# Chapter 1

## INTRODUCTION TO CALCULUS

In the English language, the rules of grammar are used to speak and write effectively. Asking for a cookie at the age of ten was much easier than when you were first learning to speak. These rules developed over time. Calculus developed in a similar way. Sir Isaac Newton and Gottfried Wilhelm von Leibniz independently organized an assortment of ideas and methods that were circulating among the mathematicians of their time. As a tool in the service of science, calculus served its purpose very well. More than two centuries passed, however, before mathematicians had identified and agreed on its underlying principles—its grammar. In this chapter, you will see some of the ideas that were brought together to form the underlying principles of calculus.

### CHAPTER EXPECTATIONS

In this chapter, you will

- simplify radical expressions, **Section 1.1**
- use limits to determine the slope and the equation of the tangent to a graph, **Section 1.2**
- pose problems and formulate hypotheses regarding rates of change, **Section 1.3, Career Link**
- calculate and interpret average and instantaneous rates of change and relate these values to slopes of secants and tangents, **Section 1.3**
- understand and evaluate limits using appropriate properties, **Sections 1.4, 1.5**
- examine continuous functions and use limits to explain why a function is discontinuous, **Sections 1.5, 1.6**

# Review of Prerequisite Skills

Before beginning this chapter, review the following concepts from previous courses:

- determining the slope of a line: $m = \frac{\Delta y}{\Delta x}$
- determining the equation of a line
- using function notation for substituting into and evaluating functions
- simplifying algebraic expressions
- factoring expressions
- finding the domain of functions
- calculating average rate of change and slopes of secant lines
- estimating instantaneous rate of change and slopes of tangent lines

## Exercise

**1.** Determine the slope of the line passing through each of the following pairs of points:

a. $(2, 5)$ and $(6, -7)$

b. $(3, -4)$ and $(-1, 4)$

c. $(0, 0)$ and $(1, 4)$

d. $(0, 0)$ and $(-1, 4)$

e. $(-2.1, 4.41)$ and $(-2, 4)$

f. $\left(\frac{3}{4}, \frac{1}{4}\right)$ and $\left(\frac{7}{4}, -\frac{1}{4}\right)$

**2.** Determine the equation of a line for the given information.

a. slope 4, $y$-intercept $-2$

b. slope $-2$, $y$-intercept 5

c. through $(-1, 6)$ and $(4, 12)$

d. through $(-2, 4)$ and $(-6, 8)$

e. vertical, through $(-3, 5)$

f. horizontal, through $(-3, 5)$

**3.** Evaluate for $x = 2$.

a. $f(x) = -3x + 5$

b. $f(x) = (4x - 2)(3x - 6)$

c. $f(x) = -3x^2 + 2x - 1$

d. $f(x) = (5x + 2)^2$

**4.** For $f(x) = \frac{x}{x^2 + 4}$, determine each of the following values:

a. $f(-10)$

b. $f(-3)$

c. $f(0)$

d. $f(10)$

**5.** Consider the function $f$ given by $f(x) = \begin{cases} \sqrt{3 - x}, & \text{if } x < 0 \\ \sqrt{3 + x}, & \text{if } x \geq 0 \end{cases}$

Calculate each of the following:

a. $f(-33)$

b. $f(0)$

c. $f(78)$

d. $f(3)$

**6.** A function $s$ is defined for $t > -3$ by $s(t) = \begin{cases} \frac{1}{t}, \text{ if } -3 < t < 0 \\ 5, \text{ if } t = 0 \\ t^3, \text{ if } t > 0 \end{cases}$

Evaluate each of the following:

a. $s(-2)$ b. $s(-1)$ c. $s(0)$ d. $s(1)$ e. $s(100)$

**7.** Expand, simplify, and write each expression in standard form.

a. $(x - 6)(x + 2)$
b. $(5 - x)(3 + 4x)$
c. $x(5x - 3) - 2x(3x + 2)$
d. $(x - 1)(x + 3) - (2x + 5)(x - 2)$
e. $(a + 2)^3$
f. $(9a - 5)^3$

**8.** Factor each of the following:

a. $x^3 - x$
b. $x^2 + x - 6$
c. $2x^2 - 7x + 6$
d. $x^3 + 2x^2 + x$
e. $27x^3 - 64$
f. $2x^3 - x^2 - 7x + 6$

**9.** Determine the domain of each of the following:

a. $y = \sqrt{x + 5}$
b. $y = x^3$
c. $y = \dfrac{3}{x - 1}$
d. $h(x) = \dfrac{x^2 + 4}{x}$
e. $y = \dfrac{6x}{2x^2 - 5x - 3}$
f. $y = \dfrac{(x - 3)(x + 4)}{(x + 2)(x - 1)(x + 5)}$

**10.** The height of a model rocket in flight can be modelled by the equation $h(t) = -4.9t^2 + 25t + 2$, where $h$ is the height in metres at $t$ seconds. Determine the average rate of change in the model rocket's height with respect to time during

a. the first second
b. the second second

**11.** Sacha drains the water from a hot tub. The hot tub holds 1600 L of water. It takes 2 h for the water to drain completely. The volume of water in the hot tub is modelled by $V(t) = 1600 - \frac{t^2}{9}$, where $V$ is the volume in litres at $t$ minutes and $0 \leq t \leq 120$.

a. Determine the average rate of change in volume during the second hour.
b. Estimate the instantaneous rate of change in volume after exactly 60 min.
c. Explain why all estimates of the instantaneous rate of change in volume where $0 \leq t \leq 120$ result in a negative value.

**12.** a. Sketch the graph of $f(x) = -2(x - 3)^2 + 4$.
b. Draw a tangent line at the point $(5, f(5))$, and estimate its slope.
c. Estimate the instantaneous rate of change in $f(x)$ when $x = 5$.

**CAREER LINK** Investigate

## CHAPTER 1: ASSESSING ATHLETIC PERFORMANCE

Differential calculus is fundamentally about the idea of instantaneous rate of change. A familiar rate of change is heart rate. Elite athletes are keenly interested in the analysis of heart rates. Sporting performance is enhanced when an athlete is able to increase his or her heart rate at a slower pace (that is, to get tired less quickly). A heart rate is described for an instant in time.

Heart rate is the instantaneous rate of change in the total number of heartbeats with respect to time. When nurses and doctors count heartbeats and then divide by the time elapsed, they are not determining the instantaneous rate of change but are calculating the average heart rate over a period of time (usually 10 s). In this chapter, the idea of the derivative will be developed, progressing from the average rate of change calculated over smaller and smaller intervals until a limiting value is reached at the instantaneous rate of change.

| Time (s) | Time (min) | Number of Heartbeats |
|---|---|---|
| 10 | 0.17 | 9 |
| 20 | 0.33 | 19 |
| 30 | 0.50 | 31 |
| 40 | 0.67 | 44 |
| 50 | 0.83 | 59 |
| 60 | 1.00 | 75 |

### Case Study—Assessing Elite Athlete Performance

The table shows the number of heartbeats of an athlete who is undergoing a cardiovascular fitness test. Complete the discussion questions to determine if this athlete is under his or her maximum desired heart rate of 65 beats per minute at precisely 30 s.

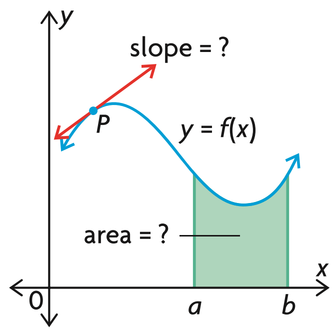
### DISCUSSION QUESTIONS

1. Graph the number of heartbeats versus time (in minutes) on graph paper, joining the points to make a smooth curve. Draw a second relationship on the same set of axes, showing the resting heart rate of 50 beats per minute. Use the slopes of the two relationships graphed to explain why the test results indicate that the person must be exercising.
2. Discuss how the average rate of change in the number of heartbeats over an interval of time could be calculated using this graph. Explain your reasoning.
3. Calculate the athlete's average heart rate over the intervals of [0 s, 60 s], [10 s, 50 s], and [20 s, 40 s]. Show the progression of these average heart rate calculations on the graph as a series of secants.
4. Use the progression of these average heart-rate secants to make a graphical prediction of the instantaneous heart rate at $t = 30$ s. Is the athlete's heart rate less than 65 beats per minute at $t = 30$ s? Estimate the heart rate at $t = 60$ s.

## What Is Calculus?

Two simple geometric problems originally led to the development of what is now called calculus. Both problems can be stated in terms of the graph of a function $y = f(x)$.

- The problem of tangents: What is the slope of the tangent to the graph of a function at a given point $P$?
- The problem of areas: What is the area under a graph of a function $y = f(x)$ between $x = a$ and $x = b$?




Interest in the problem of tangents and the problem of areas dates back to scientists such as Archimedes of Syracuse (287–212 BCE), who used his vast ingenuity to solve special cases of these problems. Further progress was made in the seventeenth century, most notably by Pierre de Fermat (1601–1665) and Isaac Barrow (1630–1677), a professor of Sir Isaac Newton (1642–1727) at the University of Cambridge, England. Professor Barrow recognized that there was a close connection between the problem of tangents and the problem of areas. However, it took the genius of both Newton and Gottfried Wilhelm von Leibniz (1646–1716) to show the way to handle both problems. Using the analytic geometry of Rene Descartes (1596–1650), Newton and Leibniz showed independently how these two problems could be solved by means of new operations on functions, called differentiation and integration. Their discovery is considered to be one of the major advances in the history of mathematics. Further research by mathematicians from many countries using these operations has created a problem-solving tool of immense power and versatility, which is known as calculus. It is a powerful branch of mathematics, used in applied mathematics, science, engineering, and economics.

We begin our study of calculus by discussing the meaning of a tangent and the related idea of rate of change. This leads us to the study of limits and, at the end of the chapter, to the concept of the derivative of a function.

# *Section 1.1*—Radical Expressions: Rationalizing Denominators

Now that we have reviewed some concepts that will be needed before beginning the introduction to calculus, we have to consider simplifying expressions with radicals in the denominator of radical expressions. Recall that a rational number is a number that can be expressed as a fraction (quotient) containing integers. So the process of changing a denominator from a radical (square root) to a rational number (integer) is called **rationalizing the denominator**. The reason that we rationalize denominators is that dividing by an integer is preferable to dividing by a radical number.

In certain situations, it is useful to rationalize the numerator. Practice with rationalizing the denominator prepares you for rationalizing the numerator.

There are two situations that we need to consider: radical expressions with one-term denominators and those with two-term denominators. For both, the numerator and denominator will be multiplied by the same expression, which is the same as multiplying by one.

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**EXAMPLE 1** **Selecting a strategy to rationalize the denominator**

Simplify $\frac{3}{4\sqrt{5}}$ by rationalizing the denominator.

**Solution**

$$\frac{3}{4\sqrt{5}} = \frac{3}{4\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$$

(Multiply both the numerator and denominator by $\sqrt{5}$)

$$= \frac{3\sqrt{5}}{4 \times 5}$$

(Simplify)

$$= \frac{3\sqrt{5}}{20}$$

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When the denominator of a radical fraction is a two-term expression, you can rationalize the denominator by multiplying by the **conjugate**.

An expression such as $\sqrt{a} + \sqrt{b}$ has the conjugate $\sqrt{a} - \sqrt{b}$.

Why are conjugates important? Recall that the linear terms are eliminated when expanding a difference of squares. For example,

$$\begin{aligned}(a - b)(a + b) &= a^2 + ab - ab - b^2 \\ &= a^2 - b^2\end{aligned}$$

If $a$ and $b$ were radicals, squaring them would rationalize them.

Consider this product: $\left(\sqrt{m} + \sqrt{n}\right)\left(\sqrt{m} - \sqrt{n}\right)$, $m$, $n$ rational

$$= \left(\sqrt{m}\right)^2 - \sqrt{mn} + \sqrt{mn} - \left(\sqrt{n}\right)^2$$

$$= m - n$$

Notice that the result is rational!

## EXAMPLE 2 Creating an equivalent expression by rationalizing the denominator

Simplify $\frac{2}{\sqrt{6} + \sqrt{3}}$ by rationalizing the denominator.

### Solution

$$\frac{2}{\sqrt{6} + \sqrt{3}} = \frac{2}{\sqrt{6} + \sqrt{3}} \times \frac{\sqrt{6} - \sqrt{3}}{\sqrt{6} - \sqrt{3}}$$

(Multiply both the numerator and denominator by $\sqrt{6} - \sqrt{3}$)

$$= \frac{2\left(\sqrt{6} - \sqrt{3}\right)}{6 - 3}$$

(Simplify)

$$= \frac{2\left(\sqrt{6} - \sqrt{3}\right)}{3}$$

## EXAMPLE 3 Selecting a strategy to rationalize the denominator

Simplify the radical expression $\frac{5}{2\sqrt{6} + 3}$ by rationalizing the denominator.

### Solution

$$\frac{5}{2\sqrt{6} + 3} = \frac{5}{2\sqrt{6} + 3} \times \frac{2\sqrt{6} - 3}{2\sqrt{6} - 3}$$

(The conjugate $2\sqrt{6} + \sqrt{3}$ is $2\sqrt{6} - \sqrt{3}$)

$$= \frac{5\left(2\sqrt{6} - 3\right)}{4\sqrt{36} - 9}$$

(Simplify)

$$= \frac{5\left(2\sqrt{6} - 3\right)}{24 - 9}$$

$$= \frac{5\left(2\sqrt{6} - 3\right)}{15}$$

(Divide by the common factor of 5)

$$= \frac{2\sqrt{6} - 3}{3}$$

The numerator can also be rationalized in the same way as the denominator was in the previous expressions.

## EXAMPLE 4 Selecting a strategy to rationalize the numerator

Rationalize the numerator of the expression $\frac{\sqrt{7} - \sqrt{3}}{2}$.

### Solution

$$\frac{\sqrt{7} - \sqrt{3}}{2} = \frac{\sqrt{7} - \sqrt{3}}{2} \times \frac{\sqrt{7} + \sqrt{3}}{\sqrt{7} + \sqrt{3}}$$

(Multiply the numerator and denominator by $\sqrt{7} + \sqrt{3}$)

$$= \frac{7 - 3}{2\left(\sqrt{7} + \sqrt{3}\right)}$$

(Simplify)

$$= \frac{4}{2\left(\sqrt{7} + \sqrt{3}\right)}$$

(Divide by the common factor of 2)

$$= \frac{2}{\sqrt{7} + \sqrt{3}}$$

## IN SUMMARY

### Key Ideas

- To rewrite a radical expression with a one-term radical in the denominator, multiply the numerator and denominator by the one-term denominator.

$$\frac{\sqrt{a}}{\sqrt{b}} = \frac{\sqrt{a}}{\sqrt{b}} \times \frac{\sqrt{b}}{\sqrt{b}}$$
$$= \frac{\sqrt{ab}}{b}$$

- When the denominator of a radical expression is a two-term expression, rationalize the denominator by multiplying the numerator and denominator by the conjugate, and then simplify.

$$\frac{1}{\sqrt{a} - \sqrt{b}} = \frac{1}{\sqrt{a} - \sqrt{b}} \times \frac{\sqrt{a} + \sqrt{b}}{\sqrt{a} + \sqrt{b}}$$
$$= \frac{\sqrt{a} + \sqrt{b}}{a - b}$$

### Need to Know

- When you simplify a radical expression such as $\frac{\sqrt{3}}{5\sqrt{2}}$, multiply the numerator and denominator by the radical only.

$$\frac{\sqrt{3}}{5\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{6}}{5(2)}$$
$$= \frac{\sqrt{6}}{10}$$

- $\sqrt{a} + \sqrt{b}$ is the conjugate $\sqrt{a} - \sqrt{b}$, and vice versa.

