 1)$?