## Exercise 1.1

### PART A

1. Write the conjugate of each radical expression.

   a. $2\sqrt{3} - 4$
   b. $\sqrt{3} + \sqrt{2}$
   c. $-2\sqrt{3} - \sqrt{2}$
   d. $3\sqrt{3} + \sqrt{2}$
   e. $\sqrt{2} - \sqrt{5}$
   f. $-\sqrt{5} + 2\sqrt{2}$

2. Rationalize the denominator of each expression. Write your answer in simplest form.

   a. $\dfrac{\sqrt{3} + \sqrt{5}}{\sqrt{2}}$
   b. $\dfrac{2\sqrt{3} - 3\sqrt{2}}{\sqrt{2}}$
   c. $\dfrac{4\sqrt{3} + 3\sqrt{2}}{2\sqrt{3}}$
   d. $\dfrac{3\sqrt{5} - \sqrt{2}}{2\sqrt{2}}$

### PART B

K 3. Rationalize each denominator.

   a. $\dfrac{3}{\sqrt{5} - \sqrt{2}}$
   b. $\dfrac{2\sqrt{5}}{2\sqrt{5} + 3\sqrt{2}}$
   c. $\dfrac{\sqrt{3} - \sqrt{2}}{\sqrt{3} + \sqrt{2}}$
   d. $\dfrac{2\sqrt{5} - 8}{2\sqrt{5} + 3}$
   e. $\dfrac{2\sqrt{3} - \sqrt{2}}{5\sqrt{2} + \sqrt{3}}$
   f. $\dfrac{3\sqrt{3} - 2\sqrt{2}}{3\sqrt{3} + 2\sqrt{2}}$

4. Rationalize each numerator.

   a. $\dfrac{\sqrt{5} - 1}{4}$
   b. $\dfrac{2 - 3\sqrt{2}}{2}$
   c. $\dfrac{\sqrt{5} + 2}{2\sqrt{5} - 1}$

C 5. a. Rationalize the denominator of $\dfrac{8\sqrt{2}}{\sqrt{20} - \sqrt{18}}$.

   b. Rationalize the denominator of $\dfrac{8\sqrt{2}}{2\sqrt{5} - 3\sqrt{2}}$.

   c. Why are your answers in parts a and b the same? Explain.

6. Rationalize each denominator.

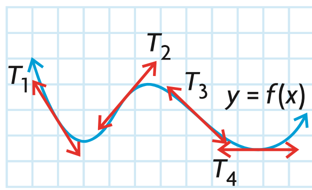
   a. $\dfrac{2\sqrt{2}}{2\sqrt{3} - \sqrt{8}}$
   b. $\dfrac{2\sqrt{6}}{2\sqrt{27} - \sqrt{8}}$
   c. $\dfrac{2\sqrt{2}}{\sqrt{16} - \sqrt{12}}$
   d. $\dfrac{3\sqrt{2} + 2\sqrt{3}}{\sqrt{12} - \sqrt{8}}$
   e. $\dfrac{3\sqrt{5}}{4\sqrt{3} - 5\sqrt{2}}$
   f. $\dfrac{\sqrt{18} + \sqrt{12}}{\sqrt{18} - \sqrt{12}}$

A 7. Rationalize the numerator of each of the following expressions:

   a. $\dfrac{\sqrt{a} - 2}{a - 4}$
   b. $\dfrac{\sqrt{x + 4} - 2}{x}$
   c. $\dfrac{\sqrt{x + h} - x}{x}$

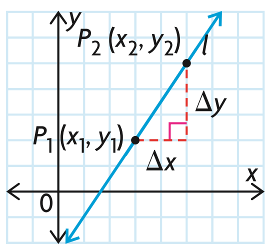
# *Section 1.2*—The Slope of a Tangent

You are familiar with the concept of a **tangent** to a curve. What geometric interpretation can be given to a tangent to the graph of a function at a point $P$? A tangent is the straight line that most resembles the graph near a point. Its slope tells how steep the graph is at the point of tangency. In the figure below, four tangents have been drawn.




The goal of this section is to develop a method for determining the slope of a tangent at a given point on a curve. We begin with a brief review of lines and slopes.

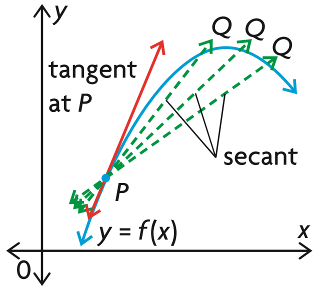
## Lines and Slopes




The slope $m$ of the line joining points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is defined as $m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$.

The equation of the line $l$ in point-slope form is $\frac{y - y_1}{x - x_1} = m$ or $y - y_1 = m(x - x_1)$. The equation in slope–$y$-intercept form is $y = mx + b$, where $b$ is the $y$-intercept of the line.

To determine the equation of a tangent to a curve at a given point, we first need to know the slope of the tangent. What can we do when we only have one point? We proceed as follows:



Consider a curve $y = f(x)$ and a point $P$ that lies on the curve. Now consider another point $Q$ on the curve. The line joining $P$ and $Q$ is called a **secant**. Think of $Q$ as a moving point that slides along the curve toward $P$, so that the slope of the secant $PQ$ becomes a progressively better estimate of the slope of the tangent at $P$.

This suggests the following definition of the slope of the tangent:

**Slope of a Tangent**

The slope of the tangent to a curve at a point $P$ is the limiting slope of the secant $PQ$ as the point $Q$ slides along the curve toward $P$. In other words, the slope of the tangent is said to be the **limit** of the slope of the secant as $Q$ approaches $P$ along the curve.

We will illustrate this idea by finding the slope of the tangent to the parabola $y = x^2$ at $P(3, 9)$.

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## INVESTIGATION 1

A. Determine the $y$-coordinates of the following points that lie on the graph of the parabola $y = x^2$:

i) $Q_1(3.5, y)$ ii) $Q_2(3.1, y)$ iii) $Q_3(3.01, y)$ iv) $Q_4(3.001, y)$

B. Calculate the slopes of the secants through $P(3, 9)$ and each of the points $Q_1$, $Q_2$, $Q_3$, and $Q_4$.

C. Determine the $y$-coordinates of each point on the parabola, and then repeat part B using the following points.

i) $Q_5(2.5, y)$ ii) $Q_6(2.9, y)$ iii) $Q_7(2.99, y)$ iv) $Q_8(2.999, y)$

D. Use your results from parts B and C to estimate the slope of the tangent at point $P(3, 9)$.

E. Graph $y = x^2$ and the tangent to the graph at $P(3, 9)$.

In this investigation, you found the slope of the tangent by finding the limiting value of the slopes of a sequence of secants. Since we are interested in points $Q$ that are close to $P(3, 9)$ on the parabola $y = x^2$ it is convenient to write $Q$ as $(3 + h, (3 + h)^2)$, where $h$ is a very small nonzero number. The variable $h$ determines the position of $Q$ on the parabola. As $Q$ slides along the parabola toward $P$, $h$ will take on values successively smaller and closer to zero. We say that "$h$ approaches zero" and use the notation "$h \rightarrow 0$."

**INVESTIGATION 2** A. Using technology or graph paper, draw the parabola $f(x) = x^2$.

B. Let $P$ be the point $(1, 1)$.

C. Determine the slope of the secant through $Q_1$ and $P(1, 1)$, $Q_2$ and $P(1, 1)$ and so on, for points $Q_1(1.5, f(1.5))$, $Q_2(1.1, f(1.1))$, $Q_3(1.01, f(1.01))$, $Q_4(1.001, f(1.001))$, and $Q_5(1.0001, f(1.0001))$.

D. Draw these secants on the same graph you created in part A.

E. Use your results to estimate the slope of the tangent to the graph of $f$ at point $P$.

F. Draw the tangent at point $P(1, 1)$.

**INVESTIGATION 3** A. Determine an expression for the slope of the secant $PQ$ through points $P(3, 9)$ and $Q(3 + h, (3 + h)^2)$.

B. Explain how you could use the expression in a part A to predict the slope of the tangent to the parabola $f(x) = x^2$ at point $P(3, 9)$.

The slope of the tangent to the parabola at point $P$ is the limiting slope of the secant line $PQ$ as point $Q$ slides along the parabola; that is, as $h \to 0$, we write "$\lim_{h \to 0}$" as the abbreviation for "limiting value as $h$ approaches 0."

Therefore, from the investigation, the slope of the tangent at a point $P$ is

$\lim_{h \to 0}$ (slope of the secant $PQ$).

**EXAMPLE 1** 

### Reasoning about the slope of a tangent as a limiting value

Determine the slope of the tangent to the graph of the parabola $f(x) = x^2$ at $P(3, 9)$.

#### Solution

Using points $P(3, 9)$ and $Q(3 + h, (3 + h)^2)$, $h \neq 0$, the slope of the secant $PQ$ is

$$\frac{\Delta x}{\Delta y} = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{(Substitute)}$$

$$= \frac{(3 + h)^2 - 9}{3 + h - 3} \quad \text{(Expand)}$$

$$= \frac{9 + 6h + h^2 - 9}{h} \quad \text{(Simplify and factor)}$$

$$= \frac{h(6 + h)}{h} \quad \text{(Divide by the common factor of } h\text{)}$$

$$= (6 + h)$$

As $h \to 0$, the value of $(6 + h)$ approaches 6, and thus $\lim_{h \to 0}(6 + h) = 6$.

We conclude that the slope of the tangent at $P(3, 9)$ to the parabola $y = x^2$ is 6.

## EXAMPLE 2 Selecting a strategy involving a series of secants to estimate the slope of a tangent

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For help graphing functions using a graphing calculator, see Technology Appendix p. 597.

a. Use your calculator to graph the parabola $y = -\frac{1}{8}(x + 1)(x - 7)$. Plot the points on the parabola from $x = -1$ to $x = 6$, where $x$ is an integer.
b. Determine the slope of the secants using each point from part a and point $P(5, 1.5)$.
c. Use the result of part b to estimate the slope of the tangent at $P(5, 1.5)$.

### Solution

a. Using the $x$-intercepts of $-1$ and 7, the equation of the axis of symmetry is $x = \frac{-1 + 7}{2} = 3$, so the $x$-coordinate of the vertex is 3.

Substitute $x = 3$ into $y = -\frac{1}{8}(x + 1)(x - 7)$.

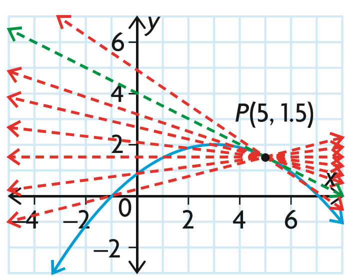
$y = -\frac{1}{8}(3 + 1)(3 - 7) = 2$

Therefore, the vertex is $(3, 2)$.

The $y$-intercept of the parabola is $\frac{7}{8}$.

The points on the parabola are $(-1, 0)$, $(0, 0.875)$, $(1, 1.5)$, $(2, 1.875)$, $(3, 2)$, $(4, 1.875)$, $(5, 1.5)$, and $(6, 0.875)$.

The parabola and the secants through each point and point $P(5, 1.5)$ are shown in red. The tangent through $P(5, 1.5)$ is shown in green.

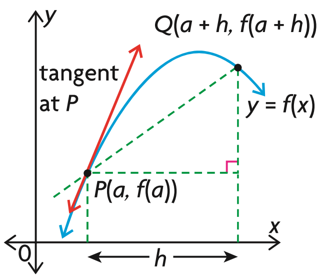



b. Using points $(-1, 0)$ and $P(5, 1.5)$, the slope is $m = \frac{1.5 - 0}{5 - (-1)} = 0.25$.
Using the other points and $P(5, 1.5)$, the slopes are 0.125, 0, $-0.125$, $-0.25$, $-0.375$, and $-0.625$, respectively.

c. The slope of the tangent at $P(5, 1.5)$ is between $-0.375$ and $-0.625$. It can be determined to be $-0.5$ using points closer and closer to $P(5, 1.5)$.

## The Slope of a Tangent at an Arbitrary Point

We can now generalize the method used above to derive a formula for the slope of the tangent to the graph of any function $y = f(x)$.




Let $P(a, f(a))$ be a fixed point on the graph of $y = f(x)$, and let $Q(x, y) = Q(x, f(x))$ represent any other point on the graph. If $Q$ is a horizontal distance of $h$ units from $P$, then $x = a + h$ and $y = f(a + h)$. Point $Q$ then has coordinates $Q(a + h, f(a + h))$.

The slope of the secant $PQ$ is $\dfrac{\Delta y}{\Delta x} = \dfrac{f(a + h) - f(a)}{a + h - a} = \dfrac{f(a + h) - f(a)}{h}$.

This quotient is fundamental to calculus and is referred to as the **difference quotient**. Therefore, the slope $m$ of the tangent at $P(a, f(a))$ is $\lim\limits_{h \to 0}$ (slope of the secant $PQ$), which may be written as $m = \lim\limits_{h \to 0} \dfrac{f(a + h) - f(a)}{h}$.

**Slope of a Tangent as a Limit**

The slope of the tangent to the graph $y = f(x)$ at point $P(a, f(a))$ is

$m = \lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x} = \lim\limits_{h \to 0} \dfrac{f(a + h) - f(a)}{h}$, if this limit exists.

---

### EXAMPLE 3 Connecting limits and the difference quotient to the slope of a tangent

a. Using the definition of the slope of a tangent, determine the slope of the tangent to the curve $y = -x^2 + 4x + 1$ at the point determined by $x = 3$.
b. Determine the equation of the tangent.
c. Sketch the graph of $y = -x^2 + 4x + 1$ and the tangent at $x = 3$.

### Solution

a. The slope of the tangent can be determined using the expression above. In this example, $f(x) = -x^2 + 4x + 1$ and $a = 3$.

Then $\quad f(3) = -(3)^2 + 4(3) + 1 = 4$

and $f(3 + h) = -(3 + h)^2 + 4(3 + h) + 1$

$= -9 - 6h - h^2 + 12 + 4h + 1$

$= -h^2 - 2h + 4$

The slope of the tangent at $(3, 4)$ is

$$m = \lim_{h \to 0} \frac{f(3 + h) - f(3)}{h}$$ (Substitute)

$$= \lim_{h \to 0} \frac{[-h^2 - 2h + 4] - 4}{h}$$ (Simplify and factor)

$$= \lim_{h \to 0} \frac{h(-h - 2)}{h}$$ (Divide by the common factor)

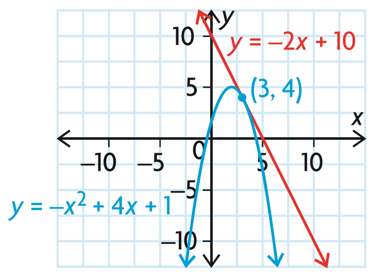
$$= \lim_{h \to 0} (-h - 2)$$ (Evaluate)

$$= -2$$

The slope of the tangent at $x = 3$ is $-2$.

b. The equation of the tangent at $(3, 4)$ is $\frac{y - 4}{x - 3} = -2$, or $y = -2x + 10$.

c. Using graphing software, we obtain




## EXAMPLE 4 Selecting a limit strategy to determine the slope of a tangent

Determine the slope of the tangent to the rational function $f(x) = \frac{3x + 6}{x}$ at point $(2, 6)$.

### Solution

Using the definition, the slope of the tangent at $(2, 6)$ is

$$m = \lim_{h \to 0} \frac{f(2 + h) - f(2)}{h}$$ (Substitute)

$$= \lim_{h \to 0} \frac{\frac{6 + 3h + 6}{2 + h} - 6}{h}$$ (Determine a common denominator)

$$= \lim_{h \to 0} \frac{\frac{6 + 3h + 6}{2 + h} - \frac{6(2 + h)}{2 + h}}{h}$$ (Simplify)

$$= \lim_{h \to 0} \frac{\frac{12 + 3h - 12 - 6h}{2 + h}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{-3h}{2 + h}}{h}$$ (Multiply by the reciprocal)

$$= \lim_{h \to 0} \frac{-3h}{2 + h} \times \frac{1}{h}$$

$$= \lim_{h \to 0} \frac{-3}{2 + h}$$ (Evaluate)

$$= -1.5$$

Therefore, the slope of the tangent to $f(x) = \frac{3x + 6}{x}$ at $(2, 6)$ is $-1.5$.

---

**EXAMPLE 5**

## Determining the slope of a line tangent to a root function

Find the slope of the tangent to $f(x) = \sqrt{x}$ at $x = 9$.

### Solution

$$f(9) = \sqrt{9} = 3$$
$$f(9 + h) = \sqrt{9 + h}$$

Using the limit of the difference quotient, the slope of the tangent at $x = 9$ is

$$m = \lim_{h \to 0} \frac{f(9 + h) - f(9)}{h}$$ (Substitute)

$$= \lim_{h \to 0} \frac{\sqrt{9 + h} - 3}{h}$$ (Rationalize the numerator)

$$= \lim_{h \to 0} \frac{\sqrt{9 + h} - 3}{h} \times \frac{\sqrt{9 + h} + 3}{\sqrt{9 + h} + 3}$$

$$= \lim_{h \to 0} \frac{(9 + h) - 9}{h(\sqrt{9 + h} + 3)}$$ (Simplify)

$$= \lim_{h \to 0} \frac{h}{h(\sqrt{9 + h} + 3)}$$ (Divide by the common factor of $h$)

$$= \lim_{h \to 0} \frac{1}{\sqrt{9 + h} + 3}$$ (Evalute)

$$= \frac{1}{6}$$

Therefore, the slope of the tangent to $f(x) = \sqrt{x}$ at $x = 9$ is $\frac{1}{6}$.

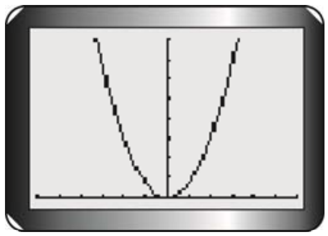
## INVESTIGATION 4

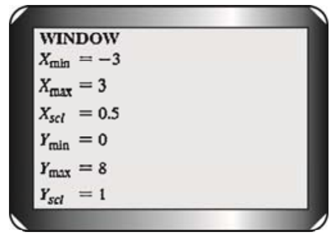
**Tech | Support**

For help graphing functions, tracing, and using the table feature on a graphing calculator, see Technology Appendices p. 597 and p. 599.

A graphing calculator can help us estimate the slope of a tangent at a point. The exact value can then be found using the definition of the slope of the tangent using the difference quotient. For example, suppose that we wish to find the slope of the tangent to $y = f(x) = x^3$ at $x = 1$.

A. Graph $Y_1 = \frac{((x + 0.01)^3 - x^3)}{0.01}$.






B. Explain why the values for the WINDOW were chosen.
Observe that the function entered in $Y_1$ is the difference quotient $\frac{f(a + h) - f(a)}{h}$ for $f(x) = x^3$ and $h = 0.01$. Remember that this approximates the slope of the tangent and not the graph of $f(x) = x^3$.

C. Use the TRACE function to find $X = 1.0212766$, $Y = 3.159756$.
This means that the slope of the secant passing through the points where $x = 1$ and $x = 1 + 0.01 = 1.01$ is about 3.2. The value 3.2 could be used as an approximation for the slope of the tangent at $x = 1$.

D. Can you improve this approximation? Explain how you could improve your estimate. Also, if you use different WINDOW values, you can see a different-sized, or differently centred, graph.

E. Try once again by setting $X_{min} = -9$, $X_{max} = 10$, and note the different appearance of the graph. Use the TRACE function to find $X = 0.90425532$, $Y = 2.4802607$, and then $X = 1.106383$, $Y = 3.7055414$. What is your guess for the slope of the tangent at $x = 1$ now? Explain why only estimation is possible.

F. Another way of using a graphing calculator to approximate the slope of the tangent is to consider $h$ as the variable in the difference quotient. For this example, $f(x) = x^3$ at $x = 1$, look at $\frac{f(a + h) - f(a)}{h} = \frac{(1 + h)^3 - 1^3}{h}$.

G. Trace values of $h$ as $h \to 0$. You can use the table or graph function of your calculator. Graphically, we say that we are looking at $\frac{(1 + h)^3 - 1}{h}$ in the neighbourhood of $h = 0$. To do this, graph $y = \frac{(1 + x)^3 - 1}{x}$ and examine the value of the function as $x \to 0$.

## IN SUMMARY

### Key Ideas

- The slope of the tangent to a curve at a point $P$ is the limit of the slopes of the secants $PQ$ as $Q$ moves closer to $P$.

  $m_{\text{tangent}} = \lim_{Q \to P} (\textit{slope of secant } PQ)$

- The slope of the tangent to the graph of $y = f(x)$ at $P(a, f(a))$ is given by

  $m_{\text{tangent}} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$.

### Need to Know

- To find the slope of the tangent at a point $P(a, f(a))$,
  - find the value of $f(a)$
  - find the value of $f(a + h)$
  - evaluate $\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$

# Exercise 1.2

## PART A

1. Calculate the slope of the line through each pair of points.
   a. $(2, 7), (-3, -8)$
   b. $\left(\frac{1}{2}, \frac{3}{2}\right), \left(\frac{7}{2}, -\frac{7}{2}\right)$
   c. $(6.3, -2.6), (1.5, -1)$
2. Determine the slope of a line perpendicular to each of the following:
   a. $y = 3x - 5$
   b. $13x - 7y - 11 = 0$
3. State the equation and sketch the graph of each line described below.
   a. passing through $(-4, -4)$ and $\left(\frac{5}{3}, -\frac{5}{3}\right)$
   b. having slope 8 and $y$-intercept 6
   c. having $x$-intercept 5 and $y$-intercept $-3$
   d. passing through $(5, 6)$ and $(5, -9)$

4. Simplify each of the following difference quotients:

a. $\dfrac{(5+h)^3-125}{h}$

b. $\dfrac{(3+h)^4-81}{h}$

c. $\dfrac{\frac{1}{1+h}-1}{h}$

d. $\dfrac{3(1+h)^2-3}{h}$

e. $\dfrac{\frac{3}{4+h}-\frac{3}{4}}{h}$

f. $\dfrac{\frac{-1}{2+h}+\frac{1}{2}}{h}$

5. Rationalize the numerator of each expression to obtain an equivalent expression.

a. $\dfrac{\sqrt{16+h}-4}{h}$

b. $\dfrac{\sqrt{h^2+5h+4}-2}{h}$

c. $\dfrac{\sqrt{5+h}-\sqrt{5}}{h}$

## PART B

6. Determine an expression, in simplified form, for the slope of the secant $PQ$.

a. $P(1, 3)$, $Q(1+h, f(1+h))$, where $f(x) = 3x^2$

b. $P(1, 3)$, $Q(1+h, (1+h)^3+2)$

c. $P(9, 3)$, $Q(9+h, \sqrt{9+h})$

**K** 7. Consider the function $f(x) = x^3$.

a. Copy and complete the following table of values. $P$ and $Q$ are points on the graph of $f(x)$.

| *P* | *Q* | Slope of Line *PQ* |
|---|---|---|
| (2, ) | (3, ) | |
| (2, ) | (2.5, ) | |
| (2, ) | (2.1, ) | |
| (2, ) | (2.01, ) | |
| (2, ) | (1, ) | |
| (2, ) | (1.5, ) | |
| (2, ) | (1.9, ) | |
| (2, ) | (1.99, ) | |

b. Use your results for part a to approximate the slope of the tangent to the graph of $f(x)$ at point $P$.

c. Calculate the slope of the secant $PQ$, where the $x$-coordinate of $Q$ is $2+h$.

d. Use your result for part c to calculate the slope of the tangent to the graph of $f(x)$ at point $P$.

e. Compare your answers for parts b and d.

f. Sketch the graph of $f(x)$ and the tangent to the graph at point $P$.

8. Determine the slope of the tangent to each curve at the given value of $x$.

a. $y = 3x^2, x = -2$  b. $y = x^2 - x, x = 3$  c. $y = x^3, x = -2$

9. Determine the slope of the tangent to each curve at the given value of $x$.

a. $y = \sqrt{x - 2}, x = 3$

b. $y = \sqrt{x - 5}, x = 9$

c. $y = \sqrt{5x - 1}, x = 2$

10. Determine the slope of the tangent to each curve at the given value of $x$.

a. $y = \frac{8}{x}, x = 2$  b. $y = \frac{8}{3 + x}, x = 1$  c. $y = \frac{1}{x + 2}, x = 3$

11. Determine the slope of the tangent to each curve at the given point.

a. $y = x^2 - 3x, (2, -2)$

b. $f(x) = \frac{4}{x}, (-2, -2)$

c. $y = 3x^3, (1, 3)$

d. $y = \sqrt{x - 7}, (16, 3)$

e. $y = \sqrt{25 - x^2}, (3, 4)$

f. $y = \frac{4 + x}{x - 2}, (8, 2)$

12. Sketch the graph of the function in question 11, part e. Show that the slope of the tangent can be found using the properties of circles.

C 13. Explain how you would approximate the slope of the tangent at a point without using the definition of the slope of the tangent.

14. Using technology, sketch the graph of $y = \frac{3}{4}\sqrt{16 - x^2}$. Explain how the slope of the tangent at $P(0, 3)$ can be found without using the difference quotient.

15. Determine the equation of the tangent to $y = x^2 - 3x + 1$ at $(3, 1)$.

16. Determine the equation of the tangent to $y = x^2 - 7x + 12$ where $x = 2$.

17. For $f(x) = x^2 - 4x + 1$, find

a. the coordinates of point $A$, where $x = 3$,

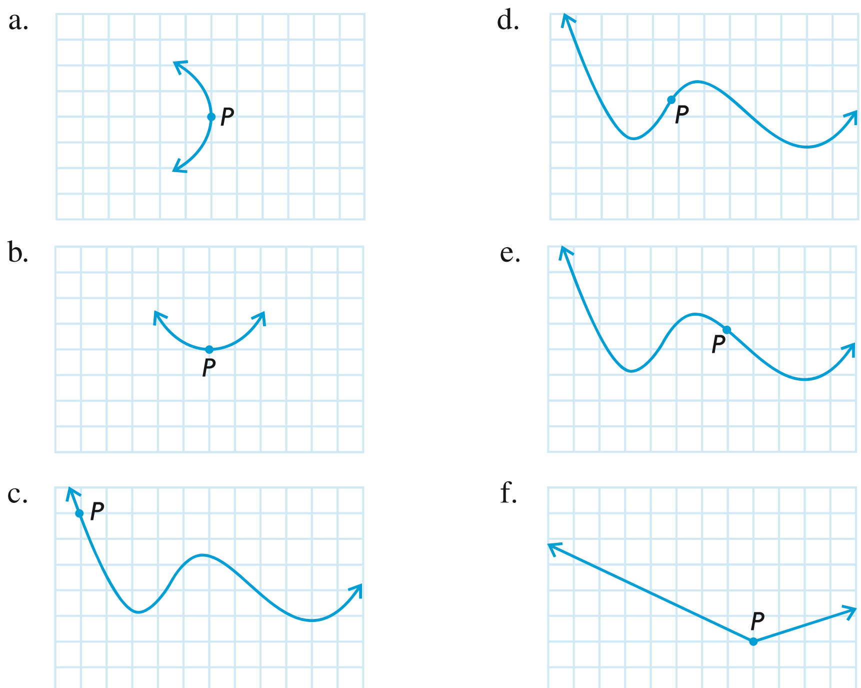
b. the coordinates of point $B$, where $x = 5$

c. the equation of the secant $AB$

d. the equation of the tangent at $A$

e. the equation of the tangent at $B$

18. Copy the following figures. Draw an approximate tangent for each curve at point $P$ and estimate its slope.




19. Find the slope of the demand curve $D(p) = \frac{20}{\sqrt{p-1}}$, $p > 1$, at point $(5, 10)$.

A 20. It is projected that, $t$ years from now, the circulation of a local newspaper will be $C(t) = 100t^2 + 400t + 5000$. Find how fast the circulation is increasing after 6 months. *Hint*: Find the slope of the tangent when $t = 0.5$.

T 21. Find the coordinates of the point on the curve $f(x) = 3x^2 - 4x$ where the tangent is parallel to the line $y = 8x$.

22. Find the points on the graph of $y = \frac{1}{3}x^3 - 5x - \frac{4}{x}$ at which the tangent is horizontal.

### PART C

23. Show that, at the points of intersection of the quadratic functions $y = x^2$ and $y = \frac{1}{2} - x^2$, the tangents to the functions are perpendicular.

24. Determine the equation of the line that passes through $(2, 2)$ and is parallel to the line tangent to $y = -3x^3 - 2x$ at $(-1, 5)$.

25. a. Determine the slope of the tangent to the parabola $y = 4x^2 + 5x - 2$ at the point whose $x$-coordinate is $a$.

    b. At what point on the parabola is the tangent line parallel to the line $10x - 2y - 18 = 0$?

    c. At what point on the parabola is the tangent line perpendicular to the line $x - 35y + 7 = 0$?

# *Section 1.3*—Rates of Change

Many practical relationships involve interdependent quantities. For example, the volume of a balloon varies with its height above the ground, air temperature varies with elevation, and the surface area of a sphere varies with the length of the radius.

These and other relationships can be described by means of a function, often of the form $y = f(x)$. The **dependent variable**, $y$, can represent quantities such as volume, air temperature, and area. The **independent variable**, $x$, can represent quantities such as height, elevation, and length.

We are often interested in how rapidly the dependent variable changes when there is a change in the independent variable. Recall that this concept is called **rate of change**. In this section, we show that an instantaneous rate of change can be calculated by finding the limit of a difference quotient in the same way that we find the slope of a tangent.

## Velocity as a Rate of Change

An object moving in a straight line is an example of a rate-of-change model. It is customary to use either a horizontal or vertical line with a specified origin to represent the line of motion. On such a line, movement to the right or upward is considered to be in the positive direction, and movement to the left (or down) is considered to be in the negative direction. An example of an object moving along a line would be a vehicle entering a highway and travelling north 340 km in 4 h.

The average velocity would be $\frac{340}{4} = 85$ km/h, since

$$\text{average velocity} = \frac{\text{change in position}}{\text{change in time}}$$

If $s(t)$ gives the position of the vehicle on a straight section of the highway at time $t$, then the average rate of change in the position of the vehicle over a time interval is average velocity $= \frac{\Delta s}{\Delta t}$.

---

**INVESTIGATION**

You are driving with a broken speedometer on a highway. At any instant, you do not know how fast the car is going. Your odometer readings are given

| ***t* (h)** | 0 | 1 | 2 | 2.5 | 3 |
|---|---|---|---|---|---|
| ***s*(*t*) (km)** | 62 | 133 | 210 | 250 | 293 |

A. Determine the average velocity of the car over each interval.
B. The speed limit is 80 km/h. Do any of your results in part A suggest that you were speeding at any time? If so, when?
C. Explain why there may be other times when you were travelling above the posted speed limit.
D. Compute your average velocity over the interval $4 \leq t \leq 7$, if $s(4) = 375$ km and $s(7) = 609$ km.
E. After 3 h of driving, you decide to continue driving from Goderich to Huntsville, a distance of 345 km. Using the average velocity from part D, how long would it take you to make this trip?

---

## EXAMPLE 1 Reasoning about average velocity

A pebble is dropped from a cliff, 80 m high. After $t$ seconds, the pebble is $s$ metres above the ground, where $s(t) = 80 - 5t^2$, $0 \leq t \leq 4$.

a. Calculate the average velocity of the pebble between the times $t = 1$ s and $t = 3$ s.
b. Calculate the average velocity of the pebble between the times $t = 1$ s and $t = 1.5$ s.
c. Explain why your answers for parts a and b are different.

### Solution

a. $$\text{average velocity} = \frac{\Delta s}{\Delta t}$$

$$s(1) = 75$$

$$s(3) = 35$$

$$\begin{aligned}\text{average velocity} &= \frac{s(3) - s(1)}{3 - 1} \\ &= \frac{35 - 75}{2} \\ &= \frac{-40}{2} \\ &= -20 \text{ m/s}\end{aligned}$$

The average velocity in this 2 s interval is $-20$ m/s.

b. $$\begin{aligned}s(1.5) &= 80 - 5(1.5)^2 \\ &= 68.75\end{aligned}$$

$$\begin{aligned}\text{average velocity} &= \frac{s(1.5) - s(1)}{1.5 - 1} \\ &= \frac{68.75 - 75}{0.5} \\ &= -12.5 \text{ m/s}\end{aligned}$$

The average velocity in this 0.5 s interval is $-12.5$ m/s.

c. Since gravity causes the velocity to increase with time, the smaller interval of 0.5 s gives a lower average velocity, as well as giving a value closer to the actual velocity at time $t = 1$.

The following table shows the results of similar calculations of the average velocity over successively smaller time intervals:

| Time Interval | Average Velocity (m/s) |
|---|---|
| $1 \le t \le 1.1$ | $-10.5$ |
| $1 \le t \le 1.01$ | $-10.05$ |
| $1 \le t \le 1.001$ | $-10.005$ |

It appears that, as we shorten the time interval, the average velocity is approaching the value $-10$ m/s. The average velocity over the time interval $1 \le t \le 1 + h$ is

$$\begin{aligned}\text{average velocity} &= \frac{s(1 + h) - s(1)}{h} \\ &= \frac{[80 - 5(1 + h)^2] - [80 - 5(1)^2]}{h} \\ &= \frac{75 - 10h - 5h^2 - 75}{h} \\ &= \frac{-10h - 5h^2}{h} \\ &= -10 - 5h, h \ne 0\end{aligned}$$

If the time interval is very short, then $h$ is small, so $5h$ is close to 0 and the average velocity is close to $-10$ m/s. The instantaneous velocity when $t = 1$ is defined to be the limiting value of these average values as $h$ approaches 0. Therefore, the velocity (the word "instantaneous" is usually omitted) at time $t = 1$ s is $v = \lim_{h \to 0}(-10 - 5h) = -10$ m/s.

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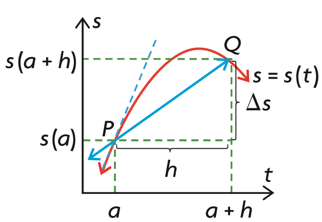



In general, suppose that the position of an object at time $t$ is given by the function $s(t)$. In the time interval from $t = a$ to $t = a + h$, the change in position is $\Delta s = s(a + h) - s(a)$.

The average velocity over this time interval is $\frac{\Delta s}{\Delta t} = \frac{s(a + h) - s(a)}{h}$, which is the same as the slope of the secant $PQ$ where $P(a, s(a))$ and $Q(a + h, s(a + h))$. The **velocity** at a particular time $t = a$ is calculated by finding the limiting value of the average velocity as $h \to 0$.

**Instantaneous Velocity**

The velocity of an object with position function $s(t)$, at time $t = a$, is

$$v(a) = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{h \to 0} \frac{s(a + h) - s(a)}{h}$$

Note that the velocity $v(a)$ is the slope of the tangent to the graph of $s(t)$ at $P(a, s(a))$. The speed of an object is the absolute value of its velocity. It indicates how fast an object is moving, whereas velocity indicates both speed and direction (relative to a given coordinate system).

## EXAMPLE 2 Selecting a strategy to calculate velocity

A toy rocket is launched straight up so that its height $s$, in metres, at time $t$, in seconds, is given by $s(t) = -5t^2 + 30t + 2$. What is the velocity of the rocket at $t = 4$?

### Solution

Since $s(t) = -5t^2 + 30t + 2$,

$$\begin{aligned} s(4 + h) &= -5(4 + h)^2 + 30(4 + h) + 2 \\ &= -80 - 40h - 5h^2 + 120 + 30h + 2 \\ &= -5h^2 - 10h + 42 \\ s(4) &= -5(4)^2 + 30(4) + 2 \\ &= 42 \end{aligned}$$

The velocity at $t = 4$ is

$$v(4) = \lim_{h \to 0} \frac{s(4 + h) - s(4)}{h}$$ (Substitute)

$$= \lim_{h \to 0} \frac{[-10h - 5h^2]}{h}$$ (Factor)

$$= \lim_{h \to 0} \frac{h(-10 - 5h)}{h}$$ (Simplify)

$$= \lim_{h \to 0} (-10 - 5h)$$ (Evaluate)

$$= -10$$

Therefore, the velocity of the rocket is 10 m/s downward at $t = 4$ s.

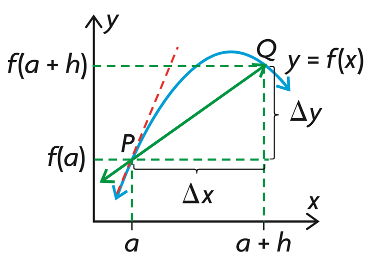
## Comparing Average and Instantaneous Rates of Change

Velocity is only one example of the concept of rate of change. In general, suppose that a quantity $y$ depends on $x$ according to the equation $y = f(x)$. As the independent variable changes from $a$ to $a + h$ ($\Delta x = a + h - a = h$), the corresponding change in the dependent variable $y$ is $\Delta y = f(a + h) - f(a)$.

**Average Rate of Change**

The difference quotient $\frac{\Delta y}{\Delta x} = \frac{f(a + h) - f(a)}{h}$ is called the average rate of change in $y$ with respect to $x$ over the interval from $x = a$ to $x = a + h$.

From the diagram, it follows that the average rate of change equals the slope of the secant $PQ$ of the graph of $f(x)$ where $P(a, f(a))$ and $Q(a + h, f(a + h))$. The instantaneous rate of change in $y$ with respect to $x$ when $x = a$ is defined to be the limiting value of the average rate of change as $h \to 0$.




**Instantaneous Rates of Change**

Therefore, we conclude that the instantaneous rate of change in $y = f(x)$ with respect to $x$ when $x = a$ is $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$, provided that the limit exists.

It should be noted that, as with velocity, the instantaneous rate of change in $y$ with respect to $x$ at $x = a$ equals the slope of the tangent to the graph of $y = f(x)$ at $x = a$.

### EXAMPLE 3 Selecting a strategy to calculate instantaneous rate of change

The total cost, in dollars, of manufacturing $x$ calculators is given by $C(x) = 10\sqrt{x} + 1000$.

a. What is the total cost of manufacturing 100 calculators?
b. What is the rate of change in the total cost with respect to the number of calculators, $x$, being produced when $x = 100$?

**Solution**

a. $C(100) = 10\sqrt{100} + 1000 = 1100$

Therefore, the total cost of manufacturing 100 calculators is \$1100.

b. The rate of change in the cost at $x = 100$ is given by

$$\lim_{h \to 0} \frac{C(100 + h) - C(100)}{h} \qquad \text{(Substitute)}$$

$$= \lim_{h \to 0} \frac{10\sqrt{100 + h} + 1000 - 1100}{h}$$

$$= \lim_{h \to 0} \frac{10\sqrt{100 + h} - 100}{h} \times \frac{10\sqrt{100 + h} + 100}{10\sqrt{100 + h} + 100} \qquad \text{(Rationalize the numerator)}$$

$$= \lim_{h \to 0} \frac{100(100 + h) - 10\,000}{h(10\sqrt{100 + h} + 100)} \qquad \text{(Expand)}$$

$$= \lim_{h \to 0} \frac{100h}{h(10\sqrt{100 + h} + 100)} \qquad \text{(Simplify)}$$

$$= \lim_{h \to 0} \frac{100}{(10\sqrt{100 + h} + 100)} \qquad \text{(Evaluate)}$$

$$= \frac{100}{(10\sqrt{100 + 0} + 100)}$$

$$= 0.5$$

Therefore, the rate of change in the total cost with respect to the number of calculators being produced, when 100 calculators are being produced, is \$0.50 per calculator.

## An Alternative Form for Finding Rates of Change

In Example 1, we determined the velocity of the pebble at $t = 1$ by taking the limit of the average velocity over the interval $1 \le t \le 1 + h$ as $h$ approaches 0.

We can also determine the velocity at $t = 1$ by considering the average velocity over the interval from 1 to a general time $t$ and letting $t$ approach the value 1.

Then, $s(t) = 80 - 5t^2$

$s(1) = 75$

$$
\begin{aligned}
v(1) &= \lim_{t \to 1} \frac{s(t) - s(1)}{t - 1} \\
&= \lim_{t \to 1} \frac{5 - 5t^2}{t - 1} \\
&= \lim_{t \to 1} \frac{5(1 - t)(1 + t)}{t - 1} \\
&= \lim_{t \to 1} -5(1 + t) \\
&= -10
\end{aligned}
$$

In general, the velocity of an object at time $t = a$ is $v(a) = \lim_{t \to a} \frac{s(t) - s(a)}{t - a}$. Similarly, the instantaneous rate of change in $y = f(x)$ with respect to $x$ when $x = a$ is $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$.

## IN SUMMARY

### Key Ideas

- The average velocity can be found in the same way that we found the slope of the secant.

  $$\text{average velocity} = \frac{\text{change in position}}{\text{change in time}}$$

- The instantaneous velocity is the slope of the tangent to the graph of the position function and is found in the same way that we found the slope of the tangent.

### Need to Know

- To find the average velocity (average rate of change) from $t = a$ to $t = a + h$, we can use the difference quotient and the position function $s(t)$

  $$\frac{\Delta s}{\Delta t} = \frac{s(a + h) - s(a)}{h}$$

- The rate of change in the position function, $s(t)$, is the velocity at $t = a$, and we can find it by computing the limiting value of the average velocity as $h \to 0$:

  $$v(a) = \lim_{h \to 0} \frac{s(a + h) - s(a)}{h}$$

# Exercise 1.3

## PART A

1. The velocity of an object is given by $v(t) = t(t - 4)^2$. At what times, in seconds, is the object at rest?

C

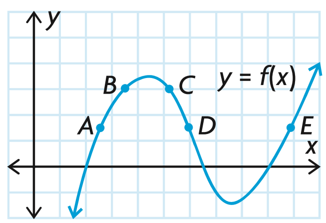
2. Give a geometrical interpretation of the following expressions, if $s$ is a position function:

   a. $\frac{s(9) - s(2)}{7}$

   b. $\lim_{h \to 0} \frac{s(6 + h) - s(6)}{h}$

3. Give a geometrical interpretation of $\lim_{h \to 0} \frac{\sqrt{4 + h} - 2}{h}$.

4. Use the graph to answer each question.




   a. Between which two consecutive points is the average rate of change in the function the greatest?

   b. Is the average rate of change in the function between $A$ and $B$ greater than or less than the instantaneous rate of change at $B$?

   c. Sketch a tangent to the graph somewhere between points $D$ and $E$ such that the slope of the tangent is the same as the average rate of change in the function between $B$ and $C$.

5. What is wrong with the statement "The speed of the cheetah was 65 km/h north"?

6. Is there anything wrong with the statement "A school bus had a velocity of 60 km/h for the morning run, which is why it was late arriving"?

## PART B

7. A construction worker drops a bolt while working on a high-rise building, 320 m above the ground. After $t$ seconds, the bolt has fallen a distance of $s$ metres, where $s(t) = 320 - 5t^2$, $0 \leq t \leq 8$.

   a. Calculate the average velocity during the first, third, and eighth seconds.

   b. Calculate the average velocity for the interval $3 \leq t \leq 8$.

   c. Calculate the velocity at $t = 2$.

**K** 8. The function $s(t) = 8t(t + 2)$ describes the distance $s$, in kilometres, that a car has travelled after a time $t$, in hours, for $0 \leq t \leq 5$.

a. Calculate the average velocity of the car during the following intervals:

i. from $t = 3$ to $t = 4$

ii. from $t = 3$ to $t = 3.1$

iii. from $t = 3$ to $t = 3.01$

b. Use your results for part a to approximate the instantaneous velocity of the car at $t = 3$.

c. Calculate the velocity at $t = 3$.

9. Suppose that a foreign-language student has learned $N(t) = 20t - t^2$ vocabulary terms after $t$ hours of uninterrupted study, where $0 \leq t \leq 10$.

a. How many terms are learned between time $t = 2$ h and $t = 3$ h?

b. What is the rate, in terms per hour, at which the student is learning at time $t = 2$ h?

**A** 10. A medicine is administered to a patient. The amount of medicine $M$, in milligrams, in 1 mL of the patient's blood, $t$ hours after the injection, is $M(t) = -\frac{1}{3}t^2 + t$, where $0 \leq t \leq 3$.

a. Find the rate of change in the amount $M$, 2 h after the injection.

b. What is the significance of the fact that your answer is negative?

11. The time $t$, in seconds, taken by an object dropped from a height of $s$ metres to reach the ground is given by the formula $t = \sqrt{\frac{s}{5}}$. Determine the rate of change in time with respect to height when the object is 125 m above the ground.

12. Suppose that the temperature $T$, in degrees Celsius, varies with the height $h$, in kilometres, above Earth's surface according to the equation $T(h) = \frac{60}{h + 2}$. Find the rate of change in temperature with respect to height at a height of 3 km.

13. A spaceship approaching touchdown on a distant planet has height $h$, in metres, at time $t$, in seconds, given by $h = 25t^2 - 100t + 100$. When does the spaceship land on the surface? With what speed does it land (assuming it descends vertically)?

14. A manufacturer of soccer balls finds that the profit from the sale of $x$ balls per week is given by $P(x) = 160x - x^2$ dollars.

a. Find the profit on the sale of 40 soccer balls per week.

b. Find the rate of change in profit at the production level of 40 balls per week.

c. Using a graphing calculator, graph the profit function and, from the graph, determine for what sales levels of $x$ the rate of change in profit is positive.

15. Use the alternate definition $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ to calculate the instantaneous rate of change of $f(x)$ at the given point or value of $x$.

a. $f(x) = -x^2 + 2x + 3, (-2,-5)$

b. $f(x) = \frac{x}{x - 1}, x = 2$

c. $f(x) = \sqrt{x + 1}, x = 24$

16. The average annual salary of a professional baseball player can be modelled by the function $S(x) = 246 + 64x - 8.9x^2 + 0.95x^3$, where $S$ represents the average annual salary, in thousands of dollars, and $x$ is the number of years since 1982. Determine the rate at which the average salary was changing in 2005.

17. The motion of an avalanche is described by $s(t) = 3t^2$, where $s$ is the distance, in metres, travelled by the leading edge of the snow at $t$ seconds.

a. Find the distance travelled from 0 s to 5 s.

b. Find the rate at which the avalanche is moving from 0 s to 10 s.

c. Find the rate at which the avalanche is moving at 10 s.

d. How long, to the nearest second, does the leading edge of the snow take to move 600 m?

## PART C

T 18. Let $(a, b)$ be any point on the graph of $y = \frac{1}{x}, x \neq 0$. Prove that the area of the triangle formed by the tangent through $(a, b)$ and the coordinate axes is 2.

19. MegaCorp's total weekly cost to produce $x$ pencils can be written as $C(x) = F + V(x)$, where $F$, a constant, represents fixed costs such as rent and utilities and $V(x)$ represents variable costs, which depend on the production level $x$. Show that the rate of change in the weekly cost is independent of fixed costs.

20. A circular oil spill on the surface of the ocean spreads outward. Find the approximate rate of change in the area of the oil slick with respect to its radius when the radius is 100 m.

21. Show that the rate of change in the volume of a cube with respect to its edge length is equal to half the surface area of the cube.

22. Determine the instantaneous rate of change in

a. the surface area of a spherical balloon (as it is inflated) at the point in time when the radius reaches 10 cm

b. the volume of a spherical balloon (as it is deflated) at the point in time when the radius reaches 5 cm

# Mid-Chapter Review

1. Calculate the product of each radical expression and its corresponding conjugate.

   a. $\sqrt{5} - \sqrt{2}$ b. $3\sqrt{5} + 2\sqrt{2}$ c. $9 + 2\sqrt{5}$ d. $3\sqrt{5} - 2\sqrt{10}$

2. Rationalize each denominator.

   a. $\dfrac{6 + \sqrt{2}}{\sqrt{3}}$ c. $\dfrac{5}{\sqrt{7} - 4}$ e. $\dfrac{5\sqrt{3}}{2\sqrt{3} + 4}$

   b. $\dfrac{2\sqrt{3} + 4}{\sqrt{3}}$ d. $\dfrac{2\sqrt{3}}{\sqrt{3} - 2}$ f. $\dfrac{3\sqrt{2}}{2\sqrt{3} - 5}$

3. Rationalize each numerator.

   a. $\dfrac{\sqrt{2}}{5}$ c. $\dfrac{\sqrt{7} - 4}{5}$ e. $\dfrac{\sqrt{3} - \sqrt{7}}{4}$

   b. $\dfrac{\sqrt{3}}{6 + \sqrt{2}}$ d. $\dfrac{2\sqrt{3} - 5}{3\sqrt{2}}$ f. $\dfrac{2\sqrt{3} + \sqrt{7}}{5}$

4. Determine the equation of the line described by the given information.

   a. slope $-\frac{2}{3}$, passing through point $(0, 6)$

   b. passing through points $(2, 7)$ and $(6, 11)$

   c. parallel to $y = 4x - 6$, passing through point $(2, 6)$

   d. perpendicular to $y = -5x + 3$, passing through point $(-1, -2)$

5. Find the slope of $PQ$, in simplified form, given $P(1, -1)$ and $Q(1 + h, f(1 + h))$, where $f(x) = -x^2$.

6. Consider the function $y = x^2 - 2x - 2$.

   a. Copy and complete the following tables of values. $P$ and $Q$ are points on the graph of $f(x)$.

| ***P*** | ***Q*** | **Slope of Line *PQ*** |
|---|---|---|
| $(-1, 1)$ | $(-2, 6)$ | $\frac{-5}{1} = -5$ |
| $(-1, 1)$ | $(-1.5, 3.25)$ | |
| $(-1, 1)$ | $(-1.1, \quad)$ | |
| $(-1, 1)$ | $(-1.01, \quad)$ | |
| $(-1, 1)$ | $(-1.001, \quad)$ | |

| ***P*** | ***Q*** | **Slope of Line *PQ*** |
|---|---|---|
| $(-1, 1)$ | $(0, \quad)$ | |
| $(-1, 1)$ | $(-0.5, \quad)$ | |
| $(-1, 1)$ | $(-0.9, \quad)$ | |
| $(-1, 1)$ | $(-0.99, \quad)$ | |
| $(-1, 1)$ | $(-0.999, \quad)$ | |

b. Use your results for part a to approximate the slope of the tangent to the graph of $f(x)$ at point $P$.

c. Calculate the slope of the secant where the $x$-coordinate of $Q$ is $-1 + h$.

d. Use your results for part c to calculate the slope of the tangent to the graph of $f(x)$ at point $P$.

e. Compare your answers for parts b and d.

7. Calculate the slope of the tangent to each curve at the given point or value of $x$.

a. $f(x) = x^2 + 3x - 5, (-3, -5)$

b. $y = \frac{1}{x}, x = \frac{1}{3}$

c. $y = \frac{4}{x - 2}, (6, 1)$

d. $f(x) = \sqrt{x + 4}, x = 5$

8. The function $s(t) = 6t(t + 1)$ describes the distance (in kilometres) that a car has travelled after a time $t$ (in hours), for $0 \leq t \leq 6$.

a. Calculate the average velocity of the car during the following intervals.

i. from $t = 2$ to $t = 3$
ii. from $t = 2$ to $t = 2.1$
iii. from $t = 2$ to $t = 2.01$

b. Use your results for part a to approximate the instantaneous velocity of the car when $t = 2$.

c. Find the average velocity of the car from $t = 2$ to $t = 2 + h$.

d. Use your results for part c to find the velocity when $t = 2$.

9. Calculate the instantaneous rate of change of $f(x)$ with respect to $x$ at the given value of $x$.

a. $f(x) = 5 - x^2, x = 2$

b. $f(x) = \frac{3}{x}, x = \frac{1}{2}$

10. An oil tank is being drained for cleaning. After $t$ minutes, there are $V$ litres of oil left in the tank, where $V(t) = 50(30 - t)^2, 0 \leq t \leq 30$.

a. Calculate the average rate of change in volume during the first 20 min.

b. Calculate the rate of change in volume at time $t = 20$.

11. Find the equation of the tangent at the given value of $x$.

a. $y = x^2 + x - 3, x = 4$

b. $y = 2x^2 - 7, x = -2$

c. $f(x) = 3x^2 + 2x - 5, x = -1$

d. $f(x) = 5x^2 - 8x + 3, x = 1$

12. Find the equation of the tangent to the graph of the function at the given value of $x$.

a. $f(x) = \frac{x}{x + 3}, x = -5$

b. $f(x) = \frac{2x + 5}{5x - 1}, x = -1$

## *Section 1.4*—The Limit of a Function

The notation $\lim_{x \to a} f(x) = L$ is read "the limit of $f(x)$ as $x$ approaches $a$ equals $L$" and means that the value of $f(x)$ can be made arbitrarily close to $L$ by choosing $x$ sufficiently close to $a$ (but not equal to $a$). But $\lim_{x \to a} f(x)$ exists if and only if the limiting value from the left equals the limiting value from the right. We shall use this definition to evaluate some limits.

*Note:* This is an intuitive explanation of the limit of a function. A more precise definition using inequalities is important for advanced work but is not necessary for our purposes.

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**INVESTIGATION 1** Determine the limit of $y = x^2 - 1$, as $x$ approaches 2.

A. Copy and complete the table of values.

| $x$ | 1 | 1.5 | 1.9 | 1.99 | 1.999 | 2 | 2.001 | 2.01 | 2.1 | 2.5 | 3 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $y = x^2 - 1$ | | | | | | | | | | | |

B. As $x$ approaches 2 from the left, starting at $x = 1$, what is the approximate value of $y$?

C. As $x$ approaches 2 from the right, starting at $x = 3$, what is the approximate value of $y$?

D. Graph $y = x^2 - 1$ using graphing software or graph paper.

E. Using arrows, illustrate that, as we choose a value of $x$ that is closer and closer to $x = 2$, the value of $y$ gets closer and closer to a value of 3.

F. Explain why the limit of $y = x^2 - 1$ exists as $x$ approaches 2, and give its approximate value.

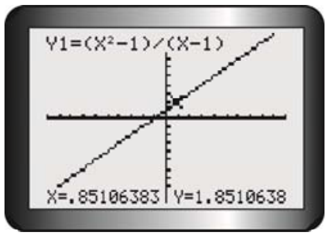
---

**EXAMPLE 1** Determine $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$ by graphing.

### Solution

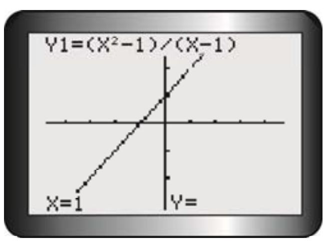
On a graphing calculator, display the graph of $f(x) = \frac{x^2 - 1}{x - 1}$, $x \neq 1$.

The graph shown on your calculator is a line $(f(x) = x + 1)$, whereas it should be a line with point (1, 2) deleted $(f(x) = x + 1, x \neq 1)$. The WINDOW used is $X_{min} = -10$, $X_{max} = 10$, $X_{scl} = 1$, and similarly for $Y$. Use the TRACE function to find $X = 0.85106383$, $Y = 1.8510638$ and $X = 1.0638298$, $Y = 2.0638298$.




Click ZOOM; select 4:ZDecimal, ENTER. Now, the graph of $f(x) = \frac{x^2 - 1}{x - 1}$ is displayed as a straight line with point (1, 2) deleted. The WINDOW has new values, too.

Use the TRACE function to find $X = 0.9$, $Y = 1.9$; $X = 1$, $Y$ has no value given; and $X = 1.1$, $Y = 2.1$.




We can estimate $\lim_{x \to 1} f(x)$. As $x$ approaches 1 from the left, written as "$x \to 1^-$", we observe that $f(x)$ approaches the value 2 from below. As $x$ approaches 1 from the right, written as $x \to 1^+$, $f(x)$ approaches the value 2 from above.

We say that the limit at $x = 1$ exists only if the value approached from the left is the same as the value approached from the right. From this investigation, we conclude that $\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$.

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## EXAMPLE 2 Selecting a table of values strategy to evaluate a limit

Determine $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$ by using a table.

### Solution

We select sequences of numbers for $x \to 1^-$ and $x \to 1^+$.

| *x* approaches 1 from the left → | | | | | | ← *x* approaches 1 from the right | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $x$ | 0 | 0.5 | 0.9 | 0.99 | 0.999 | 1 | 1.001 | 1.01 | 1.1 | 1.5 | 2 |
| $\frac{x^2 - 1}{x - 1}$ | 1 | 1.5 | 1.9 | 1.99 | 1.999 | undefined | 2.001 | 2.01 | 2.1 | 2.5 | 3 |
| $f(x) = \frac{x^2 - 1}{x - 1}$ approaches 2 from below → | | | | | | ← $f(x) = \frac{x^2 - 1}{x - 1}$ approaches 2 from above | | | | | |

This pattern of numbers suggests that $\lim_{x\to 1} \frac{x^2 - 1}{x - 1} = 2$, as we found when graphing in Example 1.

## EXAMPLE 3 Selecting a graphing strategy to evaluate a limit

**Tech Support**
For help graphing piecewise functions on a graphing calculator, see Technology Appendix p. 607.

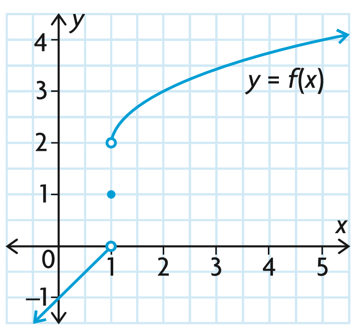
Sketch the graph of the piecewise function:

$$f(x) = \begin{cases} x - 1, \text{ if } x < 1 \\ 1, \text{ if } x = 1 \\ 2 + \sqrt{x - 1}, \text{ if } x > 1 \end{cases}$$

Determine $\lim_{x\to 1} f(x)$.

### Solution

The graph of the function $f$ consists of the line $y = x - 1$ for $x < 1$, the point $(1, 1)$ and the square root function $y = 2 + \sqrt{x - 1}$ for $x > 1$. From the graph of $f(x)$, observe that the limit of $f(x)$ as $x \to 1$ depends on whether $x < 1$ or $x > 1$. As $x \to 1^-$, $f(x)$ approaches the value of 0 from below. We write this as $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x - 1) = 0$.




Similarly, as $x \to 1^+$, $f(x)$ approaches the value 2 from above. We write this as $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \left(2 + \sqrt{x - 1}\right) = 2$. (This is the same when $x = 1$ is substituted into the expression $2 + \sqrt{x - 1}$.) These two limits are referred to as one-sided limits because, in each case, only values of $x$ on one side of $x = 1$ are considered. How-ever, the one-sided limits are unequal—$\lim_{x \to 1^-} f(x) = 0 \neq 2 = \lim_{x \to 1^+} f(x)$—or more briefly, $\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$. This implies that $f(x)$ does not approach a single value as $x \to 1$. We say "the limit of $f(x)$ as $x \to 1$ does not exist" and write "$\lim_{x \to 1} f(x)$ does not exist." This may be surprising, since the function $f(x)$ was defined at $x = 1$—that is, $f(1) = 1$. We can now summarize the ideas introduced in these examples.

**Limits and Their Existence**

We say that the number $L$ is the limit of a function $y = f(x)$ as $x$ approaches the value $a$, written as $\lim_{x \to a} f(x) = L$, if $\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$. Otherwise, $\lim_{x \to a} f(x)$ does not exist.

## IN SUMMARY

### Key Idea

- The limit of a function $y = f(x)$ at $x = a$ is written as $\lim_{x \to a} f(x) = L$, which means that $f(x)$ approaches the value $L$ as $x$ approaches the value $a$ from both the left and right side.

### Need to Know

- $\lim_{x \to a} f(x)$ may exist even if $f(a)$ is not defined.
- $\lim_{x \to a} f(x)$ can be equal to $f(a)$. In this case, the graph of $f(x)$ passes through the point $(a, f(a))$.
- If $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$, then $L$ is the limit of $f(x)$ as $x$ approaches $a$, that is $\lim_{x \to a} f(x) = L$.

# Exercise 1.4

## PART A

1. What do you think is the appropriate limit of each sequence?
   a. 0.7, 0.72, 0.727, 0.7272, . . .
   b. 3, 3.1, 3.14, 3.141, 3.1415, 3.141 59, 3.141 592, . . .

C 2. Explain a process for finding a limit.

3. Write a concise description of the meaning of the following:
   a. a right-sided limit  b. a left-sided limit  c. a (two-sided) limit

4. Calculate each limit.

a. $\lim_{x \to -5} x$

b. $\lim_{x \to 3} (x + 7)$

c. $\lim_{x \to 10} x^2$

d. $\lim_{x \to -2} (4 - 3x^2)$

e. $\lim_{x \to 1} 4$

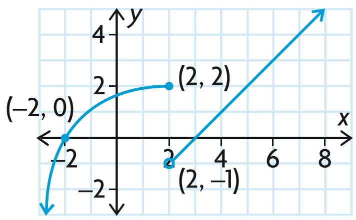
f. $\lim_{x \to 3} 2^x$

5. Determine $\lim_{x \to 4} f(x)$, where $f(x) = \begin{cases} 1, \text{ if } x \neq 4 \\ -1, \text{ if } x = 4 \end{cases}$.

## PART B

6. For the function $f(x)$ in the graph below, determine the following:

a. $\lim_{x \to -2^+} f(x)$  b. $\lim_{x \to 2^-} f(x)$  c. $\lim_{x \to 2^+} f(x)$  d. $f(2)$

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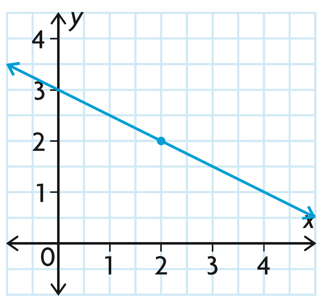
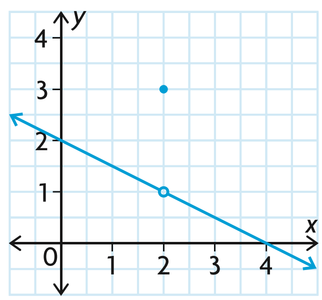
7. Use the graph to find the limit, if it exists.

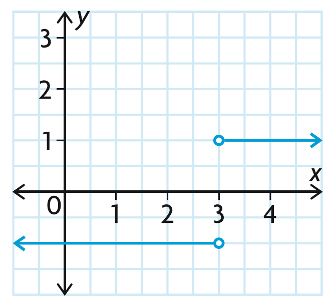
a. $\lim_{x \to 2} f(x)$




b. $\lim_{x \to 2} f(x)$




c. $\lim_{x \to 3} f(x)$




8. Evaluate each limit.

a. $\lim_{x \to -1} (9 - x^2)$

b. $\lim_{x \to 0} \sqrt{\dfrac{x + 20}{2x + 5}}$

c. $\lim_{x \to 5} \sqrt{x - 1}$

9. Find $\lim_{x \to 2} (x^2 + 1)$, and illustrate your result with a graph indicating the limiting value.

10. Evaluate each limit. If the limit does not exist, explain why.

a. $\lim_{x \to 0^+} x^4$

b. $\lim_{x \to 2^-} (x^2 - 4)$

c. $\lim_{x \to 3^-} (x^2 - 4)$

d. $\lim_{x \to 1^+} \dfrac{1}{x - 3}$

e. $\lim_{x \to 3^+} \dfrac{1}{x + 2}$

f. $\lim_{x \to 3} \dfrac{1}{x - 3}$

11. For each function, sketch the graph of the function. Determine the indicated limit if it exists.

a. $f(x) = \begin{cases} x + 2, \text{ if } x < -1 \\ -x + 2, \text{ if } x \geq -1 \end{cases}; \lim_{x \to -1} f(x)$

b. $f(x) = \begin{cases} -x + 4, \text{ if } x \leq 2 \\ -2x + 6, \text{ if } x > 2 \end{cases}; \lim_{x \to 2} f(x)$

c. $f(x) = \begin{cases} 4x, \text{ if } x \geq \frac{1}{2} \\ \frac{1}{x}, \text{ if } x < \frac{1}{2} \end{cases}; \lim_{x \to \frac{1}{2}} f(x)$

d. $f(x) = \begin{cases} 1, \text{ if } x < -0.5 \\ x^2 - 0.25, \text{ if } x \geq -0.5 \end{cases}; \lim_{x \to -0.5} f(x)$

**A** 12. Sketch the graph of any function that satisfies the given conditions.

a. $f(1) = 1, \lim_{x \to 1^+} f(x) = 3, \lim_{x \to 1^-} f(x) = 2$

b. $f(2) = 1, \lim_{x \to 2} f(x) = 0$

c. $f(x) = 1$, if $x < 1$ and $\lim_{x \to 1^+} f(x) = 2$

d. $f(3) = 0, \lim_{x \to 3^+} f(x) = 0$

13. Let $f(x) = mx + b$, where $m$ and $b$ are constants. If $\lim_{x \to 1} f(x) = -2$ and $\lim_{x \to -1} f(x) = 4$, find $m$ and $b$.

## PART C

**T** 14. Determine the real values of $a$, $b$, and $c$ for the quadratic function $f(x) = ax^2 + bx + c, a \neq 0$, that satisfy the conditions $f(0) = 0$, $\lim_{x \to 1} f(x) = 5$, and $\lim_{x \to -2} f(x) = 8$.

15. The fish population, in thousands, in a lake at time $t$, in years, is modelled by the following function:

$$p(t) = \begin{cases} 3 + \frac{1}{12}t^2, \text{ if } 0 \leq t \leq 6 \\ 2 + \frac{1}{18}t^2, \text{ if } 6 < t \leq 12 \end{cases}$$

This function describes a sudden change in the population at time $t = 6$, due to a chemical spill.

a. Sketch the graph of $p(t)$.

b. Evaluate $\lim_{t \to 6^-} p(t)$ and $\lim_{t \to 6^+} p(t)$.

c. Determine how many fish were killed by the spill.

d. At what time did the population recover to the level before the spill?

# *Section 1.5*—Properties of Limits

The statement $\lim_{x \to a} f(x) = L$ says that the values of $f(x)$ become closer and closer to the number $L$ as $x$ gets closer and closer to the number $a$ (from either side of $a$), such that $x \neq a$. This means that when finding the limit of $f(x)$ as $x$ approaches $a$, there is no need to consider $x = a$. In fact, $f(a)$ need not even be defined. The only thing that matters is the behaviour of $f(x)$ near $x = a$.

---

**EXAMPLE 1**

### Reasoning about the limit of a polynomial function

Find $\lim_{x \to 2} (3x^2 + 4x - 1)$.

#### Solution

It seems clear that when $x$ is close to 2, $3x^2$ is close to 12, and $4x$ is close to 8. Therefore, it appears that $\lim_{x \to 2} (3x^2 + 4x - 1) = 12 + 8 - 1 = 19$.

---

In Example 1, the limit was arrived at intuitively. It is possible to evaluate limits using the following properties of limits, which can be proved using the formal definition of limits. This is left for more advanced courses.

#### Properties of Limits

For any real number $a$, suppose that $f$ and $g$ both have limits that exist at $x = a$.

1. $\lim_{x \to a} k = k$, for any constant $k$
2. $\lim_{x \to a} x = a$
3. $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$
4. $\lim_{x \to a} [cf(x)] = c[\lim_{x \to a} f(x)]$, for any constant $c$
5. $\lim_{x \to a} [f(x)g(x)] = [\lim_{x \to a} f(x)][\lim_{x \to a} g(x)]$
6. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, provided that $\lim_{x \to a} g(x) \neq 0$
7. $\lim_{x \to a} [f(x)]^n = [\lim_{x \to a} f(x)]^n$, for any rational number $n$

## EXAMPLE 2 Using the limit properties to evaluate the limit of a polynomial function

Evaluate $\lim_{x \to 2} (3x^2 + 4x - 1)$.

**Solution**

$$\begin{aligned}
\lim_{x \to 2} (3x^2 + 4x - 1) &= \lim_{x \to 2} (3x^2) + \lim_{x \to 2} (4x) - \lim_{x \to 2} (1) \\
&= 3 \lim_{x \to 2} (x^2) + 4 \lim_{x \to 2} (x) - 1 \\
&= 3\left[\lim_{x \to 2} x\right]^2 + 4(2) - 1 \\
&= 3(2)^2 + 8 - 1 \\
&= 19
\end{aligned}$$

*Note:* If $f$ is a polynomial function, then $\lim_{x \to a} f(x) = f(a)$.

## EXAMPLE 3 Using the limit properties to evaluate the limit of a rational function

Evaluate $\lim_{x \to -1} \frac{x^2 - 5x + 2}{2x^3 + 3x + 1}$.

**Solution**

$$\begin{aligned}
\lim_{x \to -1} \frac{x^2 - 5x + 2}{2x^3 + 3x + 1} &= \frac{\lim_{x \to -1} (x^2 - 5x + 2)}{\lim_{x \to -1} (2x^3 + 3x + 1)} \\
&= \frac{(-1)^2 - 5(-1) + 2}{2(-1)^3 + 3(-1) + 1} \\
&= \frac{8}{-4} \\
&= -2
\end{aligned}$$

## EXAMPLE 4 Using the limit properties to evaluate the limit of a root function

Evaluate $\lim_{x \to 5} \sqrt{\frac{x^2}{x - 1}}$.

**Solution**

$$\begin{aligned}
\lim_{x \to 5} \sqrt{\frac{x^2}{x - 1}} &= \sqrt{\lim_{x \to 5} \frac{x^2}{x - 1}} \\
&= \sqrt{\frac{\lim_{x \to 5} x^2}{\lim_{x \to 5} (x - 1)}} \\
&= \sqrt{\frac{25}{4}} \\
&= \frac{5}{2}
\end{aligned}$$

Sometimes $\lim_{x\to a} f(x)$ cannot be found by direct substitution. This is particularly interesting when direct substitution results in an **indeterminate form** $\left(\frac{0}{0}\right)$. In such cases, we look for an equivalent function that agrees with $f$ for all values except at $x = a$. Here are some examples.

## EXAMPLE 5 Selecting a factoring strategy to evaluate a limit

Evaluate $\lim_{x\to 3} \frac{x^2 - 2x - 3}{x - 3}$.

### Solution

Substitution produces the indeterminate form $\frac{0}{0}$. The next step is to simplify the function by factoring and reducing to see if the limit of the reduced form can be evaluated.

$$\lim_{x\to 3} \frac{x^2 - 2x - 3}{x - 3} = \lim_{x\to 3} \frac{(x + 1)(x - 3)}{x - 3} = \lim_{x\to 3} (x + 1)$$

The reduction is valid only if $x \neq 3$. This is not a problem, since $\lim_{x\to 3}$ requires values as $x$ approaches 3, not when $x = 3$. Therefore,

$$\lim_{x\to 3} \frac{x^2 - 2x - 3}{x - 3} = \lim_{x\to 3} (x + 1) = 4.$$

## EXAMPLE 6 Selecting a rationalizing strategy to evaluate a limit

Evaluate $\lim_{x\to 0} \frac{\sqrt{x + 1} - 1}{x}$.

### Solution

A useful technique for finding a limit is to rationalize either the numerator or the denominator to obtain an algebraic form that is not indeterminate.

Substitution produces the indeterminate form $\frac{0}{0}$, so we will try rationalizing.

$$\lim_{x\to 0} \frac{\sqrt{x + 1} - 1}{x} = \lim_{x\to 0} \frac{\sqrt{x + 1} - 1}{x} \times \frac{\sqrt{x + 1} + 1}{\sqrt{x + 1} + 1}$$ (Rationalize the numerator)

$$= \lim_{x\to 0} \frac{x + 1 - 1}{x(\sqrt{x + 1} + 1)}$$

$$= \lim_{x\to 0} \frac{x}{x(\sqrt{x + 1} + 1)}$$ (Simplify)

$$= \lim_{x\to 0} \frac{1}{\sqrt{x + 1} + 1}$$ (Evaluate)

$$= \frac{1}{2}$$

**INVESTIGATION** Here is an alternate technique for finding the value of a limit.

A. Find $\lim_{x\to 1} \frac{(x - 1)}{\sqrt{x} - 1}$ by rationalizing.

B. Let $u = \sqrt{x}$, and rewrite $\lim_{x\to 1} \frac{(x - 1)}{\sqrt{x} - 1}$ in terms of $u$. We know $x = u^2$, $\sqrt{x} \geq 0$, and $u \geq 0$. Therefore, as $x$ approaches the value of 1, $u$ approaches the value of 1. Use this substitution to find $\lim_{u\to 1} \frac{(u^2 - 1)}{u - 1}$ by reducing the rational expression.

## EXAMPLE 7 Selecting a substitution strategy to evaluate a limit

Evaluate $\lim_{x\to 0} \frac{(x + 8)^{\frac{1}{3}} - 2}{x}$.

### Solution

This quotient is indeterminate $\left(\frac{0}{0}\right)$ when $x = 0$. Rationalizing the numerator $(x + 8)^{\frac{1}{3}} - 2$ is not so easy. However, the expression can be simplified by substitution. Let $u = (x + 8)^{\frac{1}{3}}$. Then $u^3 = x + 8$ and $x = u^3 - 8$. As $x$ approaches the value 0, $u$ approaches the value 2. The given limit becomes

$$\lim_{x\to 0} \frac{(x + 8)^{\frac{1}{3}} - 2}{x} = \lim_{u\to 2} \frac{u - 2}{u^3 - 8} \quad \text{(Factor)}$$

$$= \lim_{u\to 2} \frac{u - 2}{(u - 2)(u^2 + 2u + 4)} \quad \text{(Simplify)}$$

$$= \lim_{u\to 2} \frac{1}{u^2 + 2u + 4} \quad \text{(Evaluate)}$$

$$= \frac{1}{12}$$

## EXAMPLE 8 Evaluating a limit that involves absolute value

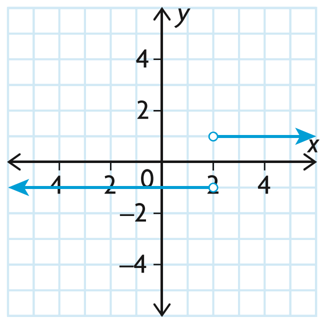
Evaluate $\lim_{x\to 2} \frac{|x - 2|}{x - 2}$. Illustrate with a graph.

### Solution

Consider the following:

$$f(x) = \frac{|x - 2|}{x - 2} = \begin{cases} \frac{(x - 2)}{x - 2}, \text{ if } x > 2 \\ \frac{-(x - 2)}{x - 2}, \text{ if } x < 2 \end{cases}$$

$$= \begin{cases} 1, \text{ if } x > 2 \\ -1, \text{ if } x < 2 \end{cases}$$




Notice that $f(2)$ is not defined. Also note and that we must consider left-hand and right-hand limits.

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (-1) = -1$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (1) = 1$$

Since the left-hand and right-hand limits are not the same, we conclude that $\lim_{x \to 2} \frac{|x - 2|}{x - 2}$ does not exist.

---

## EXAMPLE 9 Reasoning about the existence of a limit

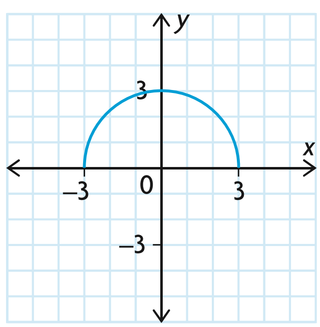
a. Evaluate $\lim_{x \to 3^-} \sqrt{9 - x^2}$

b. Explain why the limit as $x$ approaches $3^+$ cannot be determined.

c. What can you conclude about $\lim_{x \to 3} \sqrt{9 - x^2}$?

### Solution

a. The graph of $f(x) = \sqrt{9 - x^2}$ is the semicircle illustrated below.




From the graph, the left-hand limit at $x \to 3$ is 0. Therefore, $\lim_{x \to 3^-} \sqrt{9 - x^2} = 0$.

b. The function is not defined for $x > 3$.

c. $\lim_{x \to 3} \sqrt{9 - x^2}$ does not exist because the function is not defined on both sides of 3.

In this section, we learned the properties of limits and developed algebraic methods for evaluating limits. The examples in this section complement the table of values and graphing techniques introduced in previous sections.

## IN SUMMARY

### Key Ideas

- If $f$ is a polynomial function, then $\lim_{x \to a} f(x) = f(a)$.
- Substituting $x = a$ into $\lim_{x \to a} f(x)$ can yield the indeterminate form $\frac{0}{0}$. If this happens, you may be able to find an equivalent function that is the same as the function $f$ for all values except at $x = a$. Then, substitution can be used to find the limit.

### Need to Know

To evaluate a limit algebraically, you can use the following techniques:

- direct substitution
- factoring
- rationalizing
- one-sided limits
- change of variable

For any of these techniques, a graph or table of values can be used to check your result.

# Exercise 1.5

## PART A

1. Are there different answers for $\lim_{x \to 2} (3 + x)$, $\lim_{x \to 2} 3 + x$, and $\lim_{x \to 2} (x + 3)$?
2. How do you find the limit of a rational function?

C

3. Once you know $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$, do you then know $\lim_{x \to a} f(x)$? Give reasons for your answer.
4. Evaluate each limit.

a. $\lim_{x \to 2} \frac{3x}{x^2 + 2}$

b. $\lim_{x \to -1} (x^4 + x^3 + x^2)$

c. $\lim_{x \to 9} \left( \sqrt{x} + \frac{1}{\sqrt{x}} \right)^2$

d. $\lim_{x \to 2\pi} (x^3 + \pi^2 x - 5\pi^3)$

e. $\lim_{x \to 0} (\sqrt{3 + \sqrt{1 + x}})$

f. $\lim_{x \to -3} \sqrt{\frac{x - 3}{2x + 4}}$

**PART B**

5. Use a graphing calculator to graph each function and estimate the limit. Then find the limit by substitution.

a. $\lim_{x \to -2} \frac{x^3}{x - 2}$

b. $\lim_{x \to 1} \frac{2x}{\sqrt{x^2 + 1}}$

6. Show that $\lim_{t \to 1} \frac{t^3 - t^2 - 5t}{6 - t^2} = -1$.

K 7. Evaluate the limit of each indeterminate quotient.

a. $\lim_{x \to 2} \frac{4 - x^2}{2 - x}$

b. $\lim_{x \to -1} \frac{2x^2 + 5x + 3}{x + 1}$

c. $\lim_{x \to 3} \frac{x^3 - 27}{x - 3}$

d. $\lim_{x \to 0} \frac{2 - \sqrt{4 + x}}{x}$

e. $\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4}$

f. $\lim_{x \to 0} \frac{\sqrt{7 - x} - \sqrt{7 + x}}{x}$

8. Evaluate the limit by using a change of variable.

a. $\lim_{x \to 8} \frac{\sqrt[3]{x} - 2}{x - 8}$

b. $\lim_{x \to 27} \frac{27 - x}{x^{\frac{1}{3}} - 3}$

c. $\lim_{x \to 1} \frac{x^{\frac{1}{6}} - 1}{x - 1}$

d. $\lim_{x \to 1} \frac{x^{\frac{1}{6}} - 1}{x^{\frac{1}{3}} - 1}$

e. $\lim_{x \to 4} \frac{\sqrt{x} - 2}{\sqrt{x^3} - 8}$

f. $\lim_{x \to 0} \frac{(x + 8)^{\frac{1}{3}} - 2}{x}$

9. Evaluate each limit, if it exists, using any appropriate technique.

a. $\lim_{x \to 4} \frac{16 - x^2}{x^3 + 64}$

b. $\lim_{x \to 4} \frac{x^2 - 16}{x^2 - 5x + 6}$

c. $\lim_{x \to -1} \frac{x^2 + x}{x + 1}$

d. $\lim_{x \to 0} \frac{\sqrt{x + 1} - 1}{x}$

e. $\lim_{h \to 0} \frac{(x + h)^2 - x^2}{h}$

f. $\lim_{x \to 1} \left[ \left( \frac{1}{x - 1} \right) \left( \frac{1}{x + 3} - \frac{2}{3x + 5} \right) \right]$

10. By using one-sided limits, determine whether each limit exists. Illustrate your results geometrically by sketching the graph of the function.

a. $\lim_{x \to 5} \frac{|x - 5|}{x - 5}$

b. $\lim_{x \to \frac{5}{2}} \frac{|2x - 5|(x + 1)}{2x - 5}$

c. $\lim_{x \to 2} \frac{x^2 - x - 2}{|x - 2|}$

d. $\lim_{x \to -2} \frac{(x + 2)^3}{|x + 2|}$

**A** 11. Jacques Charles (1746–1823) discovered that the volume of a gas at a constant pressure increases linearly with the temperature of the gas. To obtain the data in the following table, one mole of hydrogen was held at a constant pressure of one atmosphere. The volume $V$ was measured in litres, and the temperature $T$ was measured in degrees Celsius.

| ***T* (°C)** | −40 | −20 | 0 | 20 | 40 | 60 | 80 |
|---|---|---|---|---|---|---|---|
| ***V* (L)** | 19.1482 | 20.7908 | 22.4334 | 24.0760 | 25.7186 | 27.3612 | 29.0038 |

a. Calculate first differences, and show that $T$ and $V$ are related by a linear relation.

b. Find the linear equation for $V$ in terms of $T$.

c. Solve for $T$ in terms of $V$ for the equation in part b.

d. Show that $\lim_{V \to 0^+} T$ is approximately $-273.15$. *Note:* This represents the approximate number of degrees on the Celsius scale for absolute zero on the Kelvin scale (0 K).

e. Using the information you found in parts b and d, draw a graph of $V$ versus $T$.

**T** 12. Show, using the properties of limits, that if $\lim_{x \to 5} f(x) = 3$, then $\lim_{x \to 5} \frac{x^2 - 4}{f(x)} = 7$.

13. If $\lim_{x \to 4} f(x) = 3$, use the properties of limits to evaluate each limit.

a. $\lim_{x \to 4} [f(x)]^3$

b. $\lim_{x \to 4} \frac{[f(x)]^2 - x^2}{f(x) + x}$

c. $\lim_{x \to 4} \sqrt{3f(x) - 2x}$

## PART C

14. If $\lim_{x \to 0} \frac{f(x)}{x} = 1$ and $\lim_{x \to 0} g(x)$ exists and is nonzero, then evaluate each limit.

a. $\lim_{x \to 0} f(x)$

b. $\lim_{x \to 0} \frac{f(x)}{g(x)}$

15. If $\lim_{x \to 0} \frac{f(x)}{x} = 1$ and $\lim_{x \to 0} \frac{g(x)}{x} = 2$, then evaluate each limit.

a. $\lim_{x \to 0} g(x)$

b. $\lim_{x \to 0} \frac{f(x)}{g(x)}$

16. Evaluate $\lim_{x \to 0} \frac{\sqrt{x + 1} - \sqrt{2x + 1}}{\sqrt{3x + 4} - \sqrt{2x + 4}}$.

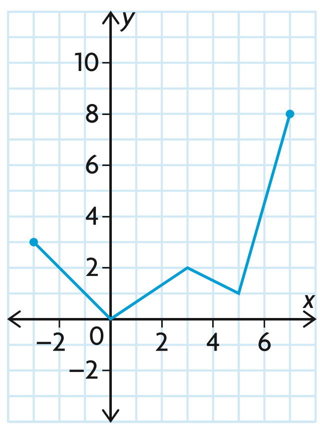
17. Does $\lim_{x \to 1} \frac{x^2 + |x - 1| - 1}{|x - 1|}$ exist? Illustrate your answer by sketching a graph of the function.

# *Section 1.6*—Continuity

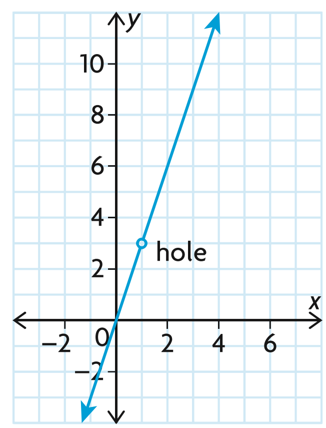
The idea of continuity may be thought of informally as the idea of being able to draw a graph without lifting one's pencil. The concept arose from the notion of a graph "without breaks or jumps or gaps."

When we talk about a function being continuous at a point, we mean that the graph passes through the point without a break. A graph that is not continuous at a point (sometimes referred to as being discontinuous at a point) has a break of some type at the point. The following graphs illustrate these ideas:

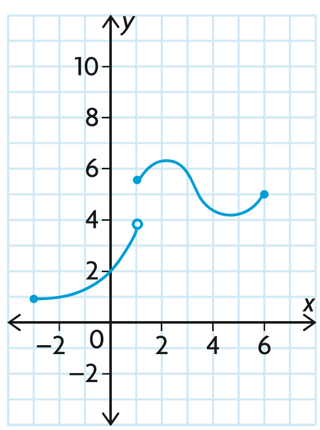
A. Continuous for all values of the domain



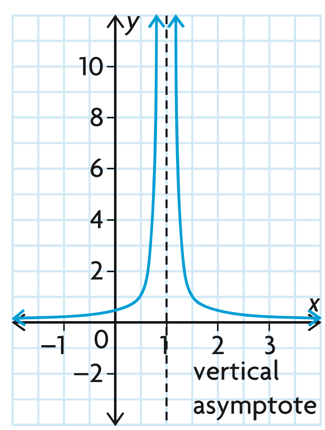

B. Discontinuous at $x = 1$ (point discontinuity)




C. Discontinuous at $x = 1$ (jump discontinuity)




D. Discontinuous at $x = 1$ (infinite discontinuity)



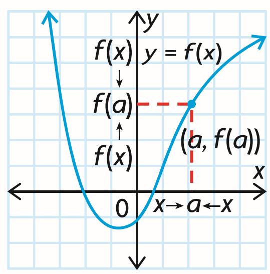

What conditions must be satisfied for a function $f$ to be continuous at $a$? First, $f(a)$ must be defined. The curves in figure B and figure D above are not continuous at $x = 1$ because they are not defined at $x = 1$.

A second condition for continuity at a point $x = a$ is that the function makes no jumps there. This means that, if "$x$ is close to $a$," then $f(x)$ must be close to $f(a)$. This condition is satisfied if $\lim_{x \to a} f(x) = f(a)$. Looking at the graph in figure C, on the previous page, we see that $\lim_{x \to 1} f(x)$ does not exist, and the function is therefore not continuous at $x = 1$.

We can now define the continuity of a function at a point.

**Continuity at a Point**

The function $f(x)$ is continuous at $x = a$ if $f(a)$ is defined and if $\lim_{x \to a} f(x) = f(a)$.




Otherwise, $f(x)$ is discontinuous at $x = a$.

The geometrical meaning of $f$ being continuous at $x = a$ can be stated as follows: As $x \to a$, the points $(x, f(x))$ on the graph of $f$ converge at the point $(a, f(a))$, ensuring that the graph of $f$ is unbroken at $(a, f(a))$.

## EXAMPLE 1 Reasoning about continuity at a point

a. Graph the following function:

$$f(x) = \begin{cases} x^2 - 3, \text{ if } x \leq -1 \\ x - 1, \text{ if } x > -1 \end{cases}$$

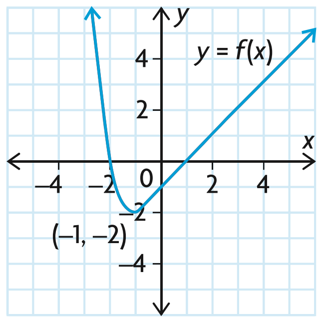
b. Determine $\lim_{x \to -1} f(x)$

c. Determine $f(-1)$.

d. Is $f$ continuous at $x = -1$? Explain.

### Solution

a.




b. From the graph, $\lim_{x \to -1} f(x) = -2$. *Note:* Both the left-hand and right-hand limits are equal.

c. $f(-1) = -2$

d. Therefore, $f(x)$ is continuous at $x = -1$, since $f(-1) = \lim_{x \to -1} f(x)$.

---

**EXAMPLE 2** **Reasoning whether a function is continuous or discontinuous at a point**

Test the continuity of each of the following functions at $x = 2$. If a function is not continuous at $x = 2$, give a reason why it is not continuous.

a. $f(x) = x^3 - x$

b. $g(x) = \dfrac{x^2 - x - 2}{x - 2}$

c. $h(x) = \dfrac{x^2 - x - 2}{x - 2}$, if $x \neq 2$ and $h(2) = 3$

d. $F(x) = \dfrac{1}{(x - 2)^2}$

e. $G(x) = \begin{cases} 4 - x^2, \text{ if } x < 2 \\ 3, \text{ if } x \geq 2 \end{cases}$

### Solution

a. The function $f$ is continuous at $x = 2$ since $f(2) = 6 = \lim_{x \to 2} f(x)$. (Polynomial functions are continuous at all real values of $x$.)

b. The function $g$ is not continuous at $x = 2$ because $g$ is not defined at this value.

c. 
$$\begin{aligned} \lim_{x \to 2} \frac{x^2 - x - 2}{x - 2} &= \lim_{x \to 2} \frac{(x - 2)(x + 1)}{(x - 2)} \\ &= \lim_{x \to 2} (x + 1) \\ &= 3 \\ &= h(2) \end{aligned}$$

Therefore, $h(x)$ is continuous at $x = 2$.

d. The function $F$ is not continuous at $x = 2$ because $F(2)$ is not defined.

e. $\lim_{x \to 2^-} G(x) = \lim_{x \to 2^-} (4 - x^2) = 0$ and $\lim_{x \to 2^+} G(x) = \lim_{x \to 2^+} (3) = 3$

Therefore, since $\lim_{x \to 2} G(x)$ does not exist, the function is not continuous at $x = 2$.

**INVESTIGATION** To test the definition of continuity by graphing, investigate the following:

A. Draw the graph of each function in Example 2.

B. Which of the graphs are continuous, contain a hole or a jump, or have a vertical asymptote?

C. Given only the defining rule of a function $y = f(x)$, such as $f(x) = \frac{8x^3 - 9x + 5}{x^2 + 300x}$, explain why the graphing technique to test for continuity on an interval may be less suitable.

D. Determine where $f(x) = \frac{8x^3 - 9x + 5}{x^2 + 300x}$ is not defined and where it is continuous.

## IN SUMMARY

### Key Ideas

- A function $f$ is continuous at $x = a$ if
  - $f(a)$ is defined
  - $\lim_{x \to a} f(x)$ exists
  - $\lim_{x \to a} f(x) = f(a)$
- A function that is not continuous has some type of break in its graph. This break is the result of a hole, jump, or vertical asymptote.

### Need to Know

- All polynomial functions are continuous for all real numbers.
- A rational function $h(x) = \frac{f(x)}{g(x)}$ is continuous at $x = a$ if $g(a) \neq 0$.
- A rational function in simplified form has a discontinuity at the zeros of the denominator.
- When the one-sided limits are not equal to each other, then the limit at this point does not exist and the function is not continuous at this point.

# Exercise 1.6

## PART A

C 1. How can looking at a graph of a function help you tell where the function is continuous?

2. What does it mean for a function to be continuous over a given domain?

3. What are the basic types of discontinuity? Give an example of each.

4. Find the value(s) of $x$ at which each function is discontinuous.

a. $f(x) = \dfrac{9 - x^2}{x - 3}$

b. $g(x) = \dfrac{7x - 4}{x}$

c. $h(x) = \dfrac{x^2 + 1}{x^3}$

d. $f(x) = \dfrac{x - 4}{x^2 - 9}$

e. $g(x) = \dfrac{13x}{x^2 + x - 6}$

f. $h(x) = \begin{cases} -x, \text{ if } x \le 3 \\ 1 - x, \text{ if } x > 3 \end{cases}$

## PART B

K 5. Determine all the values of $x$ for which each function is continuous.

a. $f(x) = 3x^5 + 2x^3 - x$

b. $g(x) = \pi x^2 - 4.2x + 7$

c. $h(x) = \dfrac{x^2 + 16}{x^2 - 5x}$

d. $f(x) = \sqrt{x + 2}$

e. $g(x) = 10^x$

f. $h(x) = \dfrac{16}{x^2 + 65}$

6. Examine the continuity of $g(x) = x + 3$ when $x = 2$.

7. Sketch a graph of the following function:

$$h(x) = \begin{cases} x - 1, \text{ if } x < 3 \\ 5 - x, \text{ if } x \ge 3 \end{cases}$$

Determine if the function is continuous everywhere.

8. Sketch a graph of the following function:

$$f(x) = \begin{cases} x^2, \text{ if } x < 0 \\ 3, \text{ if } x \ge 0 \end{cases}$$

Is the function continuous?

A 9. Recent postal rates for non-standard and oversized letter mail within Canada are given in the following table. Maximum dimensions for this type of letter mail are 380 mm by 270 mm by 20 mm.

| 100 g or Less | Between 100 g and 200 g | Between 200 g and 500 g |
|---|---|---|
| \$1.10 | \$1.86 | \$2.55 |

Draw a graph of the cost, in dollars, to mail a non-standard envelope as a function of its mass in grams. Where are the discontinuities of this function?

10. Determine whether $f(x) = \dfrac{x^2 - x - 6}{x - 3}$ is continuous at $x = 3$.

11. Examine the continuity of the following function:

$$f(x) = \begin{cases} x, \text{ if } x \le 1 \\ 1, \text{ if } 1 < x \le 2 \\ 3, \text{ if } x > 2 \end{cases}$$

12. $g(x) = \begin{cases} x + 3, \text{ if } x \neq 3 \\ 2 + \sqrt{k}, \text{ if } x = 3 \end{cases}$

Find $k$, if $g(x)$ is continuous.

13. The signum function is defined as follows:

$$f(x) = \begin{cases} -1, \text{ if } x < 0 \\ 0, \text{ if } x = 0 \\ 1, \text{ if } x > 0 \end{cases}$$

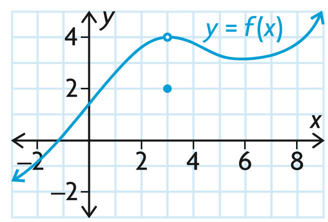
a. Sketch the graph of the signum function.

b. Find each limit, if it exists.

i. $\lim_{x \to 0^-} f(x)$ ii. $\lim_{x \to 0^+} f(x)$ iii. $\lim_{x \to 0} f(x)$

c. Is $f(x)$ continuous? Explain.

14. Examine the graph of $f(x)$.




a. Find $f(3)$.

b. Evaluate $\lim_{x \to 3^-} f(x)$.

c. Is $f(x)$ continuous on the interval $-3 < x < 8$? Explain.

15. What must be true about $A$ and $B$ for the function

$$f(x) = \begin{cases} \dfrac{Ax - B}{x - 2}, \text{ if } x \leq 1 \\ 3x, \text{ if } 1 < x < 2 \\ Bx^2 - A, \text{ if } x \geq 2 \end{cases}$$

if the function is continuous at $x = 1$ but discontinuous at $x = 2$?

## PART C

16. Find constants $a$ and $b$, such that the function

$$f(x) = \begin{cases} -x, \text{ if } -3 \leq x \leq -2 \\ ax^2 + b, \text{ if } -2 < x < 0 \\ 6, \text{ if } x = 0 \end{cases}$$

is continuous for $-3 \leq x \leq 0$.

17. Consider the following function:

$$g(x) = \begin{cases} \dfrac{x|x - 1|}{x - 1}, \text{ if } x \neq 1 \\ 0, \text{ if } x = 1 \end{cases}$$

a. Evaluate $\lim_{x \to 1^+} g(x)$ and $\lim_{x \to 1^-} g(x)$, and then determine whether $\lim_{x \to 1} g(x)$ exists.

b. Sketch the graph of $g(x)$, and identify any points of discontinuity.

## CAREER LINK WRAP-UP Investigate and Apply

### CHAPTER 1: ASSESSING ATHLETIC PERFORMANCE

An Olympic coach has developed a 6 min fitness test for her team members that sets target values for heart rates. The monitor they have available counts the total number of heartbeats, starting from a rest position at "time zero." The results for one of the team members are given in the table below.

| Time (min) | Number of Heartbeats |
|---|---|
| 0.0 | 0 |
| 1.0 | 55 |
| 2.0 | 120 |
| 3.0 | 195 |
| 4.0 | 280 |
| 5.0 | 375 |
| 6.0 | 480 |

a. The coach has established that each athlete's heart rate must not exceed 100 beats per minute at exactly 3 min. Using a graphical technique, determine if this athlete meets the coach's criterion.

b. The coach needs to know the instant in time when an athlete's heart rate actually exceeds 100 beats per minute. Explain how you would solve this problem graphically. Is a graphical solution an efficient method? Explain. How is this problem different from part a?

c. Build a mathematical model with the total number of heartbeats as a function of time ($n = f(t)$). First determine the degree of the polynomial, and then use a graphing calculator to obtain an algebraic model.

d. Solve part b algebraically by obtaining an expression for the instantaneous rate of change in the number of heartbeats (heart rate) as a function of time ($r = g(t)$) using the methods presented in the chapter. Compare the accuracy and efficiency of solving this problem graphically and algebraically.

# Key Concepts Review

We began our introduction to calculus by considering the slope of a tangent and the related concept of rate of change. This led us to the study of limits and has laid the groundwork for Chapter 2 and the concept of the derivative of a function. Consider the following brief summary to confirm your understanding of the key concepts covered in Chapter 1:

- slope of the tangent as the limit of the slope of the secant as $Q$ approaches $P$ along the curve
- slope of a tangent at an arbitrary point
- average and instantaneous rates of change, average velocity, and (instantaneous) velocity
- the limit of a function at a value of the independent variable, which exists when the limiting value from the left equals the limiting value from the right
- properties of limits and the indeterminate form $\frac{0}{0}$
- continuity as a property of a graph "without breaks or jumps or gaps"

## Formulas

- The slope of the tangent to the graph $y = f(x)$ at point $P(a, f(a))$ is
  $m = \lim_{x \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$.
- Average velocity $= \frac{\text{change in position}}{\text{change in time}}$
- The (instantaneous) velocity of an object, represented by position function $s(t)$, at time $t = a$, is $v(a) = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{h \to 0} \frac{s(a + h) - s(a)}{h}$.
- If $f$ is a polynomial function, then $\lim_{x \to a} f(x) = f(a)$.
- The function $f(x)$ is continuous at $x = a$ if $f(a)$ is defined and if $\lim_{x \to a} f(x) = f(a)$.

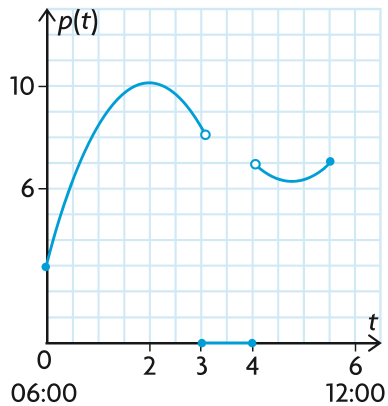
# Review Exercise

1. Consider the graph of the function $f(x) = 5x^2 - 8x$.
   a. Find the slope of the secant that joins the points on the graph given by $x = -2$ and $x = 3$.
   b. Determine the average rate of change as $x$ changes from $-1$ to 4.
   c. Find an equation for the line that is tangent to the graph of the function at $x = 1$.
2. Calculate the slope of the tangent to the given function at the given point or value of $x$.
   a. $f(x) = \dfrac{3}{x+1}, P(2, 1)$
   b. $g(x) = \sqrt{x+2}, P(-1, 1)$
   c. $h(x) = \dfrac{2}{\sqrt{x+5}}, P\left(4, \dfrac{2}{3}\right)$
   d. $f(x) = \dfrac{5}{x-2}, P\left(4, \dfrac{5}{2}\right)$
3. Calculate the slope of the graph of $f(x) = \begin{cases} 4 - x^2, \text{ if } x \leq 1 \\ 2x + 1, \text{ if } x > 1 \end{cases}$ at each of the following points:
   a. $P(-1, 3)$
   b. $P(2, 5)$
4. The height, in metres, of an object that has fallen from a height of 180 m is given by the position function $s(t) = -5t^2 + 180$, where $t \geq 0$ and $t$ is in seconds.
   a. Find the average velocity during each of the first two seconds.
   b. Find the velocity of the object when $t = 4$.
   c. At what velocity will the object hit the ground?
5. After $t$ minutes of growth, a certain bacterial culture has a mass, in grams, of $M(t) = t^2$.
   a. How much does the bacterial culture grow during the time $3 \leq t \leq 3.01$?
   b. What is its average rate of growth during the time interval $3 \leq t \leq 3.01$?
   c. What is its rate of growth when $t = 3$?
6. It is estimated that, $t$ years from now, the amount of waste accumulated $Q$, in tonnes, will be $Q(t) = 10^4(t^2 + 15t + 70), 0 \leq t \leq 10$.
   a. How much waste has been accumulated up to now?
   b. What will be the average rate of change in this quantity over the next three years?

c. What is the present rate of change in this quantity?

d. When will the rate of change reach $3.0 \times 10^5$ per year?

7. The electrical power $p(t)$, in kilowatts, being used by a household as a function of time $t$, in hours, is modelled by a graph where $t = 0$ corresponds to 06:00. The graph indicates peak use at 08:00 and a power failure between 09:00 and 10:00.




a. Determine $\lim_{t \to 2} p(t)$.

b. Determine $\lim_{t \to 4^+} p(t)$ and $\lim_{t \to 4^-} p(t)$.

c. For what values of $t$ is $p(t)$ discontinuous?

8. Sketch a graph of any function that satisfies the given conditions.

a. $\lim_{x \to -1} f(x) = 0.5$, $f$ is discontinuous at $x = -1$

b. $f(x) = -4$ if $x < 3$, $f$ is an increasing function when $x > 3$, $\lim_{x \to 3^+} f(x) = 1$

9. a. Sketch the graph of the following function:

$$f(x) = \begin{cases} x + 1, \text{ if } x < -1 \\ -x + 1, \text{ if } -1 \le x < 1 \\ x - 2, \text{ if } x > 1 \end{cases}$$

b. Find all values at which the function is discontinuous.

c. Find the limits at those values, if they exist.

10. Determine whether $f(x) = \frac{x^2 + 2x - 8}{x + 4}$ is continuous at $x = -4$.

11. Consider the function $f(x) = \frac{2x - 2}{x^2 + x - 2}$.

a. For what values of $x$ is $f$ discontinuous?

b. At each point where $f$ is discontinuous, determine the limit of $f(x)$, if it exists.

12. Use a graphing calculator to graph each function and estimate the limits, if they exist.

a. $f(x) = \frac{1}{x^2}, \lim_{x \to 0} f(x)$

b. $g(x) = x(x - 5), \lim_{x \to 0} g(x)$

c. $h(x) = \frac{x^3 - 27}{x^2 - 9}, \lim_{x \to 4} h(x)$ and $\lim_{x \to -3} h(x)$

13. Copy and complete each table, and use your results to estimate the limit. Use a graphing calculator to graph the function to confirm your result.

a. $\lim_{x \to 2} \frac{x - 2}{x^2 - x - 2}$

| $x$ | 1.9 | 1.99 | 1.999 | 2.001 | 2.01 | 2.1 |
|---|---|---|---|---|---|---|
| $\frac{x - 2}{x^2 - x - 2}$ | | | | | | |

b. $\lim_{x \to 1} \frac{x - 1}{x^2 - 1}$

| $x$ | 0.9 | 0.99 | 0.999 | 1.001 | 1.01 | 1.1 |
|---|---|---|---|---|---|---|
| $\frac{x - 1}{x^2 - 1}$ | | | | | | |

14. Copy and complete the table, and use your results to estimate the limit.

$\lim_{x \to 0} \frac{\sqrt{x + 3} - \sqrt{3}}{x}$

| $x$ | $-0.1$ | $-0.01$ | $-0.001$ | 0.001 | 0.01 | 0.1 |
|---|---|---|---|---|---|---|
| $\frac{\sqrt{x + 3} - \sqrt{3}}{x}$ | | | | | | |

Then determine the limit using an algebraic technique, and compare your answer with your estimate.

15. a. Copy and complete the table to approximate the limit of $f(x) = \frac{\sqrt{x + 2} - 2}{x - 2}$ as $x \to 2$.

| $x$ | 2.1 | 2.01 | 2.001 | 2.0001 |
|---|---|---|---|---|
| $f(x) = \frac{\sqrt{x + 2} - 2}{x - 2}$ | | | | |

b. Use a graphing calculator to graph $f$, and use the graph to approximate the limit.

c. Use the technique of rationalizing the numerator to find $\lim_{x \to 2} \frac{\sqrt{x + 2} - 2}{x - 2}$.

16. Evaluate the limit of each difference quotient. Interpret the limit as the slope of the tangent to a curve at a specific point.

a. $\lim_{h \to 0} \frac{(5 + h)^2 - 25}{h}$

b. $\lim_{h \to 0} \frac{\sqrt{4 + h} - 2}{h}$

c. $\lim_{h \to 0} \frac{\frac{1}{(4 + h)} - \frac{1}{4}}{h}$

17. Evaluate each limit using one of the algebraic methods discussed in this chapter, if the limit exists.

a. $\lim_{x \to -4} \frac{x^2 + 12x + 32}{x + 4}$

b. $\lim_{x \to a} \frac{(x + 4a)^2 - 25a^2}{x - a}$

c. $\lim_{x \to 0} \frac{\sqrt{x + 5} - \sqrt{5 - x}}{x}$

d. $\lim_{x \to 2} \frac{x^2 - 4}{x^3 - 8}$

e. $\lim_{x \to 4} \frac{4 - \sqrt{12 + x}}{x - 4}$

f. $\lim_{x \to 0} \frac{1}{x}\left(\frac{1}{2 + x} - \frac{1}{2}\right)$

18. Explain why the given limit does not exist.

a. $\lim_{x \to 3} \sqrt{x - 3}$

b. $\lim_{x \to 2} \frac{x^2 - 4}{x^2 - 4x + 4}$

c. $f(x) = \begin{cases} -5, \text{ if } x < 1 \\ 2, \text{ if } x \geq 1 \end{cases}$; $\lim_{x \to 1} f(x)$

d. $\lim_{x \to 2} \frac{1}{\sqrt{x - 2}}$

e. $\lim_{x \to 0} \frac{|x|}{x}$

f. $f(x) = \begin{cases} 5x^2, \text{ if } x < -1 \\ 2x + 1, \text{ if } x \geq -1 \end{cases}$; $\lim_{x \to -1} f(x)$

19. Determine the equation of the tangent to the curve of each function at the given value of $x$.

a. $y = -3x^2 + 6x + 4$ where $x = 1$

b. $y = x^2 - x - 1$ where $x = -2$

c. $f(x) = 6x^3 - 3$ where $x = -1$

d. $f(x) = -2x^4$ where $x = 3$

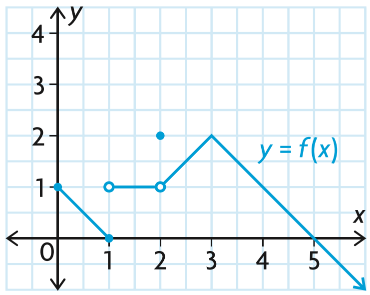
20. The estimated population of a bacteria colony is $P(t) = 20 + 61t + 3t^2$, where the population, $P$, is measured in thousands at $t$ hours.

a. What is the estimated population of the colony at 8 h?

b. At what rate is the population changing at 8 h?

# CHAPTER 1 TEST

1. Explain why $\lim_{x \to 1} \frac{1}{x - 1}$ does not exist.
2. Consider the graph of the function $f(x) = 5x^2 - 8x$. Calculate the slope of the secant that joins the points on the graph given by $x = -2$ and $x = 1$.
3. For the function shown below, determine the following:




   a. $\lim_{x \to 1} f(x)$

   b. $\lim_{x \to 2} f(x)$

   c. $\lim_{x \to 4^-} f(x)$

   d. values of $x$ for which $f$ is discontinuous

4. A weather balloon is rising vertically. After $t$ hours, its distance above the ground, measured in kilometres, is given by the formula $s(t) = 8t - t^2$.
   a. Determine the average velocity of the weather balloon from $t = 2$ h to $t = 5$ h.
   b. Determine its velocity at $t = 3$ h.
5. Determine the average rate of change in $f(x) = \sqrt{x + 11}$ with respect to $x$ from $x = 5$ to $x = 5 + h$.
6. Determine the slope of the tangent at $x = 4$ for $f(x) = \frac{x}{x^2 - 15}$.
7. Evaluate the following limits:

   a. $\lim_{x \to 3} \frac{4x^2 - 36}{2x - 6}$

   b. $\lim_{x \to 2} \frac{2x^2 - x - 6}{3x^2 - 7x + 2}$

   c. $\lim_{x \to 5} \frac{x - 5}{\sqrt{x - 1} - 2}$

   d. $\lim_{x \to -1} \frac{x^3 + 1}{x^4 - 1}$

   e. $\lim_{x \to 3} \left( \frac{1}{x - 3} - \frac{6}{x^2 - 9} \right)$

   f. $\lim_{x \to 0} \frac{(x + 8)^{\frac{1}{3}} - 2}{x}$

8. Determine constants $a$ and $b$ such that $f(x)$ is continuous for all values of $x$.

$$f(x) = \begin{cases} ax + 3, \text{ if } x > 5 \\ 8, \text{ if } x = 5 \\ x^2 + bx + a, \text{ if } x < 5 \end{cases}$